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Nonparametric Estimation of Quantile-Based Cumulative Tsallis Information Measures

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Abstract

In this paper, we study the nonparametric estimation of quantile-based cumulative Tsallis entropy under a complete sample case. The asymptotic properties of the proposed two nonparametric estimators are studied based on some regularity conditions. The performance and usefulness of the estimators are examined through simulated and real data sets.

Keywords: cumulative Tsallis entropy, quantile function, reliability measures, kernel based density estimation.

1. Introduction

If the system is out of equilibrium or the components highly depend on each other then the entropy proposed by Tsallis (see Tsallis (1988)) is found to be a better choice for measurement of uncertainty. For a non-negative continuous random variable X, Tsallis entropy of order α is defined as

$$\mathcal{T}_\alpha(X) = \tfrac{1}{\alpha-1} E\left(1-(f(X))^{\alpha-1}\right) = \tfrac{1}{\alpha-1}\left(\int_0^\infty (f(x)-f^\alpha(x))\,dx\right), \tag{1}$$

where $0<\alpha\neq 1$ and $f(.)$ denotes the probability density function (pdf) of X. Clearly, when $\alpha\to 1$, $\mathcal{T}_\alpha(X)\to\mathcal{H}(X) = -\int_0^\infty (\log f(x))\, f(x)dx$, the Shannon differential entropy (Shannon 1948). Tsallis entropy has been found applications in various fields such as statistical mechanics, thermodynamics, communication theory, image processing, reliability etc. For more details, one can refer to Cartwright (2014), Kumar (2016) and the references therein.

Sati and Gupta (2015) introduced a cumulative version of Tsallis entropy (CTE1),

$$\eta_\alpha(X) = \frac{1}{\alpha - 1}\left[1 - \int_0^\infty \bar{F}^\alpha(x)dx\right]. \tag{2}$$

obtained by replacing the probability density function (pdf) in Tsallis (1988) entropy by the survival function, as a generalized measure of uncertainty in the context of reliability and survival analysis. They obtained certain characterization results of CTE1 and showed that when $\alpha \to 1$, $\eta_\alpha(X) \to \mathscr{E}(X)$, the well known cumulative residual entropy (CRE) due to Rao, Chen, Vemuri, and Wang (2004). Later, Rajesh and Sunoj (2019) proposed a modified and alternative form to CTE1, called as cumulative Tsallis entropy (CTE2) defined as,

$$\zeta_\alpha(X) = \frac{1}{\alpha - 1}\int_0^\infty \left(\bar{F}(x) - \bar{F}^\alpha(x)\right)dx. \tag{3}$$

As $\alpha \to 1$, $\zeta_\alpha(X) \to \mathscr{E}(X)$. Although CTE1 and CTE2 are connected through $\eta_\alpha(X) = \zeta_\alpha(X) + \frac{1-E(X)}{\alpha-1}$ and they coincide when $E(X) = 1$ the two measures behave differently in the cumulative setting. Both were originally introduced as cumulative versions of Tsallis entropy and, in principle, both are expected to converge to the cumulative residual entropy $\mathscr{E}(X)$ when $\alpha \to 1$. However, Raju, Sunoj, and Rajesh (2020) illustrated that only CTE2 is showing the limiting behaviour to $\mathscr{E}(X)$ when $\alpha \to 1$. This discrepancy arises because, unlike the density-based formulation where $\int_0^\infty f(x)dx = 1$, in the survival-based framework we have $\int_0^\infty \bar{F}(x)dx \neq 1$ in general. Consequently, the expression for CTE1 contains a constant term 1 in place of $\int_0^\infty \bar{F}(x)dx$ which prevents it from serving as a valid generalization of CRE. Moreover, CTE1 can take negative values for certain distributions, contradicting the non-negativity property required for cumulative residual entropy (Rao *et al.* 2004). In contrast, CTE2 satisfies these desirable properties, making it a more appropriate cumulative analogue of Tsallis entropy. Our simulation results in Section 5 further confirm that these advantages of CTE2 persist in their quantile-based counterparts.

Recently, the quantile function (QF) approach is found to be a useful and an effective alternative to the traditional distribution function approach (see Gilchrist (2000), Nair, Sankaran, and Balakrishnan (2013)). The QF is defined as

$$Q(u) = F^{-1}(x) = \inf\{x|F(x) \geq u\},\ 0 \leq u \leq 1. \tag{4}$$

The quantile function approach provides a new methodology in the study of information measures. Nair *et al.* (2013) provide an extensive treatment of quantile-based reliability measures and the underlying properties of the quantile function. A fundamental tool in this framework is the quantile density function $q(u) = \frac{d}{du}Q(u)$, the derivative of the quantile function, which plays a crucial role in the formulation of several quantile-oriented reliability and information measures.

The quantile approach also provides simple expressions in calculating the entropies and also in cases where the distribution function do not have a closed form. Also, QF have certain unique properties that are not satisfied by the distribution function. Quantile-based study of information measures are of recent interest. For more recent works on it, we refer to Sankaran and Sunoj (2017), Sunoj, Krishnan, and Sankaran (2017), Sunoj, Krishnan, and Sankaran (2018), Kumar, Rani, and Singh (2022) and the references therein. A quantile version of CTE1 (QCTE1) was developed by Sunoj *et al.* (2017), given by

$$\tau_\alpha(X) = \frac{1}{\alpha - 1}\left\{1 - \int_0^1 (1-u)^\alpha q(u)du\right\}. \tag{5}$$

Similarly, a quantile version of CTE2 (QCTE2) was proposed by Kumar *et al.* (2022), defined by

$$\gamma_\alpha(X) = \frac{1}{\alpha-1}\left\{\int_0^1 \left[(1-u)-(1-u)^\alpha\right] q(u)du\right\}. \tag{6}$$

The quantile-based cumulative Tsallis entropy functions based on (2) and (3) can also be expressed by using a dual concept of the quantile density function defined in Capaldo, Di Crescenzo, and Pellerey (2024). Specifically, let X be a random variable with absolutely continuous survival function $\bar{F}(\cdot)$, probability density function $f(\cdot)$ and quantile function $Q(\cdot)$. If $\bar{F}^{-1}(u) = Q(1-u)$, for all $u \in [0,1]$, then the dual quantile-density function of X is defined as

$$\tilde{q}(u) = -\frac{d}{du}\bar{F}^{-1}(u) = \frac{1}{f\left(\bar{F}^{-1}(u)\right)},\ u \in [0,1],$$

see Capaldo *et al.* (2024). Therefore, by setting $u = F(x)$ in (2), using dual quantile function it follows

$$\eta_\alpha(X) = \frac{1}{\alpha-1}\left[1-\int_0^1 u^\alpha \tilde{q}(u)du\right]$$

and similarly for $\zeta_\alpha(X)$ in (3).

Various types of estimators and estimation techniques are available for different entropy measures using the quantile function. Vasicek (1976) proposed a nonparametric estimator using the quantile version of the differential entropy function. Later, this estimator was modified by Ebrahimi, Pflughoeft, and Soofi (1994). Recently, Subhash, Sunoj, Sankaran, and Rajesh (2023) studied the nonparametric estimation of quantile-based entropy function using kernel-based estimators. Although variety of works have been appeared in the literature studying different properties of CTE1 and CTE2 based on distribution and quantile functions, however, a detailed investigation of estimating their quantile versions have not been considered so far. Accordingly, the present paper considers the nonparametric estimation of QCTE1 and QCTE2, study their asymptotic properties and performance based on numerical data.

The remainder of the paper is as follows. Section 2 presents the crude estimators for the quantile-based cumulative Tsallis entropy measures QCTE1 and QCTE2 together with their asymptotic properties. Section 3 introduces the kernel-based plug-in estimators corresponding to QCTE1 and QCTE2 and discusses their consistency and asymptotic normality. Section 4 provides a comparative assessment of the proposed estimators based on simulation results. Section 5 illustrates the practical usefulness of the estimators through real data analysis, and the concluding remarks are presented in the final section.

2. The crude estimators for QCTE1 and QCTE2

In this section, we introduce crude estimators for QCTE1 and QCTE2 and study their various properties.

2.1. Definition of crude estimators of QCTE1 and QCTE2

Let $X_1, X_2, ..., X_n$ be independent and identically distributed (iid) non-negative random variables that represent the lifetimes of n components. Let $F(x)$ and $Q(u)$ be their common distribution function and quantile function respectively and $X_{1;n}, X_{2;n}, ..., X_{n;n}$ denote their order statistics. Parzen (1979) defined an empirical quantile function, given by

$$\bar{Q}(u) = X_{r;n},\ \text{for}\ \frac{r-1}{n} < u < \frac{r}{n},\ r = 1, 2, ..., n. \tag{7}$$

$\bar{Q}(u)$ in (7) is a step function with jump $\frac{1}{n}$. A smoothed version of this estimator was also developed by Parzen (1979) which takes the form,

$$\hat{Q}_n(u) = n\left(\frac{r}{n} - u\right) X_{r-1;n} + X_{r;n}\left(u - \frac{r-1}{n}\right) n \tag{8}$$

for $\frac{r-1}{n} < u < \frac{r}{n}$, $r = 1, 2, ..., n$. Differentiating (8), we obtain the empirical estimator for quantile density function,

$$\bar{q}_n(u) = n\left(X_{r;n} - X_{r-1;n}\right); \text{ for } \frac{r-1}{n} < u < \frac{r}{n}, \tag{9}$$

in terms of the spacings, which was extensively studied by Pyke (1965) in which he postulated that, the asymptotic independence and exponentiality of spacings holds for any random variable. One can also see that the spacings (9) holds a close relationship with exponential distribution. Substituting (9) in (5), we obtain,

$$\bar{\tau}^*_\alpha(X) = \frac{1}{\alpha-1}\left\{1 - \int_0^1 (1-u)^\alpha \bar{q}_n(u)du\right\}, \tag{10}$$

the empirical estimator for QCTE1. Changing the integral into summation, (10) modified to,

$$\bar{\tau}^{**}_\alpha(X) = \frac{1}{\alpha-1}\left\{1 - \sum_{i=1}^{n}(1-u_i)^\alpha \bar{q}_n(u_i)\right\} \int_{X_{(i-1)}}^{X_{(i)}} du, \tag{11}$$

where $u_i, i = 1, 2, .., n$ is the ordered uniform random variable corresponding to each $\bar{q}_n(u_i)$. The above formula works well for simulated studies as we use uniform random variables for simulating other distributions. However, in the case of real data, this technique is not directly applicable. Specifically, since u_i corresponds to ordered uniform variables in simulated settings, mapping them to the empirical survival values $\bar{F}_n(\bar{Q}_n(u_i))$ ensures consistency when the estimator is applied to real observations. This transformation is justified by the fact that any observation x can be expressed as $f(f^{-1}(x))$ for a one-to-one function. Then (11) modified to,

$$\bar{\tau}_\alpha(X) = \frac{1}{\alpha-1}\left\{1 - \sum_{i=1}^{n}(1 - \bar{F}_n(\bar{Q}_n(u_i)))^\alpha \bar{q}_n(u_i)\right\} \int_{X_{(i-1)}}^{X_{(i)}} du, \tag{12}$$

where $\bar{F}_n(t) = I(X_i \geq t)$ is the empirical estimator for $\bar{F}(t)$. Substituting (7) in (12) we get the empirical distribution of the ordered statistic $\frac{i}{n}$ for all $i = 1, 2, ...n$. Further, it is easy to determine, $X_{(i)} - X_{(i-1)} = \frac{1}{n}$ for all i. Substituting all those values in (12), we have the re-substitution estimator of the form,

$$\bar{\tau}_\alpha(X) = \frac{1}{\alpha-1}\sum_{i=1}^{n}\left\{\frac{1}{n} - \left(1 - \frac{i}{n}\right)^\alpha \left(X_{(i;n)} - X_{(i-1;n)}\right)\right\}. \tag{13}$$

2.2. Asymptotic properties of QCTE1

To study the asymptotic properties of QCTE1 $\bar{\tau}_\alpha(X)$, we set $X_{0;n} = 0$. We consider a sequence,

$$C_n(w) = \sum_{i=0}^{n-1}\left[w(i)\right]\left(X_{i+1;n} - X_{i;n}\right), \tag{14}$$

where $w : [0, 1] \to R$ is a function defined as $w(x) = (1-x)^\alpha$. Using simple substitution one can easily transform (14) into,

$$C_n(w) = \sum_{i=1}^{n} \left\{ \left(1 - \frac{i-1}{n}\right)^{\alpha} - \left(1 - \frac{i}{n}\right)^{\alpha} \right\} X_{i;n} + \left(\frac{1}{n}\right)^{\alpha} X_{n;n}. \tag{15}$$

To examine the almost sure convergence and the asymptotic normality of $C_n(w)$ in (15) we refer to the following lemma from Cali, Longobardi, Macci, and Pacchiarotti (2022).

Lemma 1. *The function $w : [0,1] \to \mathcal{R}$ is continuous and there exist $x_0 \in (0,1)$, $\beta \in (0,1)$ and $c > 0$ such that $|w(x)| \le c(1-x)^{\beta}$ for all $x \in [1-x_0, 1]$.*

It is easy to show that, $w(x) = (1-x)^{\alpha}$ satisfy this condition. In the next theorem we show the almost sure convergence of the estimator defined in (15) by setting $w(x) = (1-x)^{\alpha}$.

Theorem 1. *Assume that Lemma 1 holds. Let $\{X_n : n \ge 1\}$ be iid sequence of positive random variable in some probability space L^p for some p such that $\beta p > 1$, with a common distribution function F. Then we have,*

$$C_n(w) \xrightarrow{a.s} \int_0^1 (1-u)^{\alpha} q(u) du. \tag{16}$$

Proof. Following the lines of Theorem 9 in Rao *et al.* (2004), we have

$$C_n(w) = \int_0^{\infty} (1 - \hat{F}_n(x))^{\alpha} dz,$$

where $\hat{F}_n(x)$ is the empirical distribution function. We take $b_0 > 0$ such that $F(b_0) \ge 1 - \frac{x_0}{2}$ where $x_0 \in (0,1)$, and using Glivenko-Cantelli lemma we get,

$$F(b_0) + \frac{x_0}{2} \ge \hat{F}_n(b_0) \ge F(b_0) - \frac{x_0}{2}.$$

Thus for all $z \ge b_0$, we have,

$$\hat{F}_n(z) \ge \hat{F}_n(b_0) \ge 1 - x_0$$

which yields,

$$1 - \hat{F}_n(z) \le \frac{\gamma}{z^p}$$

where $\gamma = sup_{n \ge 1} \frac{1}{n} \sum_{i=1}^{n} X_k^i < \infty$ almost sure in L^p. Using Lemma 1, we have

$$|(1 - \hat{F}_n(z))^{\alpha}| \le c \frac{\gamma^{\beta}}{z^{\beta p}}.$$

Using the Glivenko-Cantelli Lemma and dominated convergence theorem we can easily show that,

$$\int_0^{\infty} (1 - \hat{F}_n(z))^{\alpha} dz \xrightarrow{a.s} \int_0^{\infty} (1 - F(z))^{\alpha} dz, \tag{17}$$

as $n \to \infty$. Now by setting $z = Q(u)$ the RHS of (17) reduces to (16), which completes the proof. This proof is an extension of Proposition 2.1 of Cali *et al.* (2022). □

Theorem 2. *For the iid random variables satisfying the conditions in Theorem 1*

$$\bar{\tau}_{\alpha}(X) \xrightarrow{a.s} \tau_{\alpha}(X). \tag{18}$$

Proof. Using the result, if $X \xrightarrow{a.s} c$ then for any continuous function $h(\cdot)$, $h(x) \xrightarrow{a.s} h(c)$, the proof holds. □

We can easily define an empirical estimator for $\gamma_\alpha(X)$ using the similar method applied for $\bar{\tau}_\alpha(X)$. The nonparametric empirical estimator for QCTE2 is given by,

$$\bar{\gamma}_\alpha(X) = \frac{1}{\alpha - 1}\left\{\sum_{i=1}^{n}\left(\left(1 - \frac{i}{n}\right) - \left(1 - \frac{i}{n}\right)^{\alpha}\right)(X_{i;n} - X_{i-1,n})\right\}. \tag{19}$$

Theorem 3. *For the iid random variables satisfying the conditions in Theorem 1*

$$\bar{\gamma}_\alpha(X) \xrightarrow{as} \gamma_\alpha(X). \tag{20}$$

Proof. See the proof of Theorem 2 as well as Theorem 1. □

The method we used to derive the estimators (13) and (19) are quite popular in literature. Various researchers have used the empirical method to estimate their cumulative entropy functions (see Rao *et al.* (2004), Sati and Gupta (2015), Di Crescenzo and Longobardi (2009) and Raju *et al.* (2020)). In a similar way, we have proposed empirical estimators for QCTE1 and QCTE2 that can be used in situations where distribution function do not have a closed form but their quantile function exist.

2.3. Simulation studies of empirical estimators

In this section, we examine the performance and relevance of QCTE1 and QCTE2 given in (13) and (19).

We employ simulation studies to asses the performance of the estimators defined in (13) and (19). The different sample sizes considered are $n = 20, 50, 100$ and 200 respectively. We have simulated 1000 samples from the selected model for each n. We use the power-Pareto model with parameters $(C, \lambda_1, \lambda_2)$ as our first model. This model does not have a closed form distribution function. The quantile function is given by

$$Q(u) = Cu^{\lambda_1}(1 - u)^{-\lambda_2},\ C,\ \lambda_1, \lambda_2 > 0. \tag{21}$$

The quantile function in (21) is used to model right skewed non-negative data (see Hankin and Lee (2006)). The absolute bias and MSE of $\bar{\tau}_\alpha(X)$ and $\bar{\gamma}_\alpha(X)$ corresponding to different parametric values C, λ_1, λ_2 of power-Pareto model and Tsallis index α are respectively listed in Tables 1 and 2. A close look at the tables reveal that, both absolute bias and MSE decreases with increase in sample size. This further solidifies the efficiency of the proposed empirical estimators $\bar{\tau}_\alpha(X)$ and $\bar{\gamma}_\alpha(X)$.

Another model that is also quite popular in the literature is the class of distribution function with linear hazard quantile function. The quantile function of this model is given by,

$$Q(u) = \frac{1}{a + b}\log\left(\frac{a + bu}{a(1 - u)}\right), a > 0, a + b > 0, 0 \leqslant u \leqslant 1. \tag{22}$$

This model has several fascinating properties. Some well known distributions can be obtained as the special case of this distribution. For more details one can refer to Midhu, Sankaran, and Nair (2014). The performance of the estimators can be obtained from Tables 3 and 4, in which both absolute bias and MSE decrease with the increase in sample size.

Table 1: The estimated value, absolute bias and MSE of $\bar{\tau}_\alpha(X)$ using power-Pareto distribution for specified values of $(C, \lambda_1, \lambda_2, \alpha)$

$(C, \lambda_1, \lambda_2, \alpha)$	n	$\bar{\tau}_\alpha(X)$	$Ab.Bias$	MSE
	50	0.410038	0.015515	0.056279
	100	0.479791	0.085400	0.021536
$\left(\frac{20}{9}, 2, \frac{1}{9}, 0.4\right)$	150	0.505542	0.058550	0.011866
	200	0.516672	0.048520	0.008910
	50	0.374421	0.004817	0.000247
	100	0.371021	0.001417	0.000120
$\left(\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, 3\right)$	150	0.369721	0.000117	0.000075
	200	0.369242	0.000112	0.000062
	50	0.496694	0.021176	0.004478
	100	0.484062	0.008544	0.002204
$\left(\frac{1}{2}, \frac{2}{3}, \frac{1}{2}, 2\right)$	150	0.478860	0.003342	0.001357
	200	0.477262	0.001744	0.001097
	50	0.360780	0.008700	0.000475
	100	0.357120	0.005039	0.000238
$\left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}, 3\right)$	150	0.355695	0.003614	0.000148
	200	0.355181	0.003100	0.000122

Table 2: The estimated value, absolute bias and MSE of $\bar{\gamma}_\alpha(X)$ using power-Pareto distribution for specified values of $(C, \lambda_1, \lambda_2, \alpha)$

$(C, \lambda_1, \lambda_2, \alpha)$	n	$\bar{\gamma}_\alpha(X)$	$Ab.Bias$	MSE
	50	0.937684	0.07779	0.010592
	100	0.974946	0.04053	0.003281
$\left(\frac{20}{9}, 2, \frac{1}{9}, 0.4\right)$	150	0.988910	0.02656	0.001825
	200	0.995611	0.01986	0.001176
	50	0.197881	0.046226	0.006558
	100	0.217505	0.026602	0.003657
$\left(\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, 3\right)$	150	0.229754	0.014353	0.002848
	200	0.235350	0.008756	0.002331
	50	0.186570	0.025045	0.001626
	100	0.196860	0.014755	0.000814
$\left(\frac{1}{2}, \frac{2}{3}, \frac{1}{2}, 2\right)$	150	0.202221	0.009394	0.000525
	200	0.204484	0.007131	0.000420
	50	0.399978	0.096977	0.027459
	100	0.441352	0.055603	0.015100
$\left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}, 3\right)$	150	0.466652	0.030303	0.011630
	200	0.478192	0.018763	0.009491

Table 3: The estimated value, absolute bias and MSE of $\bar{\tau}_\alpha(X)$ using linear hazard model for specified values of (a, b, α)

(a, b, α)	n	$\bar{\tau}_\alpha(X)$	$Ab.Bias$	MSE
	50	0.218407	0.244104	0.099573
	100	0.328185	0.134327	0.042086
$(0.6, 2, 0.4)$	150	0.384063	0.078448	0.024147
	200	0.412420	0.050092	0.017775
	50	0.435165	0.003736	0.000143
	100	0.433526	0.002097	0.000073
$(2, 2, 3)$	150	0.432884	0.001455	0.000046
	200	0.432674	0.001245	0.000038
	50	0.762718	0.013831	0.001772
	100	0.756860	0.007973	0.000896
$(2, 0.02, 2)$	150	0.754511	0.005624	0.000561
	200	0.753915	0.005029	0.000454
	50	0.409223	0.005269	0.000216
	100	0.406859	0.002905	0.000109
$(1, 4, 3)$	150	0.405933	0.001979	0.000068
	200	0.405596	0.001642	0.000058

Table 4: The estimated value, absolute bias and MSE of $\bar{\gamma}_\alpha(X)$ using linear hazard model for specified values of (a, b, α)

(a, b, α)	n	$\bar{\gamma}_\alpha(X)$	$Ab.Bias$	MSE
	50	0.733181	0.170446	0.044548
	100	0.812281	0.091345	0.019220
$(0.6, 2, 0.4)$	150	0.855035	0.048591	0.012017
	200	0.878930	0.024696	0.008857
	50	0.096508	0.008558	0.000224
	100	0.099999	0.005067	0.000106
$(2, 2, 3)$	150	0.101591	0.003476	0.000065
	200	0.102139	0.002927	0.000052
	50	0.218841	0.027581	0.001925
	100	0.230351	0.016071	0.000896
$(2, 0.02, 2)$	150	0.235719	0.010703	0.000543
	200	0.237784	0.008639	0.000425
	50	0.099216	0.006519	0.000153
	100	0.101766	0.003970	0.000075
$(1, 4, 3)$	150	0.102936	0.002800	0.000047
	200	0.103312	0.002423	0.000038

3. Kernel-based estimators for QCTE1 and QCTE2

In this section, we propose kernel-based estimators for QCTE1 and QCTE2, study their asymptotic properties and performance of the estimators are evaluated using simulation studies.

3.1. Kernel-based estimators of QCTE1 and QCTE2

Jones (1992) proposed a kernel-based density estimator for the quantile density function $q(u)$ as,

$$\hat{q_n}(u) = \frac{1}{f_n(Q_n(u))}, \tag{23}$$

where $f_n(x) = \frac{1}{nh_n}\sum_{i=1}^{n} K\left(\frac{x-X_i}{h_n}\right)$ is a kernel-based density estimator for $f(x)$, h_n is the bandwidth parameter and $K(.)$ is the kernel function. Then the kernel-based plug-in estimator for $\tau_\alpha(X)$ is given by,

$$\hat{\tau_\alpha}(X) = \frac{1}{\alpha-1}\left\{1-\int_0^1 (1-u)^\alpha \hat{q_n}(u)du\right\};\ 0 \leq u \leq 1,\ 0 \leq \alpha \neq 1. \tag{24}$$

3.2. Asymptotic properties

The following two theorems prove the consistency and normality properties of $\hat{\tau_\alpha}(X)$ given in (24).

Theorem 4. *Let $\hat{\tau_\alpha}(X)$ be the quantile-based estimator for CTE1 defined in equation (24) and $\hat{q_n}(u)$ be the estimator of $q(u)$ given in (23). Then the estimator $\hat{\tau_\alpha}(u)$ is consistent.*

Proof. Using the equation (24) and we have,

$$\tau_\alpha(X) - \hat{\tau_\alpha}(X) = \frac{1}{\alpha-1}\left\{1-\int_0^1 (1-u)^\alpha \left(q(u)-\hat{q_n}(u)\right)du\right\}. \tag{25}$$

Since $sup_u|q(u)-\hat{q}_n(u)| \to 0$ as $n \to \infty$ (see Soni, Dewan, and Jain (2012)) the proof is complete. □

Theorem 5. *$\sqrt{n}\left(\hat{\tau_\alpha}(X) - \tau_\alpha(X)\right)$ is asymptotically normal with mean zero and variance*

$$\sigma^2 = \frac{n}{(\alpha-1)^2} E\left(\int_0^1 (1-u)^\alpha \hat{q_n}(u)du\right)^2. \tag{26}$$

Proof. The proof is in line with Subhash *et al.* (2023) hence omitted. □

Similar to $\hat{\tau_\alpha}(u)$ defined in (24), we can define a kernel-based estimator for QCTE2 as,

$$\hat{\gamma_\alpha}(X) = \frac{1}{\alpha-1}\left\{\int_0^1 \left((1-u)-(1-u)^\alpha\right)\hat{q}(u)du\right\}. \tag{27}$$

The next two theorems prove the consistency and asymptotic normality properties of the estimator $\hat{\gamma_\alpha}(X)$ defined in (27). The proofs can be obtained following the similar steps as that of Theorems 4 and 5.

Theorem 6. *Let $\hat{\gamma_\alpha}(X)$ be the quantile-based estimator for CTE2 defined in (27) and $\hat{q_n}(u)$ be the estimator of $q(u)$ given in (23). Then the estimator $\hat{\gamma_\alpha}(X)$ is consistent.*

Theorem 7. $\sqrt{n}\left(\hat{\gamma}_\alpha(X)-\gamma_\alpha(X)\right)$ *is asymptotically normal with mean zero and variance*

$$\sigma^2=\frac{n}{(\alpha-1)^2}E\left(\int_0^1\left((1-u)-(1-u)^\alpha\right)\hat{q_n}(u)du\right)^2. \tag{28}$$

3.3. Simulation studies of the kernel-based estimators of QCTE1 and QCTE2

We carry out a simulation study to understand the performance of the proposed kernel estimators $\hat{\tau}_\alpha(X)$ and $\hat{\gamma}_\alpha(X)$ corresponding to QCTE1 and QCTE2. For comparative purposes, we have chosen the same quantile models viz., power-Pareto and class of distribution having linear hazard quantile function as we have done in the previous section based on the empirical estimators. We set Gaussian kernel as our kernel function. One of the important problems encountered while doing kernel estimation is the selection of the bandwidth parameter. We choose the bandwidth parameter h that gives minimum values for the MSE using trial and error method. The absolute bias and MSE of $\hat{\tau}_\alpha(X)$ and $\hat{\gamma}_\alpha(X)$ corresponding to the power-Pareto and linear hazard models with same parametric combination that we have used for empirical estimators are computed. Their results are shown in Tables 5, 6, 7 and 8 and it is evident that both absolute bias and MSE decrease with increase in sample size.

Table 5: The estimated value, absolute bias and MSE of $\hat{\tau}_\alpha(X)$ using power-Pareto distribution for specified values of $(C,\lambda_1,\lambda_2,\alpha)$

$(C,\lambda_1,\lambda_2,\alpha)$	n	$\hat{\tau}_\alpha(X)$	$Ab.Bias$	MSE
	50	0.813808	0.248617	0.418499
	100	0.632264	0.067073	0.084039
$\left(\frac{20}{9},2,\frac{1}{9},0.4\right)$	150	0.586005	0.020814	0.044955
	200	0.561428	0.003763	0.031603
	50	0.338958	0.030646	0.001433
	100	0.341765	0.027839	0.001000
$\left(\frac{1}{3},\frac{1}{3},\frac{2}{3},3\right)$	150	0.342566	0.027037	0.000870
	200	0.343107	0.026497	0.000813
	50	0.590155	0.065781	0.007936
	100	0.606085	0.049851	0.003801
$\left(\frac{1}{2},\frac{2}{3},\frac{1}{2},2\right)$	150	0.610511	0.045425	0.002797
	200	0.612777	0.043159	0.002437
	50	0.274294	0.045004	0.004749
	100	0.283671	0.035627	0.002315
$\left(\frac{2}{3},\frac{2}{3},\frac{1}{3},3\right)$	150	0.286453	0.032845	0.001674
	200	0.287715	0.031583	0.001476

Table 6: The estimated value, absolute bias and MSE of $\hat{\gamma}_\alpha(X)$ using power-Pareto distribution for specified values of $(C, \lambda_1, \lambda_2, \alpha)$

$(C, \lambda_1, \lambda_2, \alpha)$	n	$\hat{\gamma}_\alpha(X)$	$Ab.Bias$	MSE
	50	1.044327	0.028853	0.002318
	100	1.043217	0.027743	0.001495
$\left(\frac{20}{9}, 2, \frac{1}{9}, 0.4\right)$	150	1.043337	0.027863	0.001238
	200	1.042631	0.027157	0.001099
	50	0.249344	0.005237	0.000181
	100	0.248819	0.004712	0.000096
$\left(\frac{1}{3}, \frac{1}{3}, \frac{2}{3}, 3\right)$	150	0.248830	0.004723	0.000069
	200	0.248594	0.004487	0.000057
	50	0.218271	0.006656	0.000293
	100	0.216284	0.004669	0.000130
$\left(\frac{1}{2}, \frac{2}{3}, \frac{1}{2}, 2\right)$	150	0.215724	0.004109	0.000085
	200	0.215479	0.003864	0.000069
	50	0.488919	0.008036	0.000904
	100	0.487702	0.009253	0.000491
$\left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}, 3\right)$	150	0.487674	0.009281	0.000339
	200	0.487087	0.009868	0.000295

Table 7: The estimated value, absolute bias and MSE of $\hat{\tau}_\alpha(X)$ using linear hazard model for specified values of (a, b, α)

(a, b, α)	n	$\hat{\tau}_\alpha(X)$	$Ab.Bias$	MSE
	50	0.540494	0.077983	0.034401
	100	0.527479	0.064968	0.017681
$(0.6, 2, 0.4)$	150	0.524109	0.061598	0.012199
	200	0.519869	0.057358	0.009808
	50	0.396634	0.057358	0.001313
	100	0.399455	0.031974	0.001039
$(2, 2, 3)$	150	0.400208	0.031222	0.000985
	200	0.400592	0.030838	0.000958
	50	0.648911	0.099975	0.010973
	100	0.660264	0.088623	0.008166
$(2, 0.02, 2)$	150	0.663106	0.085781	0.007540
	200	0.664858	0.084028	0.007195
	50	0.339806	0.064148	0.004188
	100	0.340278	0.063676	0.004090
$(1, 4, 3)$	150	0.340361	0.063592	0.004066
	200	0.340556	0.063398	0.004036

Table 8: The estimated value, absolute bias and MSE of $\hat{\gamma}_\alpha(X)$ using linear hazard model for specified values of (a, b, α)

(a, b, α)	n	$\hat{\gamma}_\alpha(X)$	$Ab.Bias$	MSE
	50	1.031214	0.127587	0.016870
	100	1.030519	0.126892	0.016388
$(0.6, 2, 0.4)$	150	1.030607	0.126980	0.016308
	200	1.030188	0.126562	0.016164
	50	0.172645	0.067579	0.004694
	100	0.171777	0.066711	0.004513
$(2, 2, 3)$	150	0.171432	0.066366	0.004444
	200	0.171411	0.066345	0.004432
	50	0.253104	0.006681	0.000408
	100	0.251351	0.004929	0.000187
$(2, 0.02, 2)$	150	0.251011	0.004589	0.000127
	200	0.250766	0.004344	0.000103
	50	0.151858	0.046123	0.002421
	100	0.148818	0.043083	0.001971
$(1, 4, 3)$	150	0.147711	0.041975	0.001826
	200	0.147449	0.041714	0.001789

4. Comparing the estimators

By simple comparison of Table 1 to 8, it can be reasonably conclude that the proposed empirical and kernel estimators are in general efficient as both MSE and absolute bias decrease with increase in sample size. For both the cases, the estimated values are close to the true value as the absolute bias is negligible.

On comparing the performance of QCTE1, $\bar{\tau}_\alpha(X)$ and $\hat{\tau}_\alpha(X)$ for the power-Pareto distribution, we can see that $\bar{\tau}_\alpha(X)$ works moderately better compared to the $\hat{\tau}_\alpha(X)$. This property can be associated with the almost sure convergence of $\bar{\tau}_\alpha(X)$ with respect to the $\tau_\alpha(X)$. Now from Table 2 and 6 on QCTE2, it can be observed that both kernel and crude estimators *i.e.*, $\bar{\gamma}_\alpha(X)$ and $\hat{\gamma}_\alpha(X)$ show a considerably better performance. This allows us to conclude that, for the power-Pareto model, the crude estimator $\tau_\alpha(X)$ gives a better performance in the case of QCTE1, while both the crude and kernel shows good performance for QCTE2. We have also established the asymptotic normality of the kernel estimators using the Q-Q plot given in Figure 1.

Now for the linear hazard model, from Table 3, 4, 7 and 8, we can observe that, both estimators work nicely for the selected parameter values. However, on a close look we can see that, the crude estimators works better compared to the kernel estimators as the absolute bias and MSE of Table 3 and 4 is relatively smaller compared to the absolute bias and MSE of Table 7 and 8. This allow us to suggest crude estimators for estimating the models with linear hazard distribution.

5. Real data estimation and associated problems

One of the important applications of nonparametric estimation is to find the uncertainty associated with a data set without any knowledge on the underlying parametric model. In the case of quantile-based Shannon differential entropy function, Subhash *et al.* (2023) has given an application for a recurrent event data to study the performance of three systems based on the

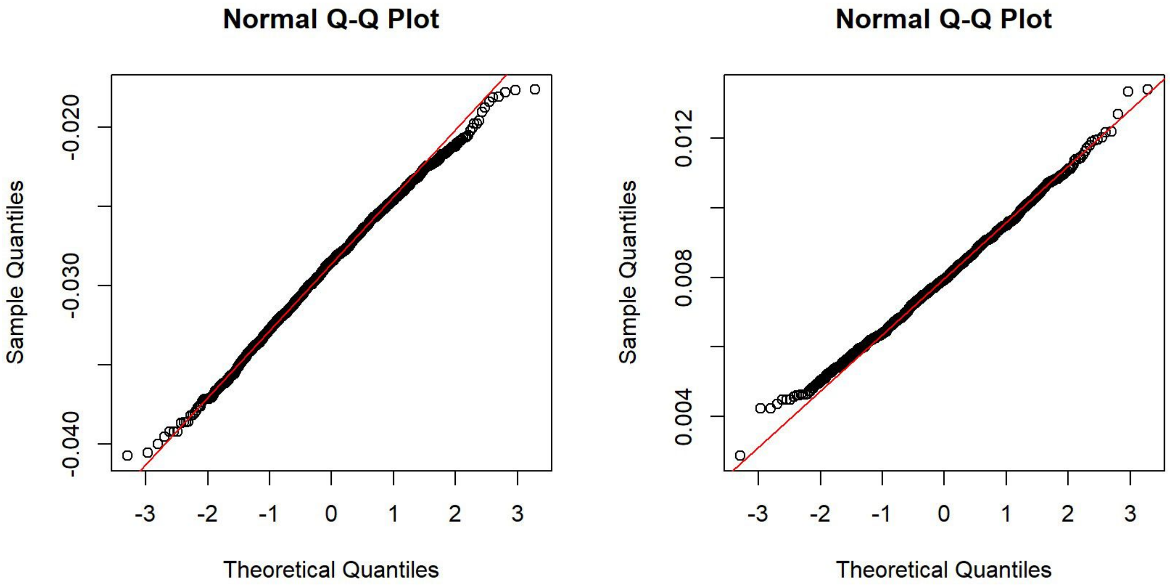

Figure 1: The qq-plot of $\hat{\tau}_\alpha(X)$ and $\hat{\gamma}_\alpha(X)$ for the power-Pareto distribution with the parameters $\left(\frac{1}{3}, \frac{1}{3}, \frac{2}{3}\right)$ and $\alpha = 2$ for $n = 500$

quantile-based entropy estimators. In a similar line, we consider a survival data set to illustrate the usefulness of the proposed estimators.

We chose the data set `veteran` from R-package `Survival`. The data set contains the survival times of lung cancer patients in days who received two treatments (standard and test). Since it is a survival data set, some observations are censored. To avoid the impact of censoring, we chose only those data points which are not censored.

The quantile-based cumulative Tsallis entropy QCTE1 and QCTE2 are calculated for the data set. The estimated values of both crude and kernel estimators are shown in Table 9 and 10 for different values of α.

Table 9: The uncertainty associated with treatment 1 and treatment 2 for CTE-1 and CTE-2 crude estimators

α	$\bar{\tau}_\alpha(X)$		$\bar{\gamma}_\alpha(X)$	
4	-7.6610	-6.0239	27.7609	31.220
2	-52.0969	-42.0710	54.1687	69.6633
1.3	-274.2438	-255.8043	79.9749	116.6439
1.001	-106166.2	-111576.1	99.3793	158.2692
0.999	1062756.2	1117502	99.46661	158.4677
0.54	383.0252	540.7247	152.013	297.8239
0.2	361.18	700.733	228.3479	561.0654

An important advantage of cumulative entropies over to the differential entropy is its non-negativity property, which quantify the uncertainty of a random phenomenon. Using Tables 9 and 10, it can be seen that for $\alpha > 1$, the estimated values of $\bar{\tau}_\alpha(X)$ takes negative values while for $\alpha < 1$, they are positive. On the other hand, the QCTE2 takes only positive values for any $\alpha \neq 1$.

Table 10: The uncertainty associated with treatment 1 and treatment 2 (in parenthesis) for CTE-1 and CTE-2 kernel estimators

α	$\hat{\tau}_\alpha(X)$		$\hat{\gamma}_\alpha(X)$	
4	-12.38717	-6.835923	26.85104	30.72522
2	-63.13723	-41.42099	57.68500	69.33786
1.3	-274.1613	-256.7673	77.70288	118.1022
1.001	-94406.93	-88352.21	98.57218	157.2551
0.999	9492.255	8883.210	99.07356	157.4534
0.54	372.3309	550.5121	147.0578	302.3461
0.2	356.8938	696.4743	225.7607	548.7732

Since Tsallis (1988) entropy is a generalized version of Shannon differential entropy, when $\alpha \to 1$, it reduces to the Shannon differential entropy. However, Raju *et al.* (2020) have numerically shown that only the cumulative Tsallis entropy due to Rajesh and Sunoj (2019) converges to the cumulative version of Shannon entropy, namely the cumulative residual entropy (CRE) of Rao *et al.* (2004) whereas the estimated values of CTE1 when $\alpha \to 1$ shows significant deviation from CRE. Similar deviation from quantile-based CRE is also exhibited by the crude and kernel estimators of QCTE1 as shown in Tables 9 and 10. To verify, we define the following crude and kernel estimators of quantile-based cumulative residual entropy (see Sankaran and Sunoj (2017)) applying the same procedures as in Section 2 and 3, given by

$$\phi_1(X) = -\sum_{i=1}^{n} \left\{ \left(1 - \frac{i}{n}\right) \log\left(1 - \frac{i}{n}\right) \left(X_{(i;n)} - X_{(i-1;n)}\right) \right\} \tag{29}$$

and

$$\phi_2(X) = -\int_0^1 \left((1-u)\log(1-u)\right) \hat{q}(u) du. \tag{30}$$

The estimated values of $\phi_1(X)$ obtained for treatment 1 and treatment 2 are 99.45821 and 158.4496 respectively. Similarly, the estimated values of $\phi_2(X)$ for the respective treatments are 98.61754 and 157.2731, clearly close to the estimated values of QCTE2 given in Tables 9 and 10. This confirm the closeness of QCTE2 to the quantile-based CRE when $\alpha \to 1$ in comparison with QCTE1. From Table 10, one can observe that $\hat{\tau}_\alpha(X)$ takes both positive and negative values while $\hat{\gamma}_\alpha(X)$ takes only positive values.

Based on a close evaluation of the crude and kernel-based estimates of QCTE2 denoted by $\bar{\gamma}_\alpha(X)$ and $\hat{\gamma}_\alpha(X)$ in Tables 9 and 10, it can be observed that the treatment 1 shows smaller uncertainty compared to treatment 2. Using the proposed estimators for QCTE2, the numerical study results allow us to conclude that the treatment 1 is relatively more efficient than treatment 2 as the former provides less uncertainty compared to latter.

6. Conclusion

In the present paper, we have considered two nonparametric estimators for quantile based-cumulative Tsallis entropy measures. The proposed estimators are useful in measuring the uncertainty contained in a random sample of observations, without making any distributional assumptions. The estimators were found to be efficient in terms of asymptotic properties and their validation using simulation study. We have also shown the usefulness of the estimators in comparing two treatments based on a survival data set.

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