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Statistical Estimation and Hypothesis Testing on Impulse Response Function

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Abstract

In this paper a time-invariant continuous linear system is considered with a real-valued impulse response function (IRF) which is defined on a bounded domain. A sample input-output cross-correlogram is taken as an estimator of the response function. The input processes are supposed to be zero-mean stationary Gaussian process and can be represented as a finite sum with uncorrelated terms. A rate of convergence of IRF estimator in the space $L_2([0, \Lambda])$ is obtained that gives a possibility to propose a nonparametric goodness-of-fit testing on IRF.

Keywords: impulse response function, cross-correlogram, large deviation probability, rate of convergence, statistical criterion, hypothesis testing,nonparametric goodness-of-fit testing .

1. Introduction

The problem of estimation of a stochastic linear system driven by impulse response function (IRF) has been a matter of active research in recent years. One of the simplest models considers a black box that receives some input and produces a corresponding output. The input may be single or multiple and there is the same choice for the output. This generates a great amount of models that can be considered. The range of applications for these models is extensive, ranging from signal processing and automatic control to econometrics (errors-in-variables models) and oceanology. For more details, see Barigozzi, Lippi, and Luciani (2021), Hannan and Deistler (1988), Soderstrom and Stoica (1989), Stern (2018) and Lütkepohl (2010), Takahito and Munehiko (2022).

The issue of the estimation of IRF is similar to inverse problem and deconvolution one that are used, for example, for restoring signal or images, signal detection (see Abramovich, Pensky, and Rozenholc (2013), Delaigle, Hall, and Meister (2008), Meister (2009), Marteau and Sapatinas (2015)).

The estimates in these models have been mostly obtained in usual statistical framework. Optimality results (in the minimax sense) have been given for various loss functions and sequence/function spaces. Many methods have been considered including kernel, local polynomial, spline, projection and wavelet methods (see, e.g.Cavalier and Raimondo (2007)).

We are interested in the estimation of the so-called impulse function from observations of responses of a SISO (single-input single-output) system to certain input signals. This problem can be considered both for linear and non-linear systems. To solve this problem, different statistical approaches were used as well as various deterministic methods that are based on a perturbation of the system by stationary stochastic processes and the further analysis of some characteristics of both input and output processes. Let us mention some publications on this problem by Bendat and Piersol (1980) and Schetzen (1980), Nelles (2020). One of the first article on this topic was Akaike (1965) where the author studied a MISO (multiple-input single-output) linear system and obtained estimates of the Fourier transform of the response function in each component.

Some methods for estimation of unknown impulse response function of linear system and the study of properties of corresponding estimators were considered in the works of Buldygin and his followers. These methods are based on constructing a sample cross-correlogram between the input stochastic process and the response of the system (see, e.g. Buldygin and Li (1997a), Buldygin and Li (1997b), Blazhievska and Zaiats (2021)).

An inequality for the supremum of the estimation error in the space of continuous functions in the case of integral-type cross-correlogram estimator was obtained by Kozachenko and Rozora (2016).

In the paper Rozora and Kozachenko (2016) a time-invariant continuous linear system with a real-valued impulse response function was considered. The input signal process was supposed to be a zero mean Gaussian stochastic process which was represented as a treamed sum with respect to orthonormal basis in $L_2(\mathbf{R})$. The case of Hermite polynomials as orthonormal basis in $L_2(\mathbf{R})$ was studied. A new method for the construction of estimator of the impulse response function was proposed ant the criteria on impulse response function were given.

In Rozora (2018) and Rozora (2020) for integral cross–correlogram estimator the algorithm of statistical hypothesis testing for response function was written applying upper estimate of overrunning by square-Gaussian random process the level specified by continuous function.

In this paper a time-invariant continuous linear system is considered with a real-valued impulse response function which is defined on bounded domain. An input-output cross-correlogram based on one single observation is taken as an estimator of the response function. The input processes are supposed to be zero-mean stationary Gaussian process and can be represented in trimmed series of Fourier decomposition. In such model the built estimator of impulse function depends on the length of averaging interval for cross-correlogram T and the level of cutting for input signal N. The estimation of large deviation probability for errors in the space $L_2(T)$ is found. These allow us to develop the hypothesis testing on the shape of the impulse response function.

The paper consists of 8 sections and the structure is as follows.

The second section covers the main definitions and general properties of the estimator. The input signal process is supposed to be a zero mean Gaussian stochastic process which is represented as a treamed sum with respect to orthonormal basis in $L_2([0,\Lambda])$.

In section 3 we suppose that the input process of the system can be represented as a series with respect to the trigonometric basis on $[0,\Lambda]$. The upper bounds for mathematical expectation, variance are found that provide asymptotically unbiasedness as $N \to \infty$ and consistency as $N, T \to \infty$.

Section 4 deals with square Gaussian random variables and processes. Inequality for $L_p(T)$ norm of a square Gaussian stochastic process is shown.

In the fifth section the convergence rate for the estimator of unknown impulse response function in the space $L_p([0,\Lambda])$ is investigated. In the sixth section the goodness of fit test is developed on the shape of the impulse response function.

Section 7 is devoted to software simulation. In one particular case the critical values of the length of averaging interval T are found for different accuracy, reliability and N (the upper limit of the summing in the model) using software environment for statistical computing and graphics R.

2. The estimator of an impulse response function and its properties

Consider a time-invariant continuous linear system with a real-valued square integrable impulse response function (IRF) $H(\tau)$ which is defined on a finite domain $\tau \in [0,\Lambda]$. This means that the response of the system to an input signal $X(t), t \in R$, has the following form:

$$Y(t) = \int_0^\Lambda H(\tau)X(t-\tau)d\tau, \quad t \in R \tag{1}$$

and $H \in L_2([0,\Lambda])$. One of the problems arising in the theory of linear systems is to estimate the function H from observations of responses of the system to certain input signals. In this paper we use cross-correlogram approach to estimate the unknown IRF H. At first, let's describe the input process that is performed due to some orthonormal basis.

Let the system of functions $\{\varphi_0(t), \varphi_k(t), \psi_k(t),\ k \geq 1,\ t \in R\}$ be an orthonormal basis in $L_2[0,\Lambda]$ and the functions $\varphi_k(t), \psi_k(t)$ are continuous and Λ-periodic. Assume that the function $\varphi_0(t)$ is a constant. This means that it should be equal to $\varphi_0(t) = \frac{1}{\sqrt{\Lambda}}$.

Consider now as input of the linear system a real-valued Gaussian stationary zero mean stochastic process $X = X_N = (X_N(u),\ u \in R)$, that can be presented as

$$X_N(u) = \sum_{k=1}^N \xi_k \varphi_k(u) + \sum_{k=1}^N \eta_k \psi_k(u), \quad u \in R, \tag{2}$$

where $N > 0$ is fixed integer number and Gaussian random variables $\xi_k, \eta_k,\ k \geq 0$, are uncorrelated with $\mathbf{E}\xi_k = \mathbf{E}\eta_k = 0$, $\mathbf{E}\xi_k^2 = \mathbf{E}\eta_k^2 = 1$.

Remark 1. *The input signal $X_N(t)$ can be considered as a model of some stochastic process that can be expanded into a series over given orthonormal basis. To read more about model construction with given accuracy and reliability we refer readers to Kozachenko, Pogorilyak, Rozora, and Tegza (2016), Kozachenko, Pashko, and Rozora (2007), Vasylyk, Rozora, Ianevych, and Lovytska (2021), Rozora, Ianevych, Pashko, and Zatula (2023), Ianevych, Rozora, and Pashko (2022).*

A covariance function $r_N(t-s) = EX_N(t)X_N(s)$ of the stationary process X_N should be equal to

$$r_N(t-s) = \sum_{k=1}^N \left(\varphi_k(t)\varphi_k(s) + \psi_k(t)\psi_k(s)\right). \tag{3}$$

If the system (1) is perturbed by the stochastic process X_N, then for the output process we obtain

$$Y_N(t) = \int_0^\Lambda H(\tau)X_N(t-\tau)d\tau,$$

Show that the covariance function of X_N will be equal to $r_N(t-s)$

$$\begin{aligned}\mathbf{E}X_N(t)X_N(s) &= \mathbf{E}\left(\sum_{k=1}^N \xi_k\varphi_k(t) + \sum_{k=1}^N \eta_k\psi_k(t)\right)\left(\sum_{k=1}^N \xi_k\varphi_k(s) + \sum_{k=1}^N \eta_k\psi_k(s)\right)\\ &= \sum_{k=1}^N \left(\varphi_k(t)\varphi_k(s) + \psi_k(t)\psi_k(s)\right).\end{aligned}$$

By the condition (3) we obtain that the covariance function of stochastic process X_N is equal to $\mathbf{E}X_N(t)X_N(s) = r_N(t-s)$. Under a_0 we define the output of the system on the constant signal. This means that

$$a_0 = \frac{1}{\sqrt{\Lambda}} \int_0^\Lambda H(t)dt. \tag{4}$$

Set

$$H^*(\tau) = H(\tau) - a_0. \tag{5}$$

As an estimator of the difference of IRF and a_0 $H^*(\tau)$ we will consider an integral cross-correlogram

$$\hat{H}(\tau) = \hat{H}_{N,T,\Lambda}(\tau) = \hat{H}_{N,T,\Lambda}(\tau) = \frac{1}{T}\int_0^T Y_N(t)X_N(t-\tau)dt, \tag{6}$$

where $T > 0$ is a parameter for averaging.

Remark 2. *The integral in (1) is considered as the mean-square Riemann integral. The integral in (1) exists if and only if there exists the Riemann integral (see Gikhman and Skorokhod (1996))*

$$\int_0^\Lambda \int_0^\Lambda H(\tau)r_N(s-\tau)H(s)dsd\tau. \tag{7}$$

The covariance function of the process X_N is given by (3). Since $\{\varphi_k(t), \psi_k(t),\, k \geq 0\}$ is an orthonormal basis in $L_2([0,\Lambda])$, $H \in L_2([0,\Lambda])$, then the integral (7) exists. Therefore, there exists also the integral in (1).

Denote now

$$a_k = \int_0^\Lambda H(t)\varphi_k(t)dt, \qquad b_k = \int_0^\Lambda H(t)\psi_k(t)dt. \tag{8}$$

Since $H \in L_2([0,\Lambda])$, then the function H can be expanded into the series by orthonormal basis $\{\varphi_k(t), \psi_k(t)\; k \geq 0\}$ on the domain $[0,\Lambda]$. We obtain

$$H(t) = \sum_{k=0}^{\infty} a_k\varphi_k(t) + \sum_{k=1}^{\infty} b_k\psi_k(t), \tag{9}$$

where a_k and b_k are from (8).

Remark 3. *According to the Riesz–Fischer theorem (see Beals (2004)) the series (9) converges in the mean squared sense. If the function $H(t)$ and derivative $H'(t)$ are continuous then the convergence in* (9) *can be considered in pointwise and uniform senses.*

Lemma 1. *Kozachenko and Rozora (2023) The following relations hold true:*

$$\boldsymbol{E}\hat{H}_{N,T,\Lambda}(\tau) = \int_0^\Lambda H(v)r_N(v-\tau)dv = \sum_{k=1}^{N}\left(\varphi_k(\tau)a_k + \psi_k(\tau)b_k\right), \quad \tau \in [0,\Lambda], \tag{10}$$

and

$$H^*(\tau) - \boldsymbol{E}\hat{H}_{N,T,\Lambda}(\tau) = \sum_{k=N+1}^{\infty}\left(\varphi_k(\tau)a_k + \psi_k(\tau)b_k\right), \quad \tau \in [0,\Lambda]. \tag{11}$$

Lemma 2. *Kozachenko and Rozora (2023) The joint moments of $\hat{H}_{N,T,\Lambda}$ is equal to*

$$\begin{aligned} \boldsymbol{E}\hat{H}_{N,T,\Lambda}(\tau)\hat{H}_{N,T,\Lambda}(\theta) &= \int_0^\Lambda H(u)r_N(\tau-u)du \cdot \int_0^\Lambda H(v)r_N(\theta-v)dv \\ &+ \frac{1}{T^2}\int_0^T\int_0^T \left[\int_0^\Lambda\int_0^\Lambda H(v)H(u)r_N(t-s+u-v)dudv \cdot r_N(t-s+\theta-\tau)\right. \\ &+ \left.\int_0^\Lambda H(v)r_N(t-s+\theta-v)dv \cdot \int_0^\Lambda H(u)r_N(s-t+\tau-u)du\right] dtds, \end{aligned} \tag{12}$$

where $r_N(t-s) = EX_N(t)X_N(s)$ is a covariance of X_N, the coefficients a_k and b_k are defined in (8).

The variance of the estimator $\hat{H}_{N,T,\Lambda}$ is equal to

$$\begin{aligned} Var\hat{H}_{N,T,\Lambda}(\tau) &= \frac{1}{T^2}\int_0^T\int_0^T\left[\int_0^\Lambda\int_0^\Lambda H(v)H(u)r_N(t-s+u-v)dudv\cdot r_N(t-s)\right.\\ &+ \left.\int_0^\Lambda H(v)r_N(t-s+\tau-v)dv\cdot\int_0^\Lambda H(u)r_N(s-t+\tau-u)du\right]dtds \end{aligned} \quad (13)$$

3. Fourier series based approach

Let now consider the system of functions

$$\left\{\frac{1}{\sqrt{\Lambda}}, \sqrt{\frac{2}{\Lambda}}\cos\left(\frac{2k\pi t}{\Lambda}\right), \sqrt{\frac{2}{\Lambda}}\sin\left(\frac{2k\pi t}{\Lambda}\right), k\geq 1\right\} \quad (14)$$

that is orthonormal basis in $L_2([0,\Lambda])$. Under notation of previous section

$$\varphi_0(t) = \frac{1}{\sqrt{\Lambda}}, \quad \varphi_k(t) = \sqrt{\frac{2}{\Lambda}}\cos\left(\frac{2k\pi t}{\Lambda}\right), \quad \psi_k(t) = \sqrt{\frac{2}{\Lambda}}\sin\left(\frac{2k\pi t}{\Lambda}\right), \ k\geq 1$$

and the coefficients a_k, b_k are equal to

$$a_k = \int_0^\Lambda H(\tau)\varphi_k(\tau)d\tau = \sqrt{\frac{2}{\Lambda}}\int_0^\Lambda H(\tau)\cos\left(\frac{2k\pi\tau}{\Lambda}\right)d\tau \quad k\geq 1, \quad (15)$$

$$b_k = \int_0^\Lambda H(\tau)\psi_k(\tau)d\tau = \sqrt{\frac{2}{\Lambda}}\int_0^\Lambda H(\tau)\sin\left(\frac{2k\pi\tau}{\Lambda}\right)d\tau \quad k\geq 1. \quad (16)$$

Suppose now that the input signal processes of the system (1) are zero mean stationary Gaussian stochastic processes that are formed by (14). This means that the process $X_N(u)$ is given by:

$$X_N(u) = \sqrt{\frac{2}{\Lambda}}\sum_{k=1}^N\left(\xi_k\cos\left(\frac{2k\pi u}{\Lambda}\right)+\eta_k\sin\left(\frac{2k\pi u}{\Lambda}\right)\right), \quad u\in R. \quad (17)$$

It follows from (3) that the covariance function of stationary Gaussian process X_N can be written as

$$\begin{aligned} r_N(t-s) &= \sum_{k=1}^N(\varphi_k(t)\varphi_k(s)+\psi_k(t)\psi_k(s))\\ &= \frac{2}{\Lambda}\sum_{k=1}^N\left(\cos\left(\frac{2k\pi t}{\Lambda}\right)\cos\left(\frac{2k\pi s}{\Lambda}\right)+\sin\left(\frac{2k\pi t}{\Lambda}\right)\sin\left(\frac{2k\pi s}{\Lambda}\right)\right) \quad (18)\\ &= \frac{2}{\Lambda}\left(\sum_{k=1}^N\cos\left(\frac{2k\pi(t-s)}{\Lambda}\right)\right). \end{aligned}$$

Consider the following conditions:

Condition A. The function $H(\tau)$ is two times differentiable on $[0,\Lambda]$. The functions $H(\tau)$ and $H'(\tau)$ are continuous on $[0,\Lambda]$ and

$$\begin{aligned} I_0 &= I_0(\Lambda) = \int_0^\Lambda|H(\tau)|d\tau<\infty,\\ I_1 &= I_1(\Lambda) = \left(\int_0^\Lambda|H'(\tau)|^2d\tau\right)^{1/2}<\infty,\\ I_2 &= I_2(\Lambda) = \int_0^\Lambda|H''(\tau)|d\tau<\infty. \end{aligned}$$

Condition B. The following relation holds true

$$H(0) = H(\Lambda).$$

Remark 4. *Condition* $\boldsymbol{B}$ *means that the effect of the impulse on the domain* $[0, \Lambda]$ *should be completely ended. In case when the condition isn't fulfilled the rotation of the graph by the angle* $\arctan \frac{H(\Lambda)-H(0)}{\Lambda}$ *can be applied.*

Let's denote

$$d = |H'(0)| + |H'(\Lambda)|. \tag{19}$$

Lemma 3. *Kozachenko and Rozora (2023) Assume that the conditions A, B are satisfied. Then*

$$|H^*(\tau) - \boldsymbol{E}\hat{H}_{N,T,\Lambda}(\tau)| \leq \frac{\Lambda(d + I_2(\Lambda))}{2\pi^2 N}, \tag{20}$$

$$Var\hat{H}_{N,T,\Lambda}(\tau) \leq \frac{\Lambda^3(\Lambda+2)I_1^2}{\pi^4 T^2}\left(2 - \frac{1}{N}\right)^2, \tag{21}$$

where $I_1(\Lambda)$ *and* $I_2(\Lambda)$ *are defined in condition A.*

4. Square Gaussian random variables and processes

In this section the definition and some properties of square Gaussian random variables and processes are presented. Let (Ω, L, P) be a probability space and let $(\mathbf{T}, \rho)$ be a compact metric space with metric ρ.

Definition 1. Buldygin and Kozachenko (2000) Let $\Xi = \{\xi_t,\, t \in \mathbf{T}\}$ be a family of joint Gaussian random variables for which $E\xi_t = 0$ (e.g., ξ_t, $t \in \mathbf{T}$, is a Gaussian stochastic process). The space $SG_\Xi(\Omega)$ is the space of square Gaussian random variables if any element $\eta \in SG_\Xi(\Omega)$ can be presented as

$$\eta = \bar{\xi}^\top A\bar{\xi} - \mathbf{E}\bar{\xi}^\top A\bar{\xi}, \tag{22}$$

where $\bar{\xi}^\top = (\xi_1, \xi_2, \ldots, \xi_n)$, $\xi_k \in \Xi$, $k = 1, \ldots, n$, A is a real-valued matrix or the element $\eta \in SG_\Xi(\Omega)$ is the square mean limit of the sequence (22)

$$\eta = l.i.m._{n\to\infty}(\bar{\xi}_n^\top A\bar{\xi}_n - \mathbf{E}\bar{\xi}_n^\top A\bar{\xi}_n).$$

Definition 2. Buldygin and Kozachenko (2000) A stochastic process $\xi(t) = \{\xi(t),\, t \in \mathbf{T}\}$ is square Gaussian if for any $t \in \mathbf{T}$ a random variable $\xi(t)$ belongs to the space $SG_\Xi(\Omega)$.

There are shown in the book by Buldygin and Kozachenko (2000) that

- $SG_\Xi(\Omega)$ is a Banach space with respect to the norm $\|\zeta\| = \sqrt{\mathbf{E}\zeta^2}$;
- $SG_\Xi(\Omega)$ is a subspace of the Orlicz space $L_U(\Omega)$ generated by the function
$$U(x) = \exp|x| - 1;$$
- the norm $\|\zeta\|_{L_U(\Omega)}$ on $SG_\Xi(\Omega)$ is equivalent to the norm $\|\zeta\|$.

Example 1. Consider a family of Gaussian centered stochastic processes
$\xi_1(t), \xi_2(t), \ldots, \xi_n(t)$, $t \in \mathbf{T}$. Let the matrix $A(t)$ be symmetric. Then

$$X(t) = \bar{\xi}^\top(t)A(t)\bar{\xi}(t) - \mathbf{E}\bar{\xi}^\top(t)A(t)\bar{\xi}(t),$$

where $\bar{\xi}^\top(t) = (\xi_1(t), \xi_2(t), \ldots, \xi_n(t))$, is a square Gaussian stochastic process.

For more information about properties of square Gaussian random processes see Buldygin and Kozachenko (2000), Kozachenko and Moklyachuk (1999), Kozachenko and Moklyachuk (2000), Kozachenko *et al.* (2007), Kozachenko *et al.* (2016), Kozachenko and Stus (1998).

The following theorem can be found in article Kozachenko and Troshki (2015).

Theorem 1. *Kozachenko and Troshki (2015) Let* $\{\boldsymbol{T}, A, \mu\}$ *be a measurable space, where* $\boldsymbol{T}$ *is a parametric set, and let* $\xi = \{\xi(t), t \in \boldsymbol{T}\}$ *be a measurable square Gaussian stochastic process. Assume that the Lebesgue integral* $\int_{\boldsymbol{T}} (\boldsymbol{E}\xi^2(t))^{\frac{p}{2}} d\mu(t)$ *is well defined for* $p \geq 1$. *Then the integral* $\int_{\boldsymbol{T}} (\boldsymbol{E}\xi^2(t))^p d\mu(t)$ *exists with probability 1, and*

$$P\left\{\int_{\boldsymbol{T}} |\xi(t)|^p d\mu(t) > x\right\} \leq 2\sqrt{1 + \frac{x^{1/p}\sqrt{2}}{C_p^{\frac{1}{p}}}} \exp\left\{-\frac{x^{1/p}}{\sqrt{2}C_p^{\frac{1}{p}}}\right\}, \tag{23}$$

for all $x \geq (\frac{p}{\sqrt{2}} + \sqrt{(\frac{p}{2}+1)p})^p C_p$, *where* $C_p = \int_{\boldsymbol{T}} (\boldsymbol{E}\xi^2(t))^{\frac{p}{2}} d\mu(t)$.

5. On the rate of convergence of the estimator of impulse response function

This section is devoted to the investigation of the rate of convergence of estimators of unknown IRF in the space of the space $L_2([0, \Lambda])$.

The next Lemma is trivial.

Lemma 4. *Stochastic process* $\hat{Z}_{N,T,\Lambda}(\tau) = \hat{H}_{N,T,\Lambda}(\tau) - E\hat{H}_{N,T,\Lambda}(\tau)$, $\tau > 0$, *is a square Gaussian one.*

Consider a difference of the estimator $\hat{H}_{N,T,\Lambda}(\tau)$ and the impulse function $H(\tau)$ nonmetering the response on the constant signal a_0

$$H^*(\tau) - \hat{H}_{N,T,\Lambda}(\tau), \quad \tau > 0.$$

Denote

$$h^*_{N,\Lambda} = \frac{\Lambda(d + I_2(\Lambda))}{2\pi^2 N}.$$

Then from (20) it follows that

$$|E\hat{H}_{N,T,\Lambda}(\tau) - H^*(\tau)| \leq h^*_{N,\Lambda}, \quad \tau \in [0, \Lambda].$$

Remark 5. *Since* $h^*_{N,\Lambda} \to 0$ *as* $N \to \infty$ *then the estimator* $\hat{H}_{N,T,\Lambda}(\tau)$ *is asymptotically unbiased.*

Put

$$\gamma_0(N, T, \Lambda) = \gamma_0 = \frac{\Lambda\sqrt{\Lambda(\Lambda+2)}I_1}{\pi^2 T}\left(2 - \frac{1}{N}\right). \tag{24}$$

From (21) we have that

$$\sup_{\tau \in [0,\Lambda]} \sqrt{Var\hat{Z}_{N,T,\Lambda}(\tau)} \leq \gamma_0$$

Since $\gamma_0 \to 0$ as $T, N \to \infty$ then a sufficient condition for consistency of $\hat{H}_{N,T,\Lambda}(\tau)$ is fulfilled.

Consider a parametric set $\mathbf{T} = [0, \Lambda]$ and let $\{[0, \Lambda], \mathbf{F}, \mu\}$ be a metric space with Euclidean measure μ.

Theorem 2. *Assume that the conditions* **A**, **B** *are satisfied. Then for*

$$\varepsilon \geq \left((\frac{p}{\sqrt{2}} + \sqrt{(\frac{p}{2}+1)p})\Lambda^{\frac{1}{p}}\gamma_0 + \Lambda h^*_{N,\Lambda}\right)^p \tag{25}$$

the estimate

$$P\left\{\int_0^\Lambda |H^*(\tau) - \hat{H}_{N,T,\Lambda}(\tau)|^p d\tau > \varepsilon\right\} \leq 2\sqrt{1 + \frac{(\varepsilon^{\frac{1}{p}} - \Lambda h^*_{N,\Lambda})\sqrt{2}}{\Lambda^{\frac{1}{p}}\gamma_0}} \exp\left\{-\frac{\varepsilon^{\frac{1}{p}} - \Lambda h^*_{N,\Lambda}}{\sqrt{2}\Lambda^{\frac{1}{p}}\gamma_0}\right\} \tag{26}$$

holds true.

Remark 6. *Emphasize that the lower bound on ε in (25) can be made arbitrarily small. Really, the values γ_0 and $h^*_{N,\Lambda}$ go to 0 as $T, N \to \infty$. The bigger T and N are, the smaller ε is.*

Proof. We show first that the result of Theorem 1 can be applied to the process $\hat{Z}_{N,T,\Lambda}(\tau),\ \tau > 0$. Really, by Lemma 4 $\hat{Z}_{N,T,\Lambda}(\tau)$ is a square Gaussian stochastic process. Prove that the Lebesgue integral

$$\int_0^\Lambda (\mathbf{E}\hat{Z}^2_{N,T,\Lambda}(\tau))^{\frac{p}{2}} d\mu(\tau)$$

is correctly defined. From inequality (21) follows that

$$\begin{aligned}\int_0^\Lambda (\mathbf{E}\hat{Z}^2_{N,T,\Lambda}(\tau))^{\frac{p}{2}} d\mu(\tau) &= \int_0^\Lambda (Var\hat{Z}_{N,T,\Lambda}(\tau))^{\frac{p}{2}} d\tau \\ &< \Lambda\left(\frac{4I_1(\Lambda)\cdot(N+1)}{\Lambda}\right)^{\frac{p}{2}}.\end{aligned}$$

Therefore, inequality (23) for the process $\hat{Z}_{N,T,\Lambda}(\tau)$ as $x \geq (\frac{p}{\sqrt{2}} + \sqrt{(\frac{p}{2}+1)p})^p C_p$ can be rewritten as

$$P\left\{\int_0^\Lambda |\hat{Z}_{N,T,\Lambda}(\tau)|^p d\mu(\tau) > x\right\} \leq 2\sqrt{1 + \frac{x^{1/p}\sqrt{2}}{C_p^{\frac{1}{p}}}} \exp\left\{-\frac{x^{1/p}}{\sqrt{2}C_p^{\frac{1}{p}}}\right\}, \tag{27}$$

where $C_p = \int_0^\Lambda (\mathbf{E}\hat{Z}^2_{N,T,\Lambda}(\tau))^{\frac{p}{2}} d\tau$.
From the Minkowski inequality follows that

$$\begin{aligned}\left(\int_0^\Lambda |H^*(\tau) - \hat{H}_{N,T,\Lambda}(\tau)|^p d\mu(\tau)\right)^{\frac{1}{p}} &= \left(\int_0^\Lambda |H(\tau) - a_0 \pm \mathbf{E}\hat{H}_{N,T,\Lambda}(\tau) - \hat{H}_{N,T,\Lambda}(\tau)|^p d\mu(\tau)\right)^{\frac{1}{p}} \\ &\leq \left(\int_0^\Lambda |H^*(\tau) - \mathbf{E}\hat{H}_{N,T,\Lambda}(\tau)|^p d\mu(\tau)\right)^{\frac{1}{p}} \\ &+ \left(\int_0^\Lambda |\mathbf{E}\hat{H}_{N,T,\Lambda}(\tau) - \hat{H}_{N,T,\Lambda}(\tau)|^p d\mu(\tau)\right)^{\frac{1}{p}} \\ &\leq h^*_{N,\Lambda}\cdot\Lambda + \left(\int_0^\Lambda |\hat{Z}_{N,T,\Lambda}(\tau)|^p d\mu(\tau)\right)^{\frac{1}{p}}.\end{aligned}$$

If $\varepsilon > (h^*_{N,\Lambda}\Lambda)^p$ then from the relationship above we have that

$$\begin{aligned}\left\{\int_0^\Lambda |H^*(\tau) - \hat{H}_{N,T,\Lambda}(\tau)|^p d\mu(\tau) > \varepsilon\right\} &= \left\{\left(\int_0^\Lambda |H^*(\tau) - \hat{H}_{N,T,\Lambda}(\tau)|^p d\mu(\tau)\right)^{\frac{1}{p}} > \varepsilon^{\frac{1}{p}}\right\} \\ &\subset \left\{\Lambda h^*_{N,\Lambda} + \left(\int_0^\Lambda |\hat{Z}_{N,T,\Lambda}(\tau)|^p d\mu(\tau)\right)^{\frac{1}{p}} > \varepsilon^{\frac{1}{p}}\right\} \\ &= \left\{\int_0^\Lambda |\hat{Z}_{N,T,\Lambda}(\tau)|^p d\mu(\tau) > \left(\varepsilon^{\frac{1}{p}} - \Lambda h^*_{N,\Lambda}\right)^p\right\}.\end{aligned}$$

Substituting $x = \left(\varepsilon^{\frac{1}{p}} - Ah^*_{N,\Lambda}\right)^p$ in (27), we obtain

$$P\left\{\int_0^\Lambda |H^*(\tau) - \hat{H}_{N,T,\Lambda}(\tau)|^p d\mu(\tau) > \varepsilon\right\} \leq 2\sqrt{1 + \frac{(\varepsilon^{\frac{1}{p}} - \Lambda h^*_{N,\Lambda})\sqrt{2}}{C_p^{\frac{1}{p}}}} \exp\left\{-\frac{\varepsilon^{\frac{1}{p}} - \Lambda h^*_{N,\Lambda}}{\sqrt{2}C_p^{\frac{1}{p}}}\right\}. \tag{28}$$

Since (21) implies $\sup_{\tau\in[0,\Lambda]} Var\hat{Z}_{N,T,\Lambda}(\tau) \le (\gamma_0)^2$, where γ_0 is from (24), then

$$C_p = \int_0^\Lambda (\mathbf{E}\hat{Z}^2_{N,T,\Lambda}(\tau))^{\frac{p}{2}} d\tau \le \Lambda\gamma_0^p.$$

To complete the proof it's enough to substitute the above value C_p in (28). Hence,

$$P\left\{\int_0^\Lambda |H^*(\tau) - \hat{H}_{N,T,\Lambda}(\tau)|^p d\mu(\tau) > \varepsilon\right\} \le 2\sqrt{1 + \frac{(\varepsilon^{\frac{1}{p}} - \Lambda h^*_{N,\Lambda})\sqrt{2}}{\Lambda^{\frac{1}{p}}\gamma_0}} \exp\left\{-\frac{\varepsilon^{\frac{1}{p}} - \Lambda h^*_{N,\Lambda}}{\sqrt{2}\Lambda^{\frac{1}{p}}\gamma_0}\right\}.$$

□

6. Goodness-of-fit testing of IRF

Using Theorem 2 it is possible to test hypothesis on the shape of IRF.

Let the null hypothesis $\mathbf{H_0}$ state that an IRF is $H(\tau)$, $\tau \in [0,\Lambda]$, and the alternative $\mathbf{H_a}$ implies the opposite statement.

Denote

$$g(\varepsilon) = g(\varepsilon, N) = 2\sqrt{1 + \frac{(\varepsilon^{\frac{1}{p}} - \Lambda h^*_{N,\Lambda})\sqrt{2}}{\Lambda^{\frac{1}{p}}\gamma_0}} \exp\left\{-\frac{\varepsilon^{\frac{1}{p}} - \Lambda h^*_{N,\Lambda}}{\sqrt{2}\Lambda^{\frac{1}{p}}\gamma_0}\right\}. \tag{29}$$

From Theorem 2 follows that if

$$\varepsilon > t_{N,T,\Lambda}(p) = \left((\frac{p}{\sqrt{2}} + \sqrt{(\frac{p}{2}+1)p})\Lambda^{\frac{1}{p}}\gamma_0 + \Lambda h^*_{N,\Lambda}\right)^p,$$

then

$$P\left\{\int_0^\Lambda |H(\tau) - \hat{H}_{N,T,\Lambda}(\tau)|^p d\tau > \varepsilon\right\} \le g(\varepsilon).$$

Let ε_δ be a solution of the equation $g(\varepsilon_\delta) = \delta$, $0 < \delta < 1$. Put

$$\varepsilon^*_\delta = \max\{\varepsilon_\delta, t_{N,T,\Lambda}(p)\}. \tag{30}$$

It is clear that $g(\varepsilon^*_\delta) \le \delta$ and

$$P\left\{\sup_{\tau\in[0,\Lambda]} |H(\tau) - \hat{H}_{N,T,\Lambda}(\tau)| > \varepsilon^*_\delta\right\} \le \delta.$$

From Theorem 2 it follows that to test the hypothesis $\mathbf{H_0}$, we can use the following testing.

Theorem 3. *For a given level of confidence $1-\delta$, $\delta \in (0,1)$, the hypothesis $\mathbf{H_0}$ is rejected if*

$$\int_0^\Lambda |H(\tau) - \hat{H}_{N,T,\Lambda}(\tau)|^p d\tau > \varepsilon^*_\delta,$$

otherwise the hypothesis $\mathbf{H_0}$ is accepted, where ε^*_δ is from (30).

7. Simulation study

We consider a particular case when $\Lambda = 10$, $\alpha = 1$ and $p = 2$. An IRF is supposed to be $H(\tau) = \tau(e^{-\tau} - e^{-\Lambda})$. Obviously, that the Conditions A, B are fulfilled for $H(\tau)$ and $I_0 = 0.9972$, $I_1 = 0.5$, $I_2 = 1.2703$, $d = 1.0004$. The function $H^*(\tau) = H(\tau) - a_0$ is equal to

$$H^*(\tau) = \tau(e^{-\tau} - e^{-\Lambda}) - 0.3153.$$

Table 1: The minimal values of T for fixed value of $N = 1000$ with given significant level δ and accuracy ε for the function $g(\varepsilon, N) = \delta$

Accuracy, ε	T when $\delta = 0.3$	T when $\delta = 0.2$	T when $\delta = 0.1$
0.001	7022 (0.4816)	8185 (0.2900)	10125 (0.1112)
0.005	2386 (0.3312)	2781 (0.2716)	3441 (0.4302)
0.01	1597 (0.4617)	1861 (0.2328)	2302 (0.1395)
0.025	964 (0.3019)	1123 (0.1912)	1389 (0.4428)
0.05	666 (0.1292)	776 (0.3743)	960 (0.4132)
0.075	539 (0.4674)	628 (0.3397)	776 (0.4467)
0.1	464 (0.3408)	540 (0.3954)	668 (0.4956)
0.2	324 (0.2712)	378 (0.0615)	468 (0.4715)
0.3	263 (0.4898)	307 (0.0977)	380 (0.1051)

Table 2: The minimal values of T for fixed value of $N = 5000$ with given significant level δ and accuracy ε for the function $g(\varepsilon, N) = \delta$

Accuracy, ε	T when $\delta = 0.3$	T when $\delta = 0.2$	T when $\delta = 0.1$
0.001	4820 (0.4189)	5618 (0.2035)	6950 (0.0059)
0.005	2066 (0.1427)	2408 (0.0915)	2979 (0.0723)
0.01	1447 (0.2671)	1686 (0.1688)	2086 (0.1263)
0.025	907 (0.1428)	1057 (0.2759)	1308 (0.0977)
0.05	639 (0.3153)	744 (0.3877)	921 (0.1558)
0.075	520 (0.4909)	607 (0.3672)	750 (0.4346)
0.1	450 (0.2473)	525 (0.2362)	649 (0.1586)
0.2	318 (0.3090)	370 (0.2692)	458 (0.0412)
0.3	259 (0.1477)	302 (0.0369)	374 (0.3654)

Table 3: The minimal values of T for fixed value of $N = 10000$ with given significant level δ and accuracy ε for the function $g(\varepsilon, N) = \delta$

Accuracy, ε	T when $\delta = 0.3$	T when $\delta = 0.2$	T when $\delta = 0.1$
0.001	4639 (0.3200)	5406 (0.3866)	6688 (0.0339)
0.005	2032 (0.0760)	2368 (0.3868)	2930 (0.1890)
0.01	1430 (0.0315)	1667 (0.3702)	2062 (0.2969)
0.025	901 (0.4603)	1050 (0.4200)	1298 (0.3821)
0.05	635 (0.4138)	741 (0.4245)	916 (0.1283)
0.075	518 (0.3214)	604 (0.1041)	747 (0.3065)
0.1	449 (0.3741)	523 (0.1260)	647 (0.1791)
0.2	317 (0.1124)	369 (0.3328)	457 (0.1171)
0.3	259 (0.3848)	301 (0.4163)	373 (0.1332)

Table 4: The minimal values of T for fixed value of $N = 50000$ with given significant level δ and accuracy ε for the function $g(\varepsilon, N) = \delta$

Accuracy, ε	T when $\delta = 0.3$	T when $\delta = 0.2$	T when $\delta = 0.1$
0.001	4503 (0.1214)	5248 (0.1099)	6492 (0.1701)
0.005	2006 (0.3767)	2338 (0.4438)	2892 (0.3280)
0.01	1417 (0.1647)	1651 (0.3230)	2043 (0.2322)
0.025	895 (0.3265)	1044 (0.4960)	1291 (0.1343)
0.05	633 (0.1787)	738 (0.4461)	912 (0.3905)
0.075	517 (0.4012)	602 (0.0965)	745 (0.1770)
0.1	447 (0.3373)	521 (0.3721)	645 (0.0370)
0.2	316 (0.2478)	369 (0.4128)	456 (0.0396)
0.3	258 (0.1909)	301 (0.0783)	372 (0.2550)

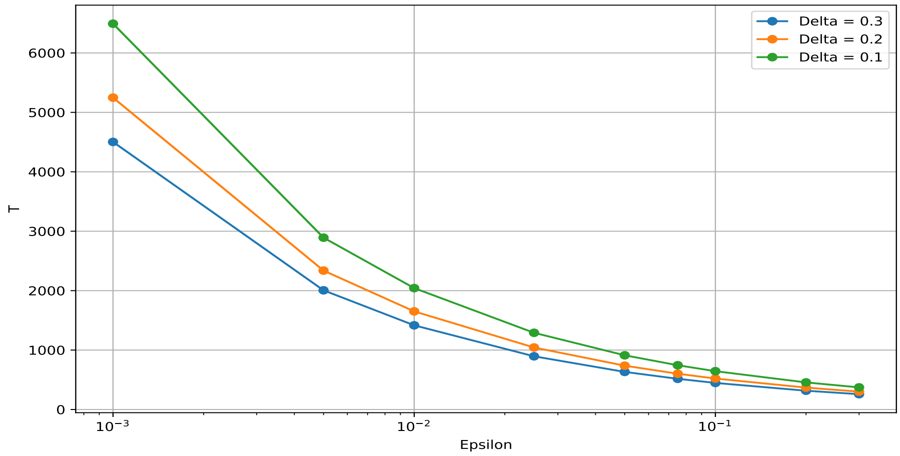

Figure 1: The values of T for fixed value of $N = 50000$

We use the free software environment for statistical computing and graphics R to find T for different values of accuracy and reliability and the level of cutting N of the input process from the conditions of Theorem 2 for L_p metric.

In tables 1-4 the minimal values of T for fixed value of cutting point N with given significant level δ and accuracy ε for the function $g(\varepsilon, N) = \delta$ are calculated and the values in parentheses indicate the standard deviation of the estimator $\hat{H}_{N,T,\Lambda}(\tau)$ with the corresponding values N and T. The values T in tables 1-4 show that the more accurate result we want to obtain the larger term T should be.

Figure 1 demonstrates the dependency of accuracy ε and parameter for averaging T in case of tree different significant level and fixed cutting point $N = 50000$. As we can see the functions are decreasing.

8. Conclusions

In this paper we considered a time-invariant continuous linear system in which the IRF is defined on the domain $[0, \Lambda]$. The input signal process was supposed to be a zero mean Gaussian stochastic process which was represented as a series with respect to an orthonormal basis in $L_2([0, \Lambda]$. A particular case where the orthonormal basis was given by trigonometric functions was studied in details. Some characteristics of the estimator of impulse function such as mathematical expectation, variance were described that . We also investigated the convergence rate for the estimator of unknown IRF in the space $L_p([0,\Lambda])$. For this reason the theory of square Gaussian random variables and processes was applied, namely we used inequality for $L_p(T)$ norm of square Gaussian stochastic process. It gave us an opportunity to construct the goodness of fit test on IRF. In one particular case the values T were found for different accuracy and reliability and N (the upper limit of the summing in the model) using software environment for statistical computing and graphics R.

In future studies we are planning to consider the cases involving discretely observed processes, which are more realistic scenario in functional data analysis as well.

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