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Nonparametric Relative Error Regression for LTRC Data

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Abstract

In the present work, we propose a new kernel estimator of the regression function based on the minimization of the mean squared relative error, when the response variable is subject to both random left truncation and right censoring (LTRC). Such variables typically appear in a medical or an engineering life test studies. Under classical conditions we establish the uniform consistency with a rate and the asymptotic normality for the estimator. The performance of the regression function estimator is evaluated on simulated data sets.

Keywords: asymptotic normality, Kernel estimator, nonparametric regression, relative error, strong uniform consistency rate, truncated−censored data.

1. Introduction

Survival analysis is a branch of statistics where the variable of interest, often a lifetime, represents the time elapsed between two events. In many cases, it may be impossible to observe this variable in its entirety. Among the various forms of incomplete data, random left truncation and right censoring (LTRC) are common in several fields, particularly in engineering and medicine. A detailed example is provided by Su and Wang (2012), who analyzed data from a multicenter HIV study conducted in Italy. Similarly, in engineering, Hong, Meeker, and McCalley (2009) used LTRC data to predict the remaining lifetime of electrical transformers. Let $(Y, T, W) \in \mathbb{R}_+^* \times \mathbb{R} \times \mathbb{R}$ denote a random vector which is defined on a common probability space $(\Omega, \mathscr{A}, \mathbb{P})$, where Y represents the lifetime variable of interest with continuous distribution function (df) F, T is the left truncation random variable with a df L and W is the right censoring random variable with a df G. Here, we assume that Y, T and W are mutually independent. On the one hand, and due to the right censoring mechanism, one observe $Z = min(Y, W)$ and the censoring indicator $\delta = \mathbb{1}_{\{Y \leq W\}}$, on the other hand, the left truncation mechanism requires that the probability of no truncation

$$\mu := \mathbb{P}(Z \geq T) \tag{1}$$

be strictly positive. Hence, the observation of Z is possible only if $Z \geq T$ and the observed triplet is (Z, T, δ). Let H denote the df of Z , then $H = 1 - (1 - F)(1 - G)$.
As we observe Z only if $Z \geq T$, then our results will be stated with respect to the conditional probability $\mathbf{P} = \mathbb{P}(.|Z \geq T)$, related to the space of observations.
Regarding the estimation method, we adopted a nonparametric approach which stands out for its flexibility, allowing for the modeling of complex relationships without strict assumptions about the functional form of the regression. This flexibility is particularly emphasized in the work of Yu and Lin (2008), where a nonparametric regression method based on local kernel estimating equations was successfully applied to analyze correlated failure time data. Compared to parametric models, which can be sensitive to violations of structural assumptions, nonparametric methods provide a robust and effective alternative, particularly for incomplete data such as LTRC. Let $\mathbf{X}$ be an $\mathbb{R}^d$-valued random vector of covariates with a joint probability density function (pdf) v.
We assume that the variables $\mathbf{X}$ and Y satisfy

$$Y = r(\mathbf{X}) + \epsilon$$

where $r(.)$ is an unknown function and ϵ is a random error variable independent of $\mathbf{X}$. In classical predictions, we estimate the operator r by minimizing the Mean Squared Error (MSE) :

$$\mathbb{E}\left[(Y - r(\mathbf{X}))^2|\mathbf{X}\right],$$

and we obtain as predictor the quantity $\mathbb{E}[Y|\mathbf{X}]$. However this loss function is unsuitable when the data contains some outliers, which is a relatively common case in practice. Therefore, in this work, we consider an alternative to estimate the function r, which is based on the minimization of the Mean Squared Relative Error (MSRE), defined for $Y > 0$, as

$$\mathbb{E}\left[\left(\frac{Y - r(\mathbf{X})}{Y}\right)^2|\mathbf{X}\right].$$

This choice is motivated by the higher resistance of the Mean Squared Relative Error (MSRE) to outliers, as it evaluates proportional rather than absolute errors. While M-estimators are also designed to be robust, they often result in implicit solutions that require iterative procedures for computation. In contrast, the Relative Error Regression (RER) approach provides an explicit expression for the regression function, simplifying both theoretical analysis and practical implementation. Park and Stefanski (1998) showed that minimizing this loss function leads to predicting the quantity

$$\frac{\mathbb{E}[Y^{-1}|\mathbf{X}]}{\mathbb{E}[Y^{-2}|\mathbf{X}]},$$

provided the first two conditional inverse moments of Y given X are finite. By way of notation, we write $\overline{r}_l(\mathbf{x}) := \mathbb{E}[Y^{-l}|\mathbf{X} = \mathbf{x}], l = 1, 2$, and we define the regression function as

$$r(\mathbf{x}) = \frac{\mathbb{E}[Y^{-1}|\mathbf{X} = \mathbf{x}]}{\mathbb{E}[Y^{-2}|\mathbf{X} = \mathbf{x}]} = \frac{\overline{r}_1(\mathbf{x})}{\overline{r}_2(\mathbf{x})}.$$

Jones, Park, Shin, Vines, and Jeong (2008) introduced and studied local constant and local linear nonparametric regression estimators when it is appropriate to assess performance in terms of mean squared relative error of prediction and they studied its asymptotic properties. Demongeot, Hamie, Laksaci, and Rachdi (2016) focused on the case where the explanatory variables are of functional type, they established the strong uniform consistency of a kernel estimator as well as its asymptotic normality. When the covariates are functional and the data are truncated, Altendji, Demongeot, Laksaci, and Rachdi (2018) constructed a new estimator of the RER function and investigated its almost sure consistency as well as its asymptotic normality. Khardani and Slaoui (2019) defined a new estimator of the RER when the response random variable is right censored, they established the asymptotic properties over a compact set, while Khardani (2019) considered this problem for twice censored data. Bouhadjera, Ould Saïd, and Remita (2019) proposed a new kernel estimator based on synthetic data of the mean squared relative error for the regression function. The authors establish the uniform almost sure convergence with rate over a compact set and its asymptotic normality, Fetitah, Attouch, Khardani, and Righi (2021) extended the RER framework to functional time series data under random censorship, proving strong consistency and asymptotic properties for their estimator. More recently, Khardani, Nefzi, and Thabet (2024) studied the relative error prediction from censored data under a mixing condition, establishing strong consistency results for the RER estimator. Bouhadjera, Ould Saïd, and Remita (2022) studied a local linear nonparametric regression estimator for censorship model and they established the uniform almost sure consistency result with rate over a compact set for the new estimate. In this paper, we adapt the previously developed estimators to our specific context by constructing a new estimator. The details of this construction are presented in Section 3. Subsequently, we provide strong uniform consistency rates and establish the asymptotic normality of our RER estimator under the LTRC model. The remainder of the paper is organized as follows. In Section 2, we recall some notations and we present the estimators under LTRC model. The main results, along with the assumptions, are listed in Section 3. In Section 4, we evaluate the performance of the estimator on simulated data. The proofs of the results are relegated to Section 5.

2. Model and notations

Let $\{(\mathbf{X}_i, Y_i); i = 1, ..., N\}$ be a sequence of independently and identically distributed (iid) random vectors distributed as $(\mathbf{X}, Y) \in \mathbb{R}^d \times \mathbb{R}_+^*$. Let $\{T_i; i = 1, ..., N\}$ and $\{W_i; i = 1, \ldots, N\}$ be two iid sequences of rv's sampled from G and L, respectively. In what follows, the star notation $(^\star)$ relates to any characteristic of the actually observed data (conditionally on $n = \sum_{i=1}^N \mathbb{1}_{\{Z_i \geq T_i\}}$). Then without possible confusion, we still denote by $\{(\mathbf{X}_i, Z_i, T_i, \delta_i); i = 1, \ldots, n\}$ the observed vectors from the original N-sample. It is clear that n is random but known and N is unknown but deterministic.
Furthermore, throughout this study, $\mathbf{E}$ and $\mathbb{E}$ will denote the expectation operators related to $\mathbf{P}$ and $\mathbb{P}$, respectively. Then conditionally on n, estimation results are stated considering $n \to \infty$ which remain true with respect to the probability $\mathbb{P}$ since $n \leq N$.
Now let us define

$$C(y) = \mathbf{P}(T \leq y \leq Z) = \mathbb{P}(T \leq y \leq Z | Z \geq T), \tag{2}$$

which can be estimated empirically by

$$C_n(y) = \frac{1}{n} \sum_{i=1}^n \mathbb{1}_{\{T_i \leq y \leq Z_i\}}.$$

Note that from (2), an elementary algebra calculus gives

$$C(y) = \frac{1}{\mu} L(y) \bar{F}(y) \bar{G}(y), \tag{3}$$

with $\bar{Q}(\cdot) := 1 - Q(\cdot)$, for any df Q and μ is that defined in (1).
Tsai, Jewell, and Wang (1987) proposed the well-known product limit estimator (PLE) F_n for F defined by

$$1 - F_n(y) = 1 - \prod_{i: Z_i \leq y} \left(1 - \frac{1}{n C_n(Z_i)}\right)^{\delta_i} =: \bar{F}_n(y),$$

the same authors showed that the estimator is asymptotically normal when the data are iid. Gijbels and Wang (1993) explained that this estimator can be seen as an average of iid random variables. Recall that Guessoum and Tatachak (2020) established asymptotic consistency results for the same estimators for associated data. the concomitant PLE of the df G, is defined by

$$1 - G_n(y) = 1 - \prod_{i:Z_i \le y} \left(1 - \frac{1}{nC_n(Z_i)}\right)^{1-\delta_i} =: \bar{G}_n(y)$$

while the PLE for the truncating law L proposed by Lynden-Bell (1971) is

$$L_n(t) = \prod_{i:T_i > t} \left(1 - \frac{1}{nC_n(T_i)}\right).$$

Remark 1. *When there is no left truncation* ($T = 0$)*,* F_n *reduces to the Kaplan-Meier PLE, and when there is no right censoring* ($W = \infty$) F_n *simplifies to the Lynden-Bell PLE.*

For any df Q, let $a_Q := \inf\{u : Q(u) > 0\}$ and $b_Q := \sup\{u : Q(u) < 1\}$ denote the endpoints of its support. In the present model, Gijbels and Wang (1993) highlighted that F can be estimated only if $a_L < a_H$, $b_L < b_H$, where $a_H = a_F \wedge a_G$, $b_H = b_F \wedge b_G$. Throughout this paper, we assume that (T, W) as independent from $(\mathbf{X}, Y)$. On the other hand, the ratio which represent the proportion of (no) truncated data $\frac{n}{N}$ can not be used to estimate μ since N is unknown. Hence, from (3) and following the idea in He and Yang (1998), an estimator for μ can be defined, namely

$$\mu_n = \frac{L_n(y)\bar{F}_n(y)\bar{G}_n(y)}{C_n(y)}, \tag{4}$$

for any argument y such that $C_n(y) \neq 0$. Under this model, the conditional sub-distribution function of $(\mathbf{X}, Z)$ is

$$\begin{aligned} F^{\star 1}_{\mathbf{X},Z}(\mathbf{x}, z) &= \mathbf{P}(\mathbf{X} \le \mathbf{x}, Z \le z, \delta = 1) \\ &= \frac{1}{\mu}\mathbb{P}(\mathbf{X} \le \mathbf{x}, Y \le z, Y \le W, Y \ge T) \\ &= \frac{1}{\mu}\int_{\mathbf{u}\le\mathbf{x}}\int_{a_L \le s \le z} L(s)\bar{G}(s) f_{\mathbf{X},Y}(\mathbf{u}, s)\mathbf{du}ds. \end{aligned}$$

where $f_{\mathbf{X},Y}(.,.)$ is the joint density function of (X, Y). By differentiation, we obtain

$$f_{\mathbf{X},Y}(\mathbf{x}, z) = \frac{\mu}{L(z)\bar{G}(z)} f^{\star 1}_{\mathbf{X},Z}(\mathbf{x}, z) \qquad z > a_L. \tag{5}$$

3. Definition of the estimator

To construct an estimator for $r(\cdot)$, let us define the function

$$\Psi_\phi(\mathbf{x}) := \int \phi(z) f_{\mathbf{X},Y}(\mathbf{x}, z)dz,$$

where $\phi(\cdot)$ is any function from $\mathbb{R}$ to $\mathbb{R}$ for which $\Psi_\phi(\cdot)$ is well defined. Further, $\phi(\cdot)$ will be explicitly defined if necessary. Using (5) and the kernel estimator of the sub-density $f^{\star 1}_{\mathbf{X},Z}(.,.)$, we have, as an estimator for $\Psi_\phi(\cdot)$,

$$\widetilde{\Psi}_{\phi n}(\mathbf{x}) := \frac{1}{nh_n^d}\sum_{i=1}^{n} \frac{\mu\delta_i}{L(Z_i)\overline{G}(Z_i)}\phi(Z_i)K_d\left(\frac{\mathbf{x} - \mathbf{X}_i}{h_n}\right), \tag{6}$$

for the $i's$ such that $L(Z_i) \neq 0$ and $\bar{G}(Z_i) \neq 0$. Here $K_d : \mathbb{R}^d \to \mathbb{R}^+$ is a multivariate kernel such that for any $\mathbf{t}_\ell = (t_\ell^1, \ldots, t_\ell^d)^\top \in \mathbb{R}^d$ and a real-valued univariate kernel K

$$\frac{1}{h_n^d} K_d\left(\frac{\mathbf{t}_\ell}{h_n}\right) := \prod_{k=1}^{d} \frac{1}{h_n} K\left(\frac{t_\ell^k}{h_n}\right).$$

Furthermore, h_n is a sequence of positive constants tending to zero when n tends to infinity, known as a bandwidth sequence which is assumed to be the same regardless of the k-th direction.

If we take $\phi(Z) = 1$ in (6) we can define an estimator $\tilde{v}_n(\cdot)$ of $v(\cdot)$

$$\tilde{v}_n(\mathbf{x}) = \frac{\mu}{nh_n^d} \sum_{i=1}^{n} \frac{\delta_i}{L(Z_i)\bar{G}(Z_i)} K_d\left(\frac{\mathbf{x} - \mathbf{X}_i}{h_n}\right).$$

By taking $\phi(Z) = Z^{-l}, l = 1, 2$ and $Z > 0$ in (6) we define an estimator for $r_l(\cdot)$, namely

$$\tilde{r}_{ln}(\mathbf{x}) = \frac{\mu}{nh_n^d} \sum_{i=1}^{n} \frac{\delta_i Z_i^{-l}}{L(Z_i)\bar{G}(Z_i)} K_d\left(\frac{\mathbf{x} - \mathbf{X}_i}{h_n}\right).$$

Consequently, an estimator for $\overline{r}_l(x)$ can be defined as

$$\tilde{\overline{r}}_{ln}(\mathbf{x}) = \frac{\tilde{r}_{ln}(\mathbf{x})}{\tilde{v}_n(\mathbf{x})} \mathbb{1}_{\{\tilde{v}_n(\mathbf{x}) \neq 0\}}.$$

Hence, an estimator for $r(x)$ can be given by

$$\tilde{r}_n(\mathbf{x}) = \frac{\frac{\mu}{nh_n^d} \sum_{i=1}^{n} \frac{\delta_i Z_i^{-1}}{L(Z_i)\bar{G}(Z_i)} K_d\left(\frac{\mathbf{x}-\mathbf{X}_i}{h_n}\right)}{\frac{\mu}{nh_n^d} \sum_{i=1}^{n} \frac{\delta_i Z_i^{-2}}{L(Z_i)\bar{G}(Z_i)} K_d\left(\frac{\mathbf{x}-\mathbf{X}_i}{h_n}\right)} =: \frac{\tilde{r}_{1n}(\mathbf{x})}{\tilde{r}_{2n}(\mathbf{x})}. \tag{7}$$

Note that from (3), the estimators $\tilde{r}_{ln}, l = 1, 2$ can be rewritten as

$$\tilde{r}_{ln}(\mathbf{x}) = \frac{1}{nh_n^d} \sum_{i=1}^{n} \frac{\delta_i Z_i^{-l} \bar{F}(Z_i)}{C(Z_i)} K_d\left(\frac{\mathbf{x} - \mathbf{X}_i}{h_n}\right) \tag{8}$$

Remark 2. *Since $\tilde{r}_n(\cdot)$ in (7) depends upon unknown quantities which make it useless in practice, we define a useful estimator $r_n(\cdot)$ by replacing G, L, μ, F and C by their estimates G_n, L_n, μ_n, F_n and C_n, respectively, i.e.*

$$r_n(\boldsymbol{x}) = \frac{\frac{\mu_n}{nh_n^d} \sum_{i=1}^{n} \frac{\delta_i Z_i^{-1}}{L_n(Z_i)\bar{G}_n(Z_i)} K_d\left(\frac{\boldsymbol{x}-\boldsymbol{X}_i}{h_n}\right)}{\frac{\mu_n}{nh_n^d} \sum_{i=1}^{n} \frac{\delta_i Z_i^{-2}}{L_n(Z_i)\bar{G}_n(Z_i)} K_d\left(\frac{\boldsymbol{x}-\boldsymbol{X}_i}{h_n}\right)} =: \frac{r_{1n}(\boldsymbol{x})}{r_{2n}(\boldsymbol{x})}. \tag{9}$$

Likewise, using (4) we have

$$r_{ln}(\boldsymbol{x}) = \frac{1}{nh_n^d} \sum_{i=1}^{n} \frac{\delta_i Z_i^{-l} \bar{F}_n(Z_i)}{C_n(Z_i)} K_d\left(\frac{\boldsymbol{x} - \boldsymbol{X}_i}{h_n}\right), l = 1, 2.$$

Remark 3. *The construction of the estimators in (7) was inspired by the estimators in the censored case cited in Bouhadjera et al. (2019), as well as by those in the truncated case presented in Altendji et al. (2018).*

Remark 4. *The following function will be of a great interest in the sequel and it is a special case of the function $\Psi_\phi(\cdot)$. Set $\phi_0(y) = \frac{\mu y^{-2l}}{L(y)\bar{G}(y)}$, we obtain*

$$\Psi_{\phi_0}(\boldsymbol{x}) = \int \frac{\mu y^{-2l}}{L(y)\bar{G}(y)} f_{\boldsymbol{X},Y}(\boldsymbol{x}, y) dy =: r_l^*(\boldsymbol{x}), l = 1, 2.$$

4. Main results

This section addresses the main results of the paper. We first introduce some related notations and commonly used assumptions.
Firstly, let Ω_0 be a compact subset of $\Gamma_0 = \{\mathbf{x} \in \mathbb{R}^d; \inf_{\mathbf{x}} v(\mathbf{x}) =: \gamma > 0\}$ and then, assume that $0 = a_L < a_H$ and $b_L < b_H$.

Remark 5. *Note that the restriction $a_L < a_H$ implies $L(Y) \geq L(a_H) > 0$ which guarantees $L_n(Y_i) \neq 0$ eventually, and then, the given estimators are well-defined.*

Throughout this paper, the letter C will denote a finite positive constant which is allowed to change from line to line.
The asymptotic results are derived under the following conditions.

4.1. Assumptions

(A1) For all $d \geq 1$, the bandwidth satisfies

- **(i)** $nh_n^d \to \infty$ and $\frac{\log^5 n}{nh_n^d} \to 0$ as $n \to \infty$;
- **(ii)** $h_n^d \log\log n = o(1)$;
- **(iii)** $nh_n^{d+4} \to \infty$.

(A2) The kernel K_d satisfies

- **(i)** K_d is a bounded pdf, compactly supported and Lipschitz continuous multivariate function.
- **(ii)** $\int_{\mathbb{R}^d} u_k K_d(\mathbf{u})\mathbf{du} = 0$ for $k = 1, \dots, d$.

(A3) The functions $r_l(\mathbf{x})$ and $r_l^*(\mathbf{x})$ are bounded, of class C^2 in Ω_0 with bounded first and second derivatives.

(A4) There exists $\nu > 2$ such that $\displaystyle\int_{\mathbb{R}} \frac{y^{-l\nu}}{(L(y)\bar{G}(y))^{(\nu-1)}} f(\mathbf{x}, y)\, dy < +\infty$.

Comments on the assumptions

Assumptions ($\mathbf{A}$1) and ($\mathbf{A}$2) are quite usual in kernel estimation setting. Assumption ($\mathbf{A}$3) is classical in studying bias terms and variance calculus under LTRC structure. Assumption ($\mathbf{A}$4) is needed for get asymptotic normality.

Proposition 1. *Under assumptions ($\boldsymbol{A}$1)(i),($\boldsymbol{A}$2)-($\boldsymbol{A}$3) we have*

$$\sup_{\boldsymbol{x}\in\Omega_0} |\tilde{r}_{ln}(\boldsymbol{x}) - \boldsymbol{E}[\tilde{r}_{ln}(\boldsymbol{x})]| = O\left(\sqrt{\frac{\log n}{nh_n^d}}\right) \ \mathbf{P}\text{-a.s., as n} \to \infty.$$

Theorem 1. *Under assumptions ($\boldsymbol{A}$1)(i), ($\boldsymbol{A}$2)-($\boldsymbol{A}$3) we have*

$$\sup_{\boldsymbol{x}\in\Omega_0} |r_n(\boldsymbol{x}) - r(\boldsymbol{x})| = O\left(\sqrt{\frac{\log n}{nh_n^d}} + \sqrt{\frac{\log\log n}{n}} + h_n^2\right) = O\left(\sqrt{\frac{\log n}{nh_n^d}} + h_n^2\right) \ \mathbf{P}\text{-a.s., as n} \to \infty,$$

The proof of Theorem 1 is mainly based on the decomposition in (18) below. It uses in part Proposition 1 and Lemmas 1 and 2. These lemmas deal with upper bounding $|\tilde{r}_{ln}(\mathbf{x}) - r_{ln}(\mathbf{x})|$ and the bias term $|\mathbf{E}[\tilde{r}_{ln}(\mathbf{x}, y)) - r_l(\mathbf{x}]|$, respectively.

Theorem 2. *Under assumptions (**A**1)-(**A**4), for n large enough we have*

$$\sqrt{nh_n^d}\,(r_n(\boldsymbol{x}) - r(\boldsymbol{x})) \xrightarrow{\mathscr{D}} \mathscr{N}\left(0, \sigma^2(\boldsymbol{x})\right),$$

where $\xrightarrow{\mathscr{D}}$ *denotes the convergence in distribution and* $\mathscr{N}\left(0, \sigma^2(\boldsymbol{x})\right)$ *is the centered gaussian distribution with variance*

$$\sigma^2(\boldsymbol{x}) = \mu\kappa\left[r_2^2(\boldsymbol{x})\Sigma_2(\boldsymbol{x}) - 2r_1(\boldsymbol{x})r_2(\boldsymbol{x})\Sigma_3(\boldsymbol{x}) + r_1^2(\boldsymbol{x})\Sigma_4(\boldsymbol{x})\right] r_2^{-4}(\boldsymbol{x}),$$

where

$$\kappa = \int K_d^2(\boldsymbol{u})\boldsymbol{du}\,,\ \Sigma_k(\boldsymbol{x}) = \int t^{-k}\frac{f(\boldsymbol{x},t)}{L(t)\overline{G}(t)}dt;\ k = 2,3,4.$$

Remark 6. *For simulation needs, the covariances* Σ_k*; k=2,3,4 will be estimated at* $\boldsymbol{x}$ *by*

$$\Sigma_{k,n}(\boldsymbol{x}) = \frac{\mu_n}{nh_n^d}\sum_{i=1}^{n}\frac{\delta_i K_d\left(\frac{\boldsymbol{x}-\boldsymbol{X}_i}{h_n}\right)}{(L_n(Z_i)\overline{G}_n(Z_i))^2}(Z_i)^{-k}; k = 2,3,4.$$

Remark 7. *Using Theorem 2, we can construct confidence intervals for* $r(\boldsymbol{x})$*. For that we define a plug-in-type estimator of* $\sigma^2(\boldsymbol{x})$

$${\sigma_n}^2(\boldsymbol{x}) = \mu_n\kappa\left[r_{2n}^2(\boldsymbol{x})\Sigma_{2n}(\boldsymbol{x}) - 2r_{1n}(\boldsymbol{x})r_{2n}(\boldsymbol{x})\Sigma_{3,n}(\boldsymbol{x}) + r_{1n}^2(\boldsymbol{x})\Sigma_{4,n}(\boldsymbol{x})\right] r_{2n}^{-4}(\boldsymbol{x}),$$

and we obtain the following corollary.

Corollary 4.1.1. *Under the assumptions of Theorem 2, a confidence interval of asymptotic level* $(1-\xi)$ *for* $r(\boldsymbol{x})$ *is*

$$\left[r_n(\boldsymbol{x}) - q_{(1-\frac{\xi}{2})}\sigma_n(\boldsymbol{x})\left(nh_n^d\right)^{-\frac{1}{2}},\ r_n(\boldsymbol{x}) + q_{(1-\frac{\xi}{2})}\sigma_n(\boldsymbol{x})\left(nh_n^d\right)^{-\frac{1}{2}}\right],$$

where $q_{(1-\xi/2)}$ *denotes the* $(1-\xi/2)$*-quantile of the standard normal distribution.*

5. Numerical application

This section is divided in three parts where we consider that the covariate variable is unidimensional ($d = 1$). The first one deals with the behavior of our RER estimator viz the target regression function $r(x) = \exp(x/2)$. In the second part, we illustrate the robustness of our estimator by introducing outliers in the sample and we compare with classical regression estimator obtained with the same data. The last part concerns the asymptotic normality of the estimator $r_n(x)$ for which we construct confidence intervals and we calculate the coverage probability.

Algorithm of sampling
In order to obtain an observed LTRC sequence $(X_i, Z_i, T_i, \delta_i); i = 1, \ldots, n$, we follow the steps below.

step 1 Generate X_i from $\mathcal{N}(1,1)$.

step 2 Compute $Y_i = \exp(X_i/2) + \epsilon_i$; $\epsilon_i \rightsquigarrow \mathcal{N}(0, 0.01)$.

step 3 Generate T_i and W_i from exponential distributions with parameters λ_1 and λ_2 respectively, (λ_1 and λ_2 are adjusted to get different values of truncation percentage (TP) and censoring percentage (CP)).

step 4 For $Z_i = \min(Y_i, W_i)$ and $\delta_i = \mathbb{1}_{\{Y_i \leq W_i\}}$, if $Z_i < T_i$, reject the vector $(X_i, Z_i, T_i, \delta_i)$ and go back to step 1 until $Z_i \geq T_i$. Hence we get the observed sample $(X_i, Z_i, T_i, \delta_i)$.

step 5 Repeat the above four steps to obtain a sample $(X_i, Z_i, T_i, \delta_i); i = 1, 2 \ldots, n$.

5.1. Consistency of $r_n(.)$

Procedure

The above procedure is repeated $B = 200$ times for each n-sample ($n = 50, 100, 300$). The values of λ_1 and λ_2 are adjusted to get $CP \approx 20\%, 40\%$ for fixed $TP \approx 10\%$ and $TP \approx 20\%, 40\%$ for fixed $CP \approx 10\%$. The Gaussian kernel was used along a grid of points $x_p; p = 1, \ldots, M$ belonging to the range $[-0.5; 2.5]$.

1. For each n-sample, along a grid of points $h(u); u = 1, \ldots, U$ belonging to the interval $[0.01; 2]$, we choose $h_{n,s}$ which minimizes the cross-validation criterion, commonly called *leave-one-out*, namely

$$CV_{RER}(h(u)) = \frac{1}{n}\sum_{i=1}^{n} \delta_i \left[\frac{Z_i - r_{n,s}^{(-i)}(X_i; h(u))}{Z_i}\right]^2,$$

where $r_{n,s}^{-i}(.)$ is the estimator of $r(.)$ calculated with the $(n-1)$ observations $\{(X_j, Z_j, T_j, \delta_j);\ j = 1, ..., i-1, i+1, ..., n\}$ at iteration s with $s = 1, \ldots, B$. The $B = 200$ curves of $r_{n,s}(x^{(p)}; h_{n,s})$ are plotted in Figures 1-3, where $r_{n,s}(x^{(p)}; h_{n,s})$ is the estimator of $r(x^{(p)})$ calculated with $h_{n,s}$ at iteration s.

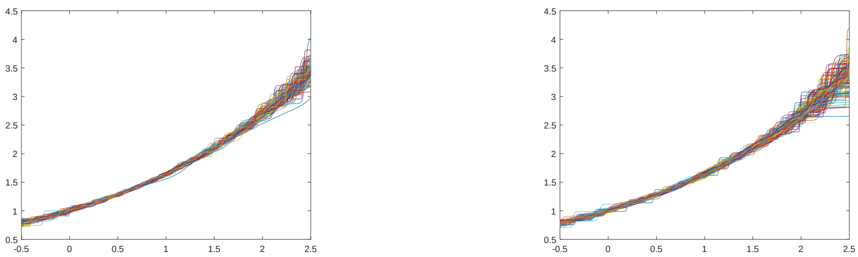


Figure 1: $B = 200$ curves (2D) of $r_{n,s}(x; h_{n,s})$ for $n = 100$ and $TP \approx 10\%$ with $CP \approx 20\%$ and 40% respectively

2. Moreover, for better visualization of the curves, the $r_{n,s}(x^{(p)}; h_{n,s}^{CV})$ are plotted separately in three dimensions in Figures 4-6.

3. Then we calculate

$$r_n(x^{(p)}) = \frac{1}{B}\sum_{s=1}^{B} r_{n,s}(x^{(p)}; h_{n,s}),$$

and for different values of TP, CP and n, the obtained estimators are plotted and compared with the target regression function $r(x) = \exp(x/2)$, in Figures 7-9.

4. To comfort this comparison, the performance of our estimator is quantified via the Global Mean Squared Error ($GMSE$) criterion (computed along $B = 200$ Monte Carlo

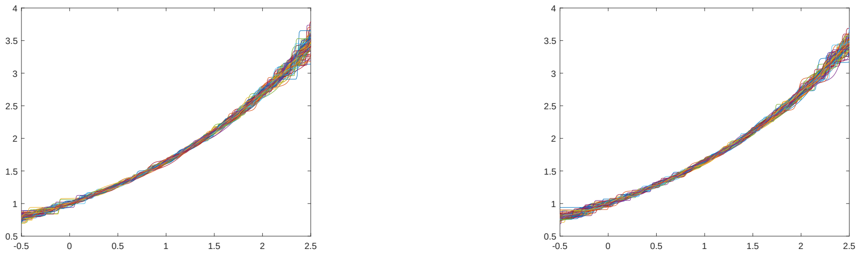


Figure 2: $B = 200$ curves (2D) of $r_{n,s}(x; h_{n,s})$ for $n = 100$ and $CP \approx 10\%$ with $TP \approx 20\%$ and 40% respectively

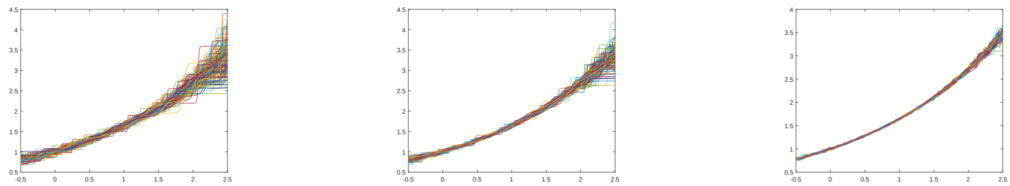


Figure 3: $B = 200$ curves (2D) of $r_{n,s}(x; h_{n,s})$ for $TP \approx 40\%$ and $CP \approx 40\%$ with $n = 50, 100$ and 300 respectively

trials), namely

$$GMSE = \frac{1}{BM} \sum_{s=1}^{B} \sum_{p=1}^{M} \left[r_{n,s}(x^{(p)}; h_{n,s}) - r(x^{(p)}) \right]^2. \qquad (10)$$

The corresponding results are summarized in Tables 1- 3, with the global bandwidth

$$h_n = \frac{1}{B} \sum_{s=1}^{B} h_{n,s}.$$

Comments on consistency simulation results

1. **Censorship effect:** From Figures 1, 4 and 7, it is clear that the quality of the estimate degrades for a greater CP which is quantified by the $GMSE$ values in Table 1. Indeed,

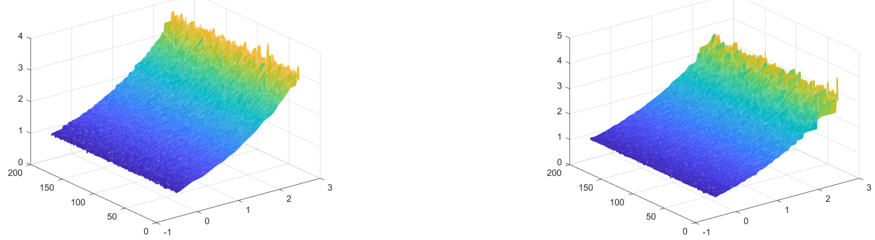


Figure 4: $B = 200$ curves (3D) of $r_{n,s}(x; h_{n,s})$ for $n = 100$ and $TP \approx 10\%$ with $CP \approx 20\%$ and 40% respectively

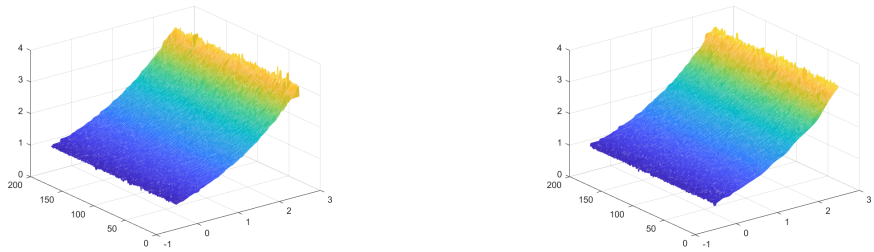


Figure 5: $B = 200$ curves (3D) of $r_{n,s}(x; h_{n,s})$ for $n = 100$ and $CP \approx 10\%$ with $TP \approx 20\%$ and 40% respectively

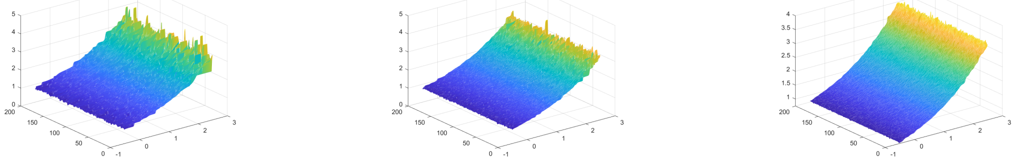


Figure 6: $B = 200$ curves (3D) of $r_{n,s}(x; h_{n,s})$ for $TP \approx 40\%$ and $CP \approx 40\%$ with $n = 50, 100$ and 300 respectively

the quality of the estimate is influenced by the CP (for a fixed TP) but it remains satisfactory nonetheless.

2. **Truncation effect:** From Table 2, we see that the effect of truncation on the estimate is slightly visible for a small size of n(=50) and attenuates when the sample size increase. In this case, for the LTRC model, the effect of truncation on the estimate is hidden by the presence of censoring as shown in Figures 2, 5 and 8.

3. **Sample size effect:** It is not surprising that the behavior of the estimation improves as the sample size n increases, as shown in Figures 3, 6 and 9. Indeed, Tables 1- 2 confirm this finding with the decrease of the GMSE when n increases.

4. **Bandwidth:** According to Tables 1- 2, the value of h_n decreases when n increases, since h_n is a sequence that tends to zero. Moreover, we see that it is influenced by truncation but not by censorship as shown in Table 3.

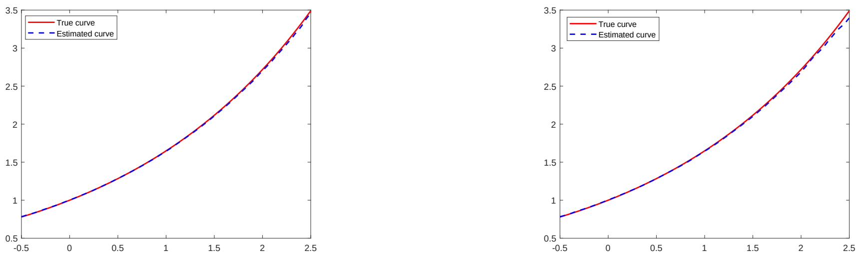


Figure 7: $r_n(x)$ vs $r(x)$ for $n = 100$ and $TP \approx 10\%$ with $CP \approx 20\%$ and 40% respectively

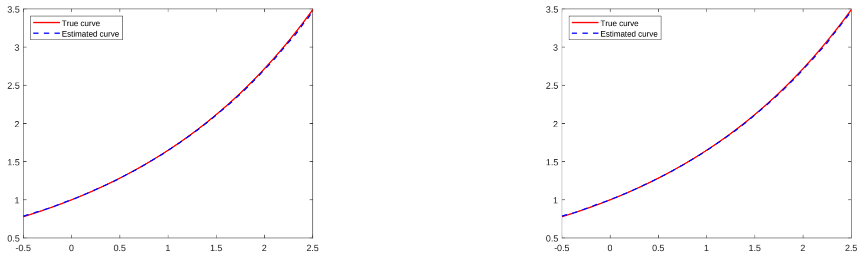


Figure 8: $r_n(x)$ vs $r(x)$ for $n = 100$ and $CP \approx 10\%$ with $TP \approx 20\%$ and 40% respectively

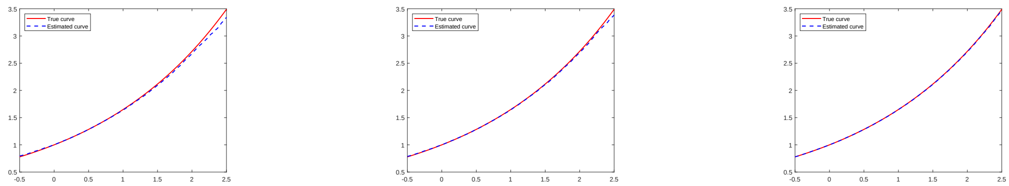


Figure 9: $r_n(x)$ vs $r(x)$ for $TP \approx 40\%$ and $CP \approx 40\%$ with $n = 50, 100$ and 300 respectively

5.2. Robustness of $r_n(.)$

In order to highlight the efficiency of the relative error regression estimator $r_n(.)$, we compare it with the classical regression (CR) estimator $m_n(.)$ proposed by Bey, Guessoum, and Tatachak (2022) and defined as follows

$$m_n(\mathbf{x}) = \frac{\sum_{i=1}^{n} \frac{\delta_i Z_i}{L_n(Z_i)\bar{G}_n(Z_i)} K_d\left(\frac{\mathbf{x}-\mathbf{X}_i}{h_n}\right)}{\sum_{i=1}^{n} \frac{\delta_i}{L_n(Z_i)\bar{G}_n(Z_i)} K_d\left(\frac{\mathbf{x}-\mathbf{X}_i}{h_n}\right)}. \tag{11}$$

For this purpose, we have followed the procedure described above and we have chosen $h_{n,s}$ which minimizes the following cross-validation criterion

$$CV_{CR}(h_n) = \frac{1}{Bn}\sum_{s=1}^{B}\sum_{i=1}^{n} \delta_i \left[Z_i - m_{n,s}^{(-i)}(X_i; h_{n,s})\right]^2.$$

The sensitivity to outliers, between the RER estimator defined by (9) and the CR estimator given in (11), is compared on a sample of size $n = 100$ with $TP \approx 20\%$ and $CP \approx 20\%$ fixed. In order to obtain the outliers, we randomly choose $A = 10$ points of the sample which will be disturbed by a multiplying factor $(MF = 5, 15, 30)$.

1. The influence of the MF on the behavior of our estimator $r_n(x)$, particularly in comparison with $m_n(x)$, is implemented through Figures 10-12 where MF=5,15,30 respectively.

Table 1: h_n and $GMSE$ of $r_n(x)$ for $TP \approx 10\%$

	$CP \approx 20\%$		$CP \approx 40\%$	
	h_n	GMSE	h_n	GMSE
n=50	0.0998	0.5648×10^{-2}	0.1112	1.4362×10^{-2}
n=100	0.0743	0.1626×10^{-2}	0.0855	0.3852×10^{-2}
n=300	0.04785	0.0224×10^{-2}	0.0521	0.0611×10^{-2}

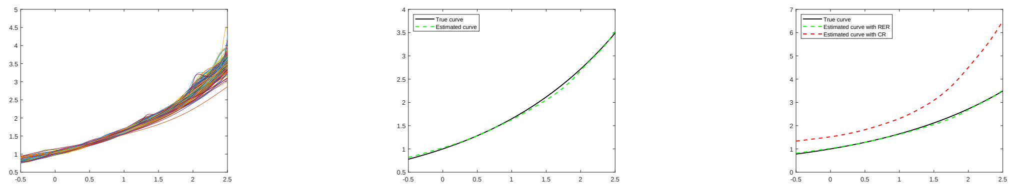
Table 2: h_n and $GMSE$ of $r_n(x)$ for $CP \approx 10\%$

	$TP \approx 20\%$		$TP \approx 40\%$	
	h_n	GMSE	h_n	GMSE
n=50	0.0907	0.4081×10^{-2}	0.0864	0.2849×10^{-2}
n=100	0.0663	0.0948×10^{-2}	0.0643	0.0806×10^{-2}
n=300	0.0442	0.0140×10^{-2}	0.0438	0.0140×10^{-2}

Table 3: h_n and $GMSE$ of $r_n(x)$ for $n = 100$

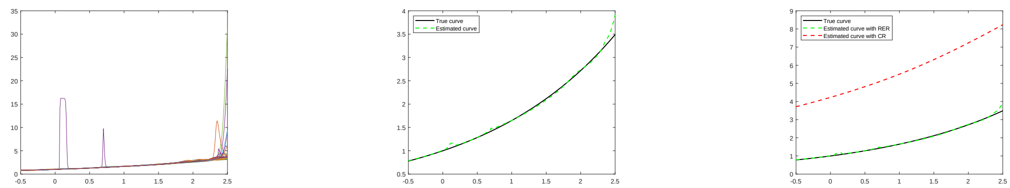
	$CP \approx 20\%$		$CP \approx 40\%$	
	h_n	GMSE	h_n	GMSE
$TP \approx 20\%$	0.0945	0.1563×10^{-2}	0.0855	0.4201×10^{-2}
$TP \approx 40\%$	0.0690	0.1074×10^{-2}	0.0765	0.4046×10^{-2}

In each of these figures, we have represented $B = 200\ r_{n,s}(x^{(p)}; h_{n,s})$ in (a), $r(x)$ with $r_n(x)$ in (b) and $r(x)$ with $r_n(x)$ and $m_n(x)$ in (c).



(a) $B = 200$ curves of $r_{n,s}(x; h_{n,s})$ (b) $r(x)$ and $r_n(x)$ (c) $r(x)$, $r_n(x)$ and $m_n(x)$

Figure 10: $TP \approx 20\%$, $CP \approx 20\%$ and $n = 100$ with $MF = 5$



(a) $B = 200$ curves of $r_{n,s}(x; h_{n,s})$ (b) $r(x)$ and $r_n(x)$ (c) $r(x)$, $r_n(x)$ and $m_n(x)$

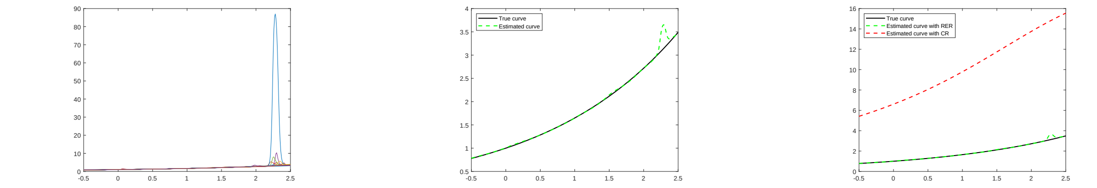
Figure 11: $TP \approx 20\%$, $CP \approx 20\%$ and $n = 100$ with $MF = 15$

2. The influence of MF on the behavior of r_n and m_n is visualized through Figure 13. In this figure, we have represented $r(x)$ with $r_n(x)$ in (a) and $r(x)$ with $m_n(x)$ in (b).

3. The comparison between the GMSE's of $r_n(x)$ and $m_n(x)$ is presented in Table 4.

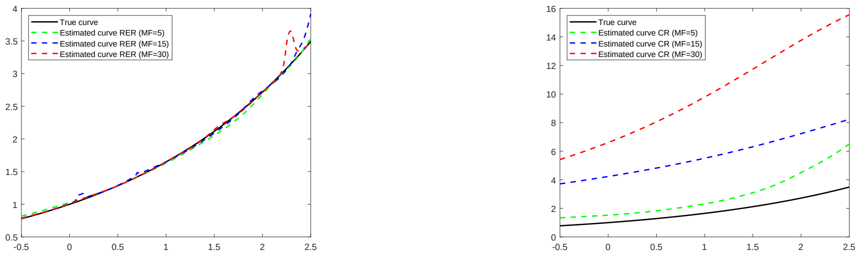
Comments on the robustness simulation results

From graphs (b) in Figures 10-12, we clearly notice that the curve of the RER estimator is very close to the theoretical curve despite the presence of outliers. On the other hand, in graphs (c) of the same figures, when the value of MF increases, the CR estimator curve departs from the theoretical curve. However, the difference between the behavior of our robust



(a) $B = 200$ curves of $r_{n,s}(x; h_{n,s})$ (b) $r(x)$ and $r_n(x)$ (c) $r(x)$, $r_n(x)$ and $m_n(x)$

Figure 12: $TP \approx 20\%$, $CP \approx 20\%$ and $n = 100$ with $MF = 30$



(a) $r(x)$ vs $r_n(x)$ (b) $r(x)$ vs $m_n(x)$

Figure 13: $TP \approx 20\%$, $CP \approx 20\%$ and $n = 100$ with MF=5(green), MF=15(blue) and MF=30 (red)

estimator $r_n(x)$ and the CR estimator $m_n(x)$ is more visible by comparing graphs (a) and (b) of Figure 13 (see the scale of the graphs). Furthermore, better visibility of the behavior of the RER estimator in the presence of outliers on graphs (a) in Figures 10-12 indicates a slight deterioration in the quality of our estimator despite its robustness. Finally, and according to Table 4, the GMSE of the RER estimator increases slightly along with MF, while that of the CR estimator increases considerably.

5.3. Asymptotic normality of $r_n(.)$

The aim of this section is to highlight the theoretical findings from theorem 2 by investigating asymptotic normality through simulations. To achieve this, we apply the Kolmogorov-Smirnov test to assess whether our data set follows a normal distribution. For $B = 500$ iid-samples $n = 50$ for, $TP = 20\%$ and $CP = 20\%$ at $x = 0.5$ the goodness-of-fit test does not reject the hypothesis that the quantity $D = \sqrt{\frac{nh_n^d}{\sigma^2(\mathbf{x})}}\,(r_n(\mathbf{x}) - r(\mathbf{x}))$ comes from a normal distribution, with a p-value equal to 0.26. Then, we draw 300 samples of sizes $n = 50, 100,$ and 300 with censorship rates (CR) approximately 10%, 40%, and truncation rates (TR) approximately 20%, 40%. From these replications, we calculate the coverage probabilities (CP) and the average lengths (AL) of 95% confidence intervals for the RER based on $r_n(0.5)$. The

Table 4: h_n and $GMSE$ of $r_n(x)$ and $m_n(x)$ for fixed $n = 300$, $TP \approx 20\%$ and $CP \approx 20\%$

	Relative Error Regression		Classical Regression	
	h_n	GMSE	h_n	GMSE
MF=5	0.2451	0.0065	0.6510	1.5764
MF=15	0.1414	0.1127	1.3340	15.4298
MF=30	0.1044	0.7947	1.1590	74.8510

different values are reported in Table 5.

To examine how the estimator behaves for various values of x, we construct 95% confidence

Table 5: The coverage probabilities and average lengths of 95% confidence intervals of $r(0.5)$

		$CR = 10\%$		$CR = 40\%$	
n	TR	AL	CP	AL	CP
50	20%	0.123	0.946	0.134	0.917
	40%	0.141	0.913	0.156	0.900
	20%	0.066	0.951	0.083	0.929
100	40%	0.068	0.944	0.074	0.927
	20%	0.031	0.957	0.037	0.944
300	40%	0.032	0.959	0.046	0.933

intervals for the RER function using different sample sizes ($n = 50$, 100, and 300) by varying x in the range $[0, 2]$ with an increment of 0.01. This is performed for ($TR = 20\%$ *and* $CR = 20\%$) and ($TR = 40\%$ *and* $CR = 40\%$), as illustrated in Figures 14 and 15.

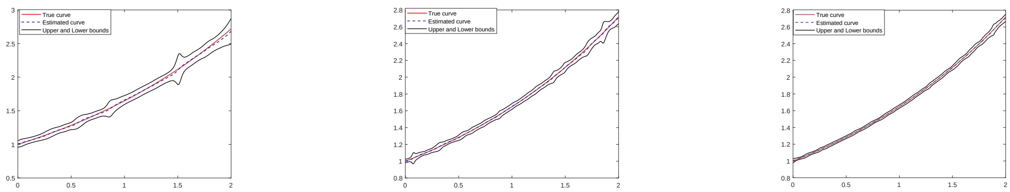


Figure 14: 95% confidence intervals of $r_n(x)$ for $TR \approx 20\%$ and $CR \approx 20\%$ with $n = 50, 100$ and 300 respectively

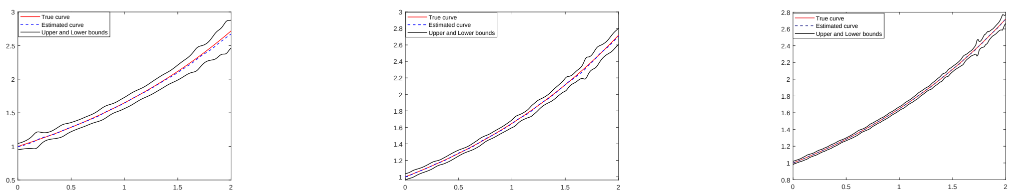


Figure 15: 95% confidence intervals of $r_n(x)$ for $TR \approx 40\%$ and $CR \approx 40\%$ with $n = 50, 100$ and 300 respectively

Comments on asymptotic normality simulation results

As seen from Table 5, the coverage probabilities of the confidence intervals are more affected by an increasing percentage of censorship. We also observe that the average lengths decrease as the sample size increases. Additionally, we can discern the effects of censorship and truncation for each fixed n.

6. Auxiliary results and proofs

6.1. Proofs for consistency results

For notational convenience we set

$$\psi_i(\mathbf{x}) = \frac{\mu\delta_i}{L(Z_i)\overline{G}(Z_i)} Z_i^{-l} K_d\left(\frac{\mathbf{x} - \mathbf{X}_i}{h_n}\right) - \mathbf{E}\left[\frac{\mu\delta_i}{L(Z_i)\overline{G}(Z_i)} Z_i^{-l} K_d\left(\frac{\mathbf{x} - \mathbf{X}_i}{h_n}\right)\right] =: \psi_i.$$

Then it is clear that we have

$$\tilde{r}_{ln}(\mathbf{x}) - \mathbf{E}\left[\tilde{r}_{ln}(\mathbf{x})\right] = \frac{1}{nh_n^d}\sum_{i=1}^{n}\psi_i.$$

Proof of Proposition 1. The proof is based on the decomposition in (13) below. Then, to upper-bound each term we use covering set techniques and Bernstein inequality.
Indeed, since Ω_0 is a compact set, it can be covered by a finite number d_n of balls B_k centered at $\mathbf{x}_k \in \mathbb{R}^d$ such that

$$\max_{1\le k\le d_n}\|\mathbf{x}-\mathbf{x}_k\| \le n^{-\frac{1}{2}}h_n^{\frac{d}{2}+1} =: \omega_n^d. \tag{12}$$

Let M_0 be a positive constant such that $d_n\omega_n^d \le M_0$.
It follows from (12)

$$\begin{aligned}\sup_{\mathbf{x}\in\Omega}|\tilde{r}_{ln}(\mathbf{x}) - \mathbf{E}[\tilde{r}_{ln}(\mathbf{x})]| \quad &\le \quad \max_{1\le k\le d_n}\sup_{\mathbf{x}\in B_k}|\tilde{r}_{ln}(\mathbf{x}) - \tilde{r}_{ln}(\mathbf{x}_k)| + \max_{1\le k\le d_n}\sup_{\mathbf{x}\in B_k}|\mathbf{E}[\tilde{r}_{ln}(\mathbf{x})] - \mathbf{E}[\tilde{r}_{ln}(\mathbf{x}_k)]| \\ &\quad + \max_{1\le k\le d_n}|\tilde{r}_{ln}(\mathbf{x}_k) - \mathbf{E}[\tilde{r}_{ln}(\mathbf{x}_k)]| \\ &=: \quad \mathscr{I}_{1n} + \mathscr{I}_{2n} + \mathscr{I}_{3n}. \end{aligned} \tag{13}$$

Note that the terms $\mathscr{I}_{1n}$ and $\mathscr{I}_{2n}$ are upper-bounded similarly. Under assumptions (**A**1)(i) and (**A**2)(i) we have

$$\begin{aligned}|\tilde{r}_{ln}(\mathbf{x}) - \tilde{r}_{ln}(\mathbf{x}_k)| \quad &\le C\frac{\mu}{nh_n^d L(a_H)\overline{G}(b_H)}\frac{\|\mathbf{x}-\mathbf{x}_k\|}{h_n} \\ &\le C\frac{\omega_n^d}{h_n^{d+1}L(a_H)\overline{G}(b_H)} \\ &= O\left(\left(nh_n^d\right)^{-1/2}\right). \end{aligned} \tag{14}$$

Now, to upper-bound the term $\mathscr{I}_{3n}$ we use a Bernstein inequality. Since $(\psi_i)_{1\le i\le n}$ is a sequence of independent and centered random variables, and if there exists a positive constant $M < \infty$ such that $|\psi_i| \le M$, we have

$$\forall\varepsilon > 0,\ \mathbf{P}\left(\left|\sum_{i=1}^{n}\psi_i(\mathbf{x}_k)\right| \ge n\varepsilon\right) \le \exp\left(-\frac{n\varepsilon^2}{2\sigma_\psi^2\left(1+\frac{M\varepsilon}{\sigma_\psi^2}\right)}\right), \tag{15}$$

where $\sigma_\psi^2 := var\left(\psi_1(\mathbf{x}_k)\right)$. It is clear that $\mathbf{E}(\psi_i(\mathbf{x}_k)) = 0$ and $|\psi_i(\mathbf{x}_k)| \le \frac{2C\|K_d\|_\infty}{L(a_H)\overline{G}(b_H)} =: M$.
Let us evaluate σ_ψ^2.

$$\begin{aligned}\sigma_\psi^2 \quad &= \quad \mathrm{var}\left(\frac{\mu\delta_1}{L(Z_1)\overline{G}(Z_1)}Z_1^{-l}K_d\left(\frac{\mathbf{x}_k-\mathbf{X}_1}{h_n}\right)\right) \\ &= \quad \left\{\mathbf{E}\left[\frac{\mu^2\delta_1^2}{L^2(Z_1)\overline{G}^2(Z_1)}(Z_1^{-l})^2K_d^2\left(\frac{\mathbf{x}_k-\mathbf{X}_1}{h_n}\right)\right] - \mathbf{E}^2\left[\frac{\mu\delta_1}{L(Z_1)\overline{G}(Z_1)}Z_1^{-l}K_d\left(\frac{\mathbf{x}_k-\mathbf{X}_1}{h_n}\right)\right]\right\} \\ &=: \quad \{\mathscr{V}_1 - \mathscr{V}_2\}. \end{aligned}$$

Then using (5), a change of variable, a Taylor expansion, assumptions (**A**1)(i), (**A**2)(i) and (**A**3) we get

$$\begin{aligned}\mathscr{V}_1 \quad &= \quad \int_{\mathbb{R}^d}\int_{\mathbb{R}}\frac{\mu^2}{L^2(z)\overline{G}^2(z)}(z^{-l})^2K_d^2\left(\frac{\mathbf{x}_k-\mathbf{u}}{h_n}\right)f^{\star 1}_{\mathbf{X},Z}(\mathbf{u},z)\mathbf{du}dz \\ &= \quad h_n^d\int_{\mathbb{R}^d}\int_{\mathbb{R}}\frac{\mu}{L(z)\overline{G}(z)}z^{-2l}K_d^2\left(s\right)f_{\mathbf{X},Y}(\mathbf{x}_k - \mathbf{s}h_n, z)\mathbf{ds}dz \\ &= \quad h_n^d\int_{\mathbb{R}^d}K_d^2(\mathbf{s})r_l^*(\mathbf{x}_k-\mathbf{s}h_n)\mathbf{ds} \\ &= \quad O(h_n^d). \end{aligned}$$

For $\mathscr{V}_2$ we proceed as for $\mathscr{V}_1$. Under assumptions (**A**1)(i), (**A**2)(i), (**A**3), a change of variable and a Taylor expansion, we get

$$\begin{aligned}\mathscr{V}_2 &= \left[\int_{\mathbb{R}^d}\int_{\mathbb{R}} \frac{\mu}{L(z)\overline{G}(z)}(z^{-l})K_d\left(\frac{\mathbf{x}_k-\mathbf{u}_1}{h_n}\right) f^{\star 1}_{\mathbf{X},Z}(\mathbf{u},z)dudz\right]^2 \\ &= O(h_n^{2d}),\end{aligned}$$

which together with $\mathscr{V}_1$ yield $\sigma_\psi^2 = O(h_n^d)$. Now, we are ready to apply Bernstein's inequality given in (15).

$$\begin{aligned}\mathbf{P}\left(\mathscr{I}_{3n} \geq \varepsilon\right) &= \mathbf{P}\left(\max_{1\leq k\leq d_n}\left|\sum_{i=1}^{n}\psi_i(\mathbf{x}_k)\right| \geq \varepsilon n h_n^d\right) \\ &\leq \sum_{i=1}^{d_n}\mathbf{P}\left(\left|\sum_{i=1}^{n}\psi_i(\mathbf{x}_k)\right| \geq \varepsilon n h_n^d\right) \\ &\leq d_n \exp\left(-\frac{(\varepsilon h_n^d)^2 n}{2\sigma_\psi^2\left(1+\frac{M h_n^d \varepsilon}{\sigma_\psi^2}\right)}\right)\end{aligned}$$

Next, for any $\varepsilon_0 > 0$, if we choose $\varepsilon = \varepsilon_0\sqrt{\frac{\log n}{nh_n^d}}$, then

$$\begin{aligned}\mathbf{P}\left(\mathscr{I}_{3n} \geq \varepsilon_0\sqrt{\frac{\log n}{nh_n^d}}\right) &\leq M_0(\omega_n^d)^{-1}\exp\left(-\frac{\varepsilon_0^2\log n}{2\left(C+M\varepsilon_0\left(\sqrt{\frac{\log n}{nh_n^d}}\right)\right)}\right) \\ &\leq C\left(n^{-\frac{1}{2}}h_n^{\frac{d}{2}+1}\right)^{-1} n^{-\left(\frac{\varepsilon_0^2}{2\left(C+M\varepsilon_0\left(\sqrt{\frac{\log n}{nh_n^d}}\right)\right)}\right)} \\ &= \left(nh_n^{d+2}\right)^{-\frac{1}{2}} O\left(n^{-(C\varepsilon_0^2-1)}\right). \qquad (16)\end{aligned}$$

Whence, for sufficiently large n, using assumption (**A**1)(i) and for a suitable choice of ε_0 (i.e. $\varepsilon_0^2 > \frac{2}{C}$), the term in (16) becomes of order $O(n^{-\theta_0})$ with $\theta_0 > 1$. Consequently

$$\mathbf{P}\left(\mathscr{I}_{3n} \geq \varepsilon_0\sqrt{\frac{\log n}{nh_n^d}}\right) = O(n^{-\theta_0}), \qquad (17)$$

and, according to (13),(14) and (17), it follows that

$$\sum_{n\geq 1}\mathbf{P}\left(\sup_{\mathbf{x}\in\Omega_0}|\tilde{r}_{ln}(\mathbf{x}) - \mathbf{E}[\tilde{r}_{ln}(\mathbf{x})]| \geq \varepsilon\sqrt{\frac{\log n}{nh_n^d}}\right) \leq \sum_{n\geq 1}\mathbf{P}\left(\mathscr{I}_{3n} \geq \varepsilon_0\sqrt{\frac{\log n}{nh_n^d}}\right) < \infty.$$

Thus by the Borel-Cantelli lemma, we conclude that

$$\sup_{\mathbf{x}\in\Omega_0}|\tilde{r}_{ln}(\mathbf{x}) - \mathbf{E}\left[\tilde{r}_{ln}(\mathbf{x})\right]| = O\left(\sqrt{\frac{\log n}{nh_n^d}}\right)\ \mathbf{P}\text{-a.s., as n} \to \infty.$$

The proof of Proposition 1 is complete. □

Proof of Theorem 1. A classical decomposition allows to write

$$\begin{aligned}\sup_{x\in\Omega_0}|r_n(\mathbf{x})-r(\mathbf{x})| \quad \leq \quad & \frac{1}{\gamma-\sup_{\mathbf{x}\in\Omega_0}|r_{2n}(\mathbf{x})-r_2(\mathbf{x})|}\left\{\sup_{\mathbf{x}\in\Omega_0}|r_{1n}(\mathbf{x})-\tilde{r}_{1n}(\mathbf{x})|\right.\\ & +\sup_{\mathbf{x}\in\Omega_0}|\tilde{r}_{1n}(\mathbf{x})-\mathbf{E}[\tilde{r}_{1n}(\mathbf{x})]|\\ & +\sup_{\mathbf{x}\in\Omega_0}|\mathbf{E}[\tilde{r}_{1n}(\mathbf{x})]-r_1(\mathbf{x})|\\ & \left.+r_2^{-1}(\mathbf{x})\sup_{\mathbf{x}\in\Omega_0}r_1(\mathbf{x})\times\sup_{x\in\Omega_0}|r_{2n}(\mathbf{x})-r_2(\mathbf{x})|\right\}. \qquad (18)\end{aligned}$$

In addition

$$\sup_{\mathbf{x}\in\Omega_0}|r_{2n}(\mathbf{x})-r_2(\mathbf{x})|\leq\sup_{\mathbf{x}\in\Omega_0}|r_{2n}(\mathbf{x})-\tilde{r}_{2n}(\mathbf{x})|+\sup_{\mathbf{x}\in\Omega_0}|\tilde{r}_{2n}(\mathbf{x})-\mathbf{E}[\tilde{r}_{2n}(\mathbf{x})]|+\sup_{\mathbf{x}\in\Omega_0}|\mathbf{E}[\tilde{r}_{2n}(\mathbf{x})]-r_2(\mathbf{x})|$$

The proof of Theorem 1 follows by using Proposition 1 and Lemmas 1-2 below. □

Lemma 1. *Under assumptions (**A**2), we have*

$$\sup_{\boldsymbol{x}\in\Omega_0}|r_{ln}(\boldsymbol{x})-\tilde{r}_{ln}(\boldsymbol{x})|=O\left(\sqrt{\frac{\log\log n}{n}}\right)\ \mathbf{P}\text{-a.s., as n}\to\infty.$$

Proof of Lemma 1. We have

$$\begin{aligned}|r_{ln}(\mathbf{x})-\tilde{r}_{ln}(\mathbf{x})| &= \left|\frac{1}{nh_n^d}\sum_{i=1}^n\left(\frac{\mu_n}{L_n(Z_i)\overline{G}_n(Z_i)}-\frac{\mu}{L(Z_i)\overline{G}(Z_i)}\right)\delta_iZ_i^{-l}K_d\left(\frac{\mathbf{x}-\mathbf{X}_i}{h_n}\right)\right|\\ &= \left|\frac{1}{nh_n^d}\sum_{i=1}^n\left(\frac{\overline{F}_n(Z_i)}{C_n(Z_i)}-\frac{\overline{F}(Z_i)}{C(Z_i)}\right)\delta_iZ_i^{-l}K_d\left(\frac{\mathbf{x}-\mathbf{X}_i}{h_n}\right)\right|\\ &= \left|\frac{1}{nh_n^d}\sum_{i=1}^n\left(\frac{(\overline{F}_n(Z_i)-\overline{F}(Z_i))}{C_n(Z_i)}\right.\right.\\ &\qquad \left.\left.+\frac{\overline{F}(Z_i)(C(Z_i)-C_n(Z_i))}{C_n(Z_i)C(Z_i)}\right)\delta_iZ_i^{-l}K_d\left(\frac{\mathbf{x}-\mathbf{X}_i}{h_n}\right)\right| \qquad (19)\end{aligned}$$

Thus, using the law of iterated logarithm (LIL), according to the result of Gijbels and Wang (1993), we have

$$\sup_{y\geq a_H}|F_n(y)-F(y)|=O\left(\sqrt{\frac{\log\log n}{n}}\right)\ \mathbf{P}\text{-a.s., as n}\to\infty. \qquad (20)$$

and

$$\sup_{y\geq a_H}|C_n(y)-C(y)|=O\left(\sqrt{\frac{\log\log n}{n}}\right)\ \mathbf{P}\text{-a.s., as n}\to\infty. \qquad (21)$$

Moreover, since $C(y)>0$ and $C_n(y)>0$, so there exists $M_1>0$ and $M_2>0$ such that $C(y)\geq M_1$ and $C_n(y)\geq M_2$ respectively,we obtain

$$\sup_{x\in\Omega_0}|r_{ln}(\mathbf{x})-\tilde{r}_{ln}(\mathbf{x})| \quad\leq\quad a_H^{-l}\left\{\frac{1}{M_1}\sup_{x\in\Omega_0}|F_n(\mathbf{x})-F(\mathbf{x})|+\frac{1}{M_1M_2}\sup_{x\in\Omega_0}|C_n(\mathbf{x})-C(\mathbf{x})|\right\}v_n^\star(\mathbf{x}).$$

The function $v_n^\star(\mathbf{x}):=\frac{1}{nh_n^d}\sum_{i=1}^nK_d\left(\frac{\mathbf{x}-\mathbf{X}_i}{h_n}\right)$ is a consistent kernel estimator, under the independence hypothesis, for the density of an $\mathbb{R}^d$-valued random vector $\mathbf{X}$ in the case of complete data (no truncation and no censoring), by using the strong law of large numbers (SLLN) and assumption (**A**2), we can easily prove that

$$v_n^\star(\mathbf{x}) = O(1) \tag{22}$$

Finally, by combining (20), (21) and (22) we obtain the result. □

Lemma 2. *Under assumptions (**A**2) and (**A**3) we have*

$$\sup_{x\in\Omega_0} |\boldsymbol{E}[\tilde{r}_{ln}(\boldsymbol{x})] - r_l(\boldsymbol{x})| = O(h_n^2).$$

Proof of Lemma 2. Using equality (5), an integration by parts, a change of variable and a Taylor expansion, we have

$$\begin{aligned}
\mathbf{E}\left[\tilde{r}_{ln}(\mathbf{x})\right] - r_l(\mathbf{x}) &= \mathbf{E}\left[\frac{\mu}{nh_n^d}\sum_{i=1}^{n}\frac{\delta_i}{L(Z_i)\overline{G}(Z_i)}(Z_i)^{-l}K_d\left(\frac{\mathbf{x}-\mathbf{X}_i}{h_n}\right)\right] - r_l(\mathbf{x}) \\
&= \frac{1}{h_n^d}\int_{\mathbb{R}^d} K_d\left(\frac{\mathbf{x}-\mathbf{u}}{h_n}\right)\left(\int_{\mathbb{R}} z^{(-l)} f(\mathbf{u},z)dz\right)\mathbf{du} - r_l(\mathbf{x}) \\
&= \int_{\mathbb{R}^d} K_d(\mathbf{s})\left(r_l(\mathbf{x}-\mathbf{s}h_n) - r_l(\mathbf{x})\right)\mathbf{ds}.
\end{aligned}$$

Then under assumptions (**A**2) and (**A**3) we obtain

$$\sup_{\mathbf{x}\in\Omega_0} |\mathbf{E}[\tilde{r}_{ln}(\mathbf{x})] - r_l(\mathbf{x})| = O(h_n^2).$$

The proof of lemma 2 is finished. □

Now in order to prove Theorem 2, we write

$$r_n(\mathbf{x}) = \frac{\mu_n^{-1} r_{1n}(\mathbf{x})}{\mu_n^{-1} r_{2n}(\mathbf{x})}.$$

Then, we write

$$\begin{aligned}
\mu_n^{-1} r_{ln}(\mathbf{x}) - \mu^{-1} r_l(\mathbf{x}) &= \left(\mu_n^{-1} r_{ln}(\mathbf{x}) - \mu^{-1}\tilde{r}_{ln}(\mathbf{x})\right) + \left(\mu^{-1}\tilde{r}_{ln}(\mathbf{x}) - \mathbf{E}\left[\mu^{-1}\tilde{r}_{ln}(\mathbf{x})\right]\right) \\
&+ \left(\mathbf{E}\left[\mu^{-1}\tilde{r}_{ln}(\mathbf{x})\right] - \mu^{-1} r_l(\mathbf{x})\right) \\
&=: \Lambda_{ln}(\mathbf{x}) + \Gamma_{ln}(\mathbf{x}) + \chi_{ln}(\mathbf{x}).
\end{aligned}$$

We begin by demonstrating that $\Lambda_{ln}(\mathbf{x})$ and $\chi_{ln}(\mathbf{x})$ are negligible, followed by establishing the asymptotic normality of the leading term $\Gamma_{ln}(\mathbf{x})$. The proofs are divided into the following several lemmas

Lemma 3. *Under assumptions (**A1**), (**A2**) and (**A3**), for n large enough we have*

i) $\sqrt{nh_n^d}\,|\Lambda_{ln}(\boldsymbol{x})| = o_{\boldsymbol{P}}(1)$,

ii) $\sqrt{nh_n^d}\,|\chi_{ln}(\boldsymbol{x})| = o_{\boldsymbol{P}}(1)$,

Proof of Lemma 3. i) We proceed as in the proof of Lemma 1 with the help of Assumption (**A1**)(ii). Then we get

$$\sqrt{nh_n^d}\,|\Lambda_{ln}(\mathbf{x})| = O\left(\sqrt{h_n^d \log\log n}\right) = o_{\mathbf{P}}(1).$$

ii) We use assumption (**A1**)(iii) and we follow step by step the proof of Lemma 2. Then we get

$$\sqrt{nh_n^d}\,|\chi_{ln}(\mathbf{x})| = O\left(h_n^2\sqrt{nh_n^d}\right) = o_{\mathbf{P}}(1).$$

□

Before proceeding with the asymptotic normality of the leading terms, it will be necessary to derive expressions and/or bounds for variances and covariances on several occasions below. To initiate this process, we present the following lemma:

Lemma 4. *Under assumptions (**A**1)(i),(**A**2),(**A**3), and for n large enough it holds that:*

i) $\operatorname{var}\left(\sqrt{nh_n^d}\Gamma_{1n}(\boldsymbol{x})\right) \longrightarrow \frac{\kappa}{\mu}\Sigma_2(\boldsymbol{x})$.

ii) $\operatorname{var}\left(\sqrt{nh_n^d}\Gamma_{2n}(\boldsymbol{x})\right) \longrightarrow \frac{\kappa}{\mu}\Sigma_4(\boldsymbol{x})$.

iii) $\operatorname{cov}\left(\sqrt{nh_n^d}\Gamma_{1n}(\boldsymbol{x}), \sqrt{nh_n^d}\Gamma_{2n}(\boldsymbol{x})\right) \longrightarrow \frac{\kappa}{\mu}\Sigma_3(\boldsymbol{x})$.

Proof of Lemma 4. i) According to the notations in Theorem 2, it is easily seen that

$$\begin{aligned}
\operatorname{var}\left(\sqrt{nh_n^d}\Gamma_{n1}(\mathbf{x})\right) &= \frac{1}{h_n^d}\operatorname{var}\left(\frac{\delta_1}{L(Z_1)\overline{G}(Z_1)}(Z_1)^{-1}K_d\left(\frac{\mathbf{x}-\mathbf{X}_1}{h_n}\right)\right)\\
&= \frac{1}{h_n^d}\frac{1}{\mu}\left(\int_{\mathbb{R}^d}\int_{\mathbb{R}}\frac{(z)^{-2}}{L(z)\overline{G}(z)}K_d^2\left(\frac{\mathbf{x}-\mathbf{u}}{h_n}\right)f_{X,Y}(\mathbf{u},z)d\mathbf{u}\,dz\right) - \frac{1}{h_n^d}O(h_n^{2d})\\
&\overset{n\longrightarrow\infty}{\longrightarrow} \frac{\kappa}{\mu}\Sigma_2(\mathbf{x}) + o(1).
\end{aligned} \tag{23}$$

ii) We have

$$\begin{aligned}
\operatorname{var}\left(\sqrt{nh_n^d}\Gamma_{n2}(\mathbf{x})\right) &= \frac{1}{h_n^d}\operatorname{var}\left(\frac{\delta_1}{L(Z_1)\overline{G}(Z_1)}(Z_1)^{-2}K_d\left(\frac{\mathbf{x}-\mathbf{X}_1}{h_n}\right)\right)\\
&= \frac{1}{h_n^d}\frac{1}{\mu}\left(\int_{\mathbb{R}^d}\int_{\mathbb{R}}\frac{(z)^{-4}}{L(z)\overline{G}(z)}K_d^2\left(\frac{\mathbf{x}-\mathbf{u}}{h_n}\right)f_{X,Y}(\mathbf{u},z)d\mathbf{u}\,dz\right) - \frac{1}{h_n^d}O(h_n^{2d})\\
&\overset{n\longrightarrow\infty}{\longrightarrow} \frac{\kappa}{\mu}\Sigma_4(\mathbf{x}) + o(1).
\end{aligned} \tag{24}$$

iii) Under (**A**2),(**A**3), change of variables and Taylor expansion we obtain

$$\begin{aligned}
cov\left(\sqrt{nh_n^d}\,\Gamma_{n1}(\mathbf{x}), \sqrt{nh_n^d}\,\Gamma_{n2}(\mathbf{x})\right) &= nh_n^d\,cov\left(\Gamma_{n1}(\mathbf{x}), \Gamma_{n2}(\mathbf{x})\right)\\
&= \frac{1}{h_n^d}\,cov\left(\frac{\delta_1 Z_1^{-1}}{L(Z_1)\overline{G}(Z_1)}K_d\left(\frac{\mathbf{x}-\mathbf{X}_1}{h_n}\right),\right.\\
&\qquad\left.\frac{\delta_1 Z_1^{-2}}{L(Z_1)\overline{G}(Z_1)}K_d\left(\frac{\mathbf{x}-\mathbf{X}_1}{h_n}\right)\right)\\
&= \frac{1}{h_n^d\,\mu}\left(\int_{\mathbb{R}^d}\int_{\mathbb{R}}\frac{z^{-3}}{L(z)\,\overline{G}(z)}\,K_d^2\left(\frac{\mathbf{x}-\mathbf{u}}{h_n}\right)f_{X,Y}(\mathbf{u},z)\,d\mathbf{u}\,dz\right)\\
&\quad - O(h_n^d)\\
&\overset{n\to\infty}{\longrightarrow} \frac{\kappa}{\mu}\,\Sigma_3(\mathbf{x}) + o(1).
\end{aligned} \tag{25}$$

□

Lemma 5. *Under assumptions (**A**1)(i),(**A**2)-(**A**4), and for n large enough it holds that:*

$$\sqrt{nh_n^d}\left(\Gamma_{n1}(\boldsymbol{x}),\Gamma_{n2}(\boldsymbol{x})\right)\xrightarrow{\mathscr{D}}\mathscr{N}\left(0,\mu^{-1}\kappa\Sigma(\boldsymbol{x})\right), \text{with } \Sigma=\begin{bmatrix}\Sigma_2 & \Sigma_3\\ \Sigma_3 & \Sigma_4\end{bmatrix}.$$

Proof of Lemma 5. We have to prove that for sufficiently large n, any linear combination of $\sqrt{nh_n^d}\Gamma_{n1}(x)$ and $\sqrt{nh_n^d}\Gamma_{n2}(x)$ is asymptotically Gaussian for a vector $\mathbf{a}^\top=(a_1,a_2)\in\mathbb{R}^2$ with $a_1^2+a_2^2\neq 0$,

$$\sqrt{nh_K^d}\left(a_1\Gamma_{n1}(\mathbf{x})+a_2\Gamma_{n2}(\mathbf{x})\right)\xrightarrow{\mathscr{D}}\mathscr{N}\left(0,\mu^{-1}\kappa\mathbf{a}^\top\Sigma(\mathbf{x})\mathbf{a}\right).$$

To do so, let

$$\begin{aligned}\Delta_n(x) &= \sqrt{nh_n^d}\left[a_1\Gamma_{n1}(x)+a_2\Gamma_{n2}(x)\right]\\ &= \sum_{i=1}^n\left\{a_1\frac{1}{\sqrt{nh_n^d}}\left[\frac{\delta_iZ_i^{-1}}{L_n(Z_i)\overline{G}_n(Z_i)}K_d\left(\frac{\mathbf{x}-\mathbf{X}_i}{h_n}\right)-\mathbf{E}\left(\frac{\delta_iZ_i^{-1}}{L_n(Z_i)\overline{G}_n(Z_i)}K_d\left(\frac{\mathbf{x}-\mathbf{X}_i}{h_n}\right)\right)\right]\right.\\ &\quad\left.+a_2\frac{1}{\sqrt{nh_n^d}}\left[\frac{\delta_iZ_i^{-2}}{L_n(Z_i)\overline{G}_n(Z_i)}K_d\left(\frac{\mathbf{x}-\mathbf{X}_i}{h_n}\right)-\mathbf{E}\left(\frac{\delta_iZ_i^{-2}}{L_n(Z_i)\overline{G}_n(Z_i)}K_d\left(\frac{\mathbf{x}-\mathbf{X}_i}{h_n}\right)\right)\right]\right\}\\ &=: \sum_{i=1}^n\Delta_{ni}(x)\end{aligned}$$

It is clear that the $\Delta_{ni}(x)$ are iid. We will apply Lyapunov's Central Limit Theorem to the sequence $\Delta_{ni}(x)$. Let $\rho_{ni}^\nu(x)=\mathbf{E}\left[|\Delta_{ni}(x)|^\nu\right]$, for $\nu>2$. By the C_r-inequality (see Loève (1963), p.156), we obtain

$$\begin{aligned}\rho_{ni}^\nu(\mathbf{x}) &= \mathbf{E}\left|a_1\frac{1}{\sqrt{nh_n^d}}\left\{\frac{\delta_iZ_i^{-1}}{L_n(Z_i)\overline{G}_n(Z_i)}K_d\left(\frac{\mathbf{x}-\mathbf{X}_i}{h_n}\right)-\mathbf{E}\left(\frac{\delta_iZ_i^{-1}}{L_n(Z_i)\overline{G}_n(Z_i)}K_d\left(\frac{\mathbf{x}-\mathbf{X}_i}{h_n}\right)\right)\right\}\right.\\ &\quad\left.+a_2\frac{1}{\sqrt{nh_n^d}}\left\{\frac{\delta_iZ_i^{-2}}{L_n(Z_i)\overline{G}_n(Z_i)}K_d\left(\frac{\mathbf{x}-\mathbf{X}_i}{h_n}\right)-\mathbf{E}\left(\frac{\delta_iZ_i^{-2}}{L_n(Z_i)\overline{G}_n(Z_i)}K_d\left(\frac{\mathbf{x}-\mathbf{X}_i}{h_n}\right)\right)\right\}\right|^\nu\\ &\leq 2^{\nu-1}\left(\frac{h_n^d}{n}\right)^{\frac{\nu}{2}}\left\{a_1^\nu\mathbf{E}\left[\left|\frac{1}{h_n^d}\frac{\delta_iZ_i^{-1}}{L_n(Z_i)\overline{G}_n(Z_i)}K_d\left(\frac{\mathbf{x}-\mathbf{X}_i}{h_n}\right)\right|^\nu\right]+\right.\\ &\quad\left.+a_2^\nu\mathbf{E}\left[\left|\frac{1}{h_n^d}\frac{\delta_iZ_i^{-2}}{L_n(Z_i)\overline{G}_n(Z_i)}K_d\left(\frac{\mathbf{x}-\mathbf{X}_i}{h_n}\right)\right|^\nu\right]\right\}\end{aligned}$$

which implies that, under assumption (**A**4) and the fact that $\nu>2$

$$\rho_n^\nu(\mathbf{x}):=\sum_{i=1}^n\rho_{ni}^\nu(\mathbf{x})=O\left(n^{1-\frac{\nu}{2}}h_n^{\frac{d\nu}{2}}\right)=o(1). \tag{26}$$

On the other hand, by using (23), (24) and (25), we get

$$\begin{aligned}s_n^2(\mathbf{x}):=Var(\Delta_n(\mathbf{x})) &= a_1^2Var\left(\sqrt{nh_n^d}\Gamma_{n1}(\mathbf{x})\right)+a_2^2Var\left(\sqrt{nh_n^d}\Gamma_{n2}(\mathbf{x})\right)\\ &\quad+2a_1a_2Cov\left(\sqrt{nh_n^d}\Gamma_{n1}(\mathbf{x}),\sqrt{nh_n^d}\Gamma_{n2}(\mathbf{x})\right)\\ &= a_1^2\frac{\kappa}{\mu}\Sigma_2(\mathbf{x})+a_2^2\frac{\kappa}{\mu}\Sigma_4(\mathbf{x})+2a_1a_2\frac{\kappa}{\mu}\Sigma_3(\mathbf{x})+o(1).\end{aligned}$$

thus,

$$s_n^2(\mathbf{x})\xrightarrow[n\to+\infty]{}\mu^{-1}\kappa\,\mathbf{a}^T\Sigma(\mathbf{x})\,\mathbf{a}>0, \tag{27}$$

for all $a \neq 0$.
Then, (26) and (27) give

$$\frac{1}{\left(Var\left(\Delta_n(\mathbf{x})\right)\right)^{\frac{\nu}{2}}} \sum_{i=1}^{n} \mathbf{E}\left[|\Delta_{ni}(\mathbf{x})|^{\nu}\right] = \frac{\rho_n^{\nu}(\mathbf{x})}{s_n^{\nu}(\mathbf{x})} \xrightarrow[n \to +\infty]{} 0.$$

Hence, the result is a consequence of Berry Esseen's Theorem (see Chow and Teicher (1978), p. 322). □

Proof of Theorem 2. Let $\Theta : \mathbb{R}^2 \to \mathbb{R}$ be a mapping defined, for any (x, y), by $\Theta(x, y) = \frac{x}{y}$ for $y \neq 0$. As $r(\mathbf{x}) = \frac{\mu^{-1} r_1(\mathbf{x})}{\mu^{-1} r_2(\mathbf{x})}$, remark that $r_n(\mathbf{x})$ and $r(\mathbf{x})$ are the respective images of $(\mu_n^{-1} r_{1n}(\mathbf{x}), \mu_n^{-1} r_{2n}(\mathbf{x}))$ and $(\mu^{-1} r_1(\mathbf{x}), \mu^{-1} r_2(\mathbf{x}))$ by Θ. So, from Lemmas 3-5 and from the δ-method Theorem we deduce that $\sqrt{nh_K^d}\,(r_n(\mathbf{x}) - r(\mathbf{x}))$ converges in distribution to $\mathcal{N}\left(0, \nabla\Theta^{\top} \frac{\kappa}{\mu} \Sigma(\mathbf{x}) \nabla\Theta\right)$, with

$$\nabla\Theta\left(\mu^{-1} r_1(\mathbf{x}), \mu^{-1} r_2(\mathbf{x})\right) = \mu\left(r_2^{-1}(\mathbf{x}), -r_2^{-2}(\mathbf{x}) r_1(\mathbf{x})\right)^{\top}.$$

Then an elementary calculation gives

$$\sigma^2(\mathbf{x}) = \mu\kappa\left[r_2^2(\mathbf{x})\Sigma_2(\mathbf{x}) - 2r_2(\mathbf{x})\Sigma_3(\mathbf{x}) r_1(\mathbf{x}) + \Sigma_4(\mathbf{x}) r_1^2(\mathbf{x})\right] r_2^{-4}(\mathbf{x}).$$

The proof of Theorem 2 is complete. □

Proof of Corollary 4.1.1. The proof is an immediate consequence of Theorem 2. □

7. Conclusion

In this study, we developed a new kernel estimator for the regression function, minimizing the mean squared relative error when the data are LTRC. We thoroughly investigated its asymptotic behavior, establishing almost sure uniform convergence with a rate and demonstrating asymptotic normality. The convergence rate was obtained through Bernstein's inequality, while Lyapunov's central limit theorem was applied to prove the asymptotic distribution. Simulation studies confirmed the theoretical findings, emphasizing the reliability and efficiency of the proposed estimator in handling incomplete data scenarios.

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