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Student Models for a Risky Asset with Dependence: Option Pricing and Greeks

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Abstract

We propose several new models in finance known as the Fractal Activity Time Geometric Brownian Motion (FATGBM) models with Student marginals. We summarize four models that construct stochastic processes of underlying prices with short-range and long-range dependencies. We derive solutions of option Greeks and compare with those in the Black-Scholes model. We analyse performance of delta hedging strategy using simulated time series data and verify that hedging errors are biased particularly for long-range dependence cases. We also apply underlying model calibration on S&P 500 index (SPX) and the U.S./Euro rate, and implement delta hedging on SPX options.

Keywords: option pricing, fractal activity time, student processes, dependence structure, supOU processes, delta hedging.

1. Introduction

Financial markets are difficult to forecast, while modelling financial price movements are not only useful for investment decisions, but also critical for derivative pricing. Granger (2005) proposed his view on the non-Gaussian nature of financial data and future research routes to address this issue. Following this paper, we propose supOU and related models (Barndorff-Nielsen and Shepard 2001; Barndorff-Nielsen and Leonenko 2005) for stochastic pricing with the dependence structure and introduce applications on option pricing and hedging. Stochastic models are widely adopted in asset pricing, however, still face the challenge of replicating empirical properties of financial returns, in particular, non-normal and fat-tails, dependence of squared and absolute returns (named as Taylor effect) (Granger 2005; Heyde 2009). To tackle this problem, there emerges a series of studies. Barndorff-Nielsen, Mikosch, and Resnick (2012) documented recent work of time-changed models that use Lévy processes with Student, normal inverse Gaussian (NIG), variance-gamma (VG) distributions. Since increments of Lévy processes are independent and functions of the elapsed time, all these models can fit the non-Gaussian properties but not the dependencies of squared returns. Another group of researchers focused on establishing models with random time changes (see Barndorff-

Nielsen (1997); Madan and Cherny (2010)) which provide alternative methods to construct fat-tailed distributions, however again, cannot produce dependent properties (even weak dependence). The geometric Brownian motion with fractal activity time (FATGBM) models proposed by Heyde (1999) successfully capture both non-Gaussian and dependencies properties.

In the FATGBM model, a random activity time $\{T_t : t \geq 0\}$, modelled as a positive, non-decreasing stochastic process with stationary but not necessarily independent increments, is incorporated with the standard Brownian motion $\{W_t : t \geq 0\}$:

$$S_t = S_0 e^{\mu t + \theta T_t + \sigma W_{T_t}},\ t \geq 0,\ T_0 = 0, \tag{1}$$

where we assume that $\{W_t : t \geq 0\}$ is independent of $\{T_t : t \geq 0\}$. It is the strong solution of the SDE (see Kobayashi (2011); Kerss, Leonenko, and Sikorskii (2014) and the reference therein):

$$dS_t = \mu S_t dt + \left(\theta + \frac{\sigma^2}{2}\right) S_t dT_t + \sigma S_t dW_{T_t}, \tag{2}$$

where $\mu, \theta \in \mathbb{R}$ and $\sigma > 0$ are constants. The following stochastic representation log-returns is applied:

$$\tilde{R}_t = \log \frac{S_t}{S_{t-1}} \stackrel{\text{d}}{=} \mu + \theta\tau_t + \sigma\sqrt{\tau_t} W_1,\ t = 1, 2, ..., \tag{3}$$

where $\stackrel{\text{d}}{=}$ denotes equality in distributions, and $\tau_t = T_t - T_{t-1}$ is the increments over the (small) unit time. We can also write an approximation of simple-returns:

$$R_t = \frac{S_t - S_{t-1}}{S_{t-1}} \stackrel{\text{d}}{\approxeq} \mu + \left(\theta + \frac{\sigma^2}{2}\right) \tau_t + \sigma\sqrt{\tau_t} W_1,\ t = 1, 2, ..., \tag{4}$$

where $\stackrel{\text{d}}{\approxeq}$ denotes approximate equality in distributions. It is worth noting that this approximation holds only if the time step between $t-1$ and t is small, for example, a day or shorter in practice. Moreover, when using daily or higher frequency data, the log-return and simple-return are numerically close.

Remark 1.1. Approximate equality in distribution (4) follows from (2) if dt is small.

Mathematical theories of FATGBM have been developed in recent years. The distributions of returns are exact. The stochastic volatility models, such as the Heston model and GARCH models, are kind of close to FATGBM models, but many of them do not guarantee explicit distributions of returns. In particular the famous Barndorff-Nielsen and Sheppard stochastic volatility model has only approximated hyperbolic distributions (Barndorff-Nielsen and Shepard 2001). Here we express some properties of dependence structures of the processes R_t, which are also identified in empirical finance, for example, Granger (2005). Assuming finiteness of moments, we have for integer $s \geq 1$,

$$\mathbb{C}ov(\tilde{R}_t, \tilde{R}_{t+s}) = \theta^2 \mathbb{C}ov(\tau_t, \tau_{t+s}), \tag{5}$$

$$\begin{aligned}\mathbb{C}ov(\tilde{R}_t^2, \tilde{R}_{t+s}^2) = &\left(\sigma^4 + 4\theta^2\mu^2 + 4\theta\mu\sigma^2\right) \mathbb{C}ov(\tau_t, \tau_{t+s}) + \theta^4 \mathbb{C}ov(\tau_t^2, \tau_{t+s}^2) \\ &+ \left(\theta^2\sigma^2 + 2\theta^3\mu\right)\left(\mathbb{C}ov(\tau_t, \tau_{t+s}^2) + \mathbb{C}ov(\tau_t^2, \tau_{t+s})\right)\end{aligned}, \tag{6}$$

For $\theta = 0$, this is a symmetric model and these covariance measures reduce to $\mathbb{C}ov(\tilde{R}_t, \tilde{R}_{t+s}) = 0$ and $\mathbb{C}ov(\tilde{R}_t^2, \tilde{R}_{t+s}^2) = \sigma^4 \mathbb{C}ov(\tau_t, \tau_{t+s})$. Thus, the dependence structure expressed by covariance function for the process τ_t implies those for $\tilde{R}_t^2$, irresponsive of the size of μ. Moreover, for $\theta = \mu = 0$ we have

$$\mathbb{C}ov(|\tilde{R}_t|, |\tilde{R}_{t+s}|) = \frac{2}{\pi}\sigma^2 \mathbb{C}ov(\tau_t^{1/2}, \tau_{t+s}^{1/2}). \tag{7}$$

It has been verified that FATGBM models are more suitable to fit the real financial data (Heyde 2009, 2002; Seneta 2004; Heyde and Leonenko 2005; Finlay and Seneta 2007; Leonenko, Petherick, and Sikorskii 2011; Finlay, Seneta, and Wang 2012). More importantly, financial prices seasonality and business cycles can be well explained under using the fractal active time T_t with dependence structures. Regarding model calibration and application, Kerss *et al.* (2014); Leonenko *et al.* (2011) pointed out that the marginal distribution of R_t and its dependence structure can be fitted separately. With all these nice mathematical properties, FATGBM models have advantages in derivative pricing and trading.

The conditional distribution of $\tilde{R}_t$ given $V = \tau_t \geq 0$, is normal with mean $\mu + \theta V$ and variance $\sigma^2 V$. With the idea of subordinated processes, returns can be modelled as a variety of hyperbolic distributions by choosing different distributions for τ_t. For example, Kerss *et al.* (2014), Leonenko, Petherick, and Sikorskii (2012), Finlay and Seneta (2012) and Castelli, Leonenko, and Shchestyuk (2017) modelled prices following normal inverse Gaussian, generalized hyperbolic, tempered stable, and Student distribution, respectively. Moreover, this model can be downgraded to the classic Brownian motion by defining $T_t = t$ (i.e. $\tau_t \equiv 1$). In this paper, we investigate option pricing, Greeks and delta-hedging using Student FATGBM models. Relevant theorems are documented and proved in Heyde and Leonenko (2005), Leonenko *et al.* (2011) and Castelli *et al.* (2017). If the marginal distribution of τ_t is the inverse (or reciprocal) gamma $R\Gamma(a, b)$ with density

$$f_{R\Gamma}(x; a, b) = \frac{b^a}{\Gamma(a)} \frac{1}{x^{a+1}} \exp\{-\frac{b}{x}\},\ x > 0. \tag{8}$$

and if $a = \frac{\nu}{2}$, $b = \frac{\delta^2}{2}$, the strictly stationary stochastic process $\{\tilde{R}_t,\ t = 0, 1, 2...\}$ has marginal Student distribution, known as generalized hyperbolic skew student's t-distribution. Without loss of generality, we apply $\mathbb{E}[\tau_t] = 1$, leading to $\delta^2 = \nu - 2$, since any scaling can be absorbed into μ and σ as required (assuming $\mathbb{E}[\tau_t] < \infty$). The density of $\tilde{R}_t$ is

$$f_{ST}(x) = \begin{cases} \frac{c_1(\nu)}{\tilde{\sigma}} \left(1 + \frac{(\frac{x-\mu}{\tilde{\sigma}})^2}{\nu}\right)^{-\frac{\nu+1}{2}} & \text{for } \theta = 0 \\ \frac{c_2(\nu)}{\tilde{\sigma}} \left(\frac{\tilde{\sigma}^2}{\theta^2}\left(\nu + (\frac{x-\mu}{\tilde{\sigma}})^2\right)\right)^{-\frac{\nu+1}{4}} \mathrm{e}^{\frac{x-\mu}{\tilde{\sigma}^2}\theta\frac{\nu-2}{\nu}} K_{\frac{\nu+1}{2}}\left(\frac{\nu-2}{\nu}\frac{|\theta|}{\tilde{\sigma}}\sqrt{\nu + (\frac{x-\mu}{\tilde{\sigma}})^2}\right) & \text{for } \theta \neq 0 \end{cases}, \tag{9}$$

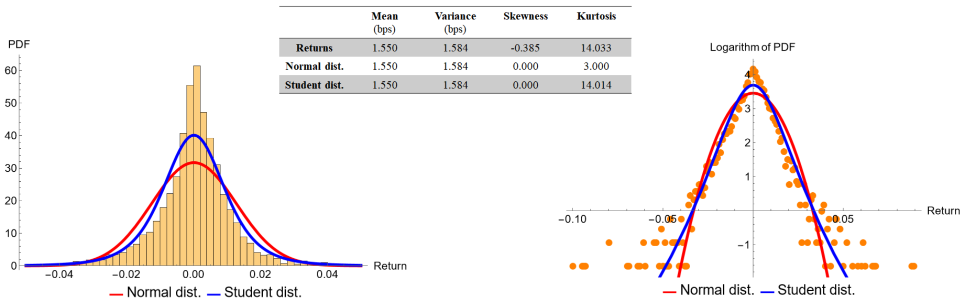
$x \in \mathbb{R}$,

where $\mu \in \mathbb{R}$ is location parameter, $\tilde{\sigma} = \sigma\delta/\sqrt{\nu} > 0$ is scaling parameter, $\nu > 2$ is degrees of freedom, $c_1(\nu) = \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\nu\pi}\Gamma(\frac{\nu}{2})}$, $c_2(\nu) = \sqrt{\frac{2}{\pi}\frac{\nu-2}{\nu}} \frac{(\frac{\nu}{2}-1)^{\frac{\nu}{2}}}{\Gamma(\frac{\nu}{2})}$, and $K_\lambda(x) = \frac{1}{2}\int_0^\infty z^{\lambda-1}\mathrm{e}^{-\frac{x}{2}(z+\frac{1}{z})}dz$ is the modified Bessel functions of the third kind. See Appendix A for more details.

Although theorems of FATGBM models are sufficiently explored, academic finance research on model applications is scarce. To the best of our knowledge, hedging strategies of FATGBM are still not investigated. That explains why, despite good mathematics properties, this modelling technique has not raised much attention in industrial practices. In this paper, we formulate the option Greeks and the basic delta hedging strategy for European options priced by FATGBM models. We show that analytical solutions of Greeks exist for these models, hence, instant risks are measurable. However, in FATGBM models, there are two sources of randomness, T_t and W_t, that construct an incomplete market. Hence, a perfect hedging does not exist (for more details, see Section 3.4). We compare the Greeks and hedging errors of the Student FATGBM model (with short- and long-range dependency) and the Black-Scholes model. We also run an application experiment by fitting the S&P 500 Index data to the model, and show the European option pricing and hedging.

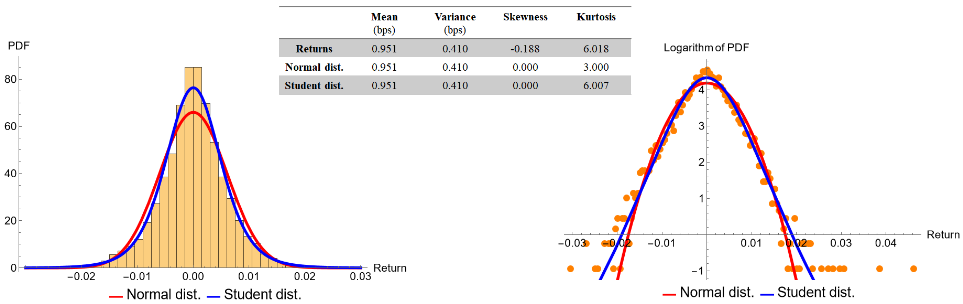
2. The facts

In this section, we present the time series and statistical properties of financial returns using examples of S&P 500 Index and U.S./Euro rate. It is well-known that financial returns are not normal. They are usually slightly skewed and fat-tailed, even heavy-tailed. We can observe these in Figure 1 and that the Student distribution gives better fitting than the normal distribution. The statistics also indicates that Student model successfully fits



	Mean (bps)	Variance (bps)	Skewness	Kurtosis
Returns	1.550	1.584	-0.385	14.033
Normal dist.	1.550	1.584	0.000	3.000
Student dist.	1.550	1.584	0.000	14.014

(a) S&P 500 Index



	Mean (bps)	Variance (bps)	Skewness	Kurtosis
Returns	0.951	0.410	-0.188	6.018
Normal dist.	0.951	0.410	0.000	3.000
Student dist.	0.951	0.410	0.000	6.007

(b) U.S./Euro rate

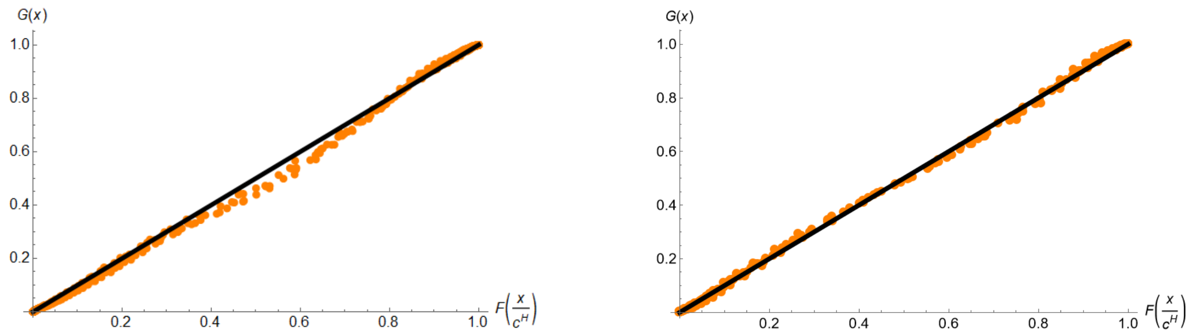
Figure 1: Fitting returns to Normal dist. and Student-T dist.

the leptokurtic property of returns. There have been studies of Student like models with skewness (Borland and Bouchaud 2004), while they are not based on the FATGBM. Indeed, skewness is sometimes ignored by financial analysts due to the lack of economic explanations. Hence, we think symmetric FATGBM models are sufficient for practitioners.

Definition 2.1 (Self-similarity)**.** Stochastic process $\{Z_t, t \geq 0\}$ is self-similar if some $H \in (0,1)$ exists such that $\forall c > 0$, $Z_{ct} \overset{\text{fdd}}{=} c^H Z_t$, $t \geq 0$, where $\overset{\text{fdd}}{=}$ denotes equality in a sense of finite dimensional distributions.

We show this property through Q-Q plot in Figure 2 of empirical returns R_{ct} and R_t, taking $c = 2, 3, ..., 10$. (Kerss *et al.* 2014; Leonenko *et al.* 2012; Castelli *et al.* 2017) prove that returns produced by FATGBM models approximately self-similar processes.

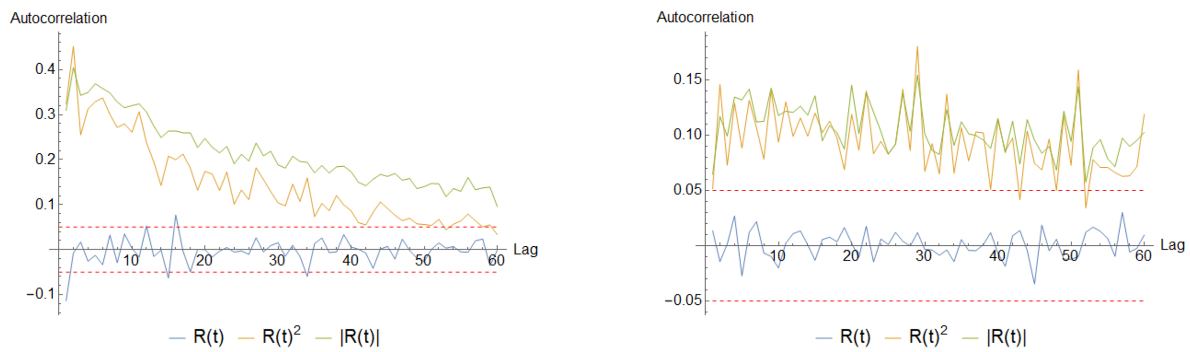
Another important empirical property is the dependence structures of absolute and squared returns. Figure 3 shows that autocorrelation of returns is weak or statistically insignificant. While both absolute and squared returns have strong autocorrelations. For the S&P 500 Index in Figure 3a, these autocorrelations decay fast, indicating short-range dependence (SRD). On the contrary, in Figure 3b, absolute and squared returns of the U.S./Euro rate have long-range dependence (LRD). In the theories of FATGBM models, it has been proved that SRD and LRD can be modelled by defining corresponding dependence structures for τ_t (Heyde 1999; Heyde and Leonenko 2005). Detailed methods particularly for Student FATGBM models are in Section 3.1.



(a) S&P 500 Index ($H = 0.500$)

(b) U.S./Euro rate ($H = 0.535$)

Figure 2: Asymptotic self-similarity of returns



(a) S&P 500 index

(b) U.S./Euro rate

Figure 3: Dependence structures of returns

3. Pricing and hedging of Student FATGBM models

Recall that there are two independent stochastic processes in FATGBM: a standard Brownian motion W_t, and the fractal activity time T_t. Also, in Student FATGBM, $\tau_t := T_t - T_{t-1} \sim R\Gamma(\frac{\nu}{2}, \frac{\delta^2}{2})$. Under two special cases we get returns follow symmetric Student distribution $T(\mu, \sigma\frac{\delta}{\sqrt{\nu}}, \nu)$. One case is $\theta = 0$, $\tilde{R}_t \sim T(\mu, \sigma\frac{\delta}{\sqrt{\nu}}, \nu)$. The other case is $\theta = -\frac{1}{2}\sigma^2$, leading to $R_t \sim T(\mu, \sigma\frac{\delta}{\sqrt{\nu}}, \nu)$. The latter is required by the martingale construction for derivative pricing, which is explained in details in Section 3.3. Specific model constructions below are symmetric Student processes; hence, they apply to either of the two cases.

3.1. Models with dependence structures

Student FATGBM models in finance have been well explored by Heyde and Leonenko (2005), Leonenko *et al.* (2011) and Castelli *et al.* (2017), including the construction of SRD and LRD for τ_t that produce dependence structures of squared returns (see Equation (6)). As we do not observe dependences structure of returns, we believe θ should be small.

Definition 3.1 (Short-range dependence)**.** A stationary process $\{X_t : t \in \mathbb{R}\}$ with $\mathbb{E}[X_t^2] < \infty$ is called short-range dependence (SRD), or short-memory, if

$$\int_0^\infty |\mathbb{C}ov\left[X_t, X_{t+s}\right]| ds < \infty \text{ and } \int_0^\infty \mathbb{C}ov\left[X_t, X_{t+s}\right] ds \neq 0.$$

Definition 3.2 (Long-range dependence)**.** A stationary process $\{X_t : t \in \mathbb{R}\}$ with $\mathbb{E}[X_t^2] < \infty$ is called long-range dependence (LRD), or long-memory, if

$$\int_0^\infty |\mathbb{C}ov\left[X_t, X_{t+s}\right]| ds = \infty \text{ and } \int_0^\infty \mathbb{C}ov\left[X_t, X_{t+s}\right] ds \neq 0.$$

We summarize construction models of SRD and LRD processes X_t with inverse gamma marginal distributions introduced in Heyde and Leonenko (2005), Finlay and Seneta (2007), Leonenko *et al.* (2011), Finlay *et al.* (2012), Leonenko *et al.* (2012), Finlay and Seneta (2012) and Castelli *et al.* (2017). Regarding all these models, define $\tilde{X}_t = \int_0^t X_t dt$. For some normalising sequence A_N that tends to infinity, by Lamperti's limit theorem (see Theorem 2.8.5 in Pipiras and Taqqu (2017)), $\frac{1}{A_N}\left(\tilde{X}_{[Nt]} - \mathbb{E}[\tilde{X}_{[Nt]}]\right)$, $t \in [0,1]$ converges in a sense of finite-dimensional distributions (or in the Skorokhod space $D[0,1]$) to certain H-self-similar process Z_t, such as fractional Brownian motion, sum of independent Rosenblatt processes or more general Hermite-type processes. Further, we find the range of the self-similarity parameter H (in Definition 2.1) in Section 5.

Model I: It is known that inverse gamma distribution is infinitely divisible and self-decomposable (Barndorff-Nielsen and Shepard 2001; Heyde and Leonenko 2005; Massing 2018). According to Barndorff-Nielsen and Shepard (2001) and Heyde and Leonenko (2005), there exists a strictly stationary Ornstein–Uhlenbeck (OU)-type process $\{X_t,\, t \geq 0\}$ that has $R\Gamma(\frac{\nu}{2}, \frac{\delta^2}{2})$ distribution and satisfies the stochastic differential equation:

$$dX_t = -\lambda X_t dt + dL(\lambda t),\, t \geq 0, \tag{10}$$

where $\{L(t) : t \geq 0\}$ is the background driving Lévy process (BDLP) for X_t. Its cumulant function is given in Heyde and Leonenko (2005).

Model II: Define $\{X_t,\, t \geq 0\}$ as the weak solution of the stochastic differential equation:

$$dX_t = -\lambda\left(X_t - \frac{\delta^2}{\nu - 2}\right) dt + \sqrt{\frac{4\lambda}{\nu - 2} X_t^2} dW_t,\, \delta > 0,\, \nu > 2, \tag{11}$$

where $\{W_t : t \geq 0\}$ is a standard Brownian motion. According to Remark 3.3 in Heyde and Leonenko (2005), this SDE has a unique Markovian weak solution, which is also ergodic (see more details in Castelli *et al.* (2017)), with invariant $R\Gamma(\frac{\nu}{2}, \frac{\delta^2}{2})$ distribution.

Both Model I and Model II produce short-memory. The autocorrelation function of $X_t \sim R\Gamma(a,b)$ exists if $b > 2$, and is given by $\rho_X(s) = e^{-\lambda s}$, $s \geq 0$. Furthermore, the discrete time process $\{X_t,\, t = 0, 1, 2...\}$ is strictly stationary and has inverse gamma marginal. According to Proposition 4 in Castelli *et al.* (2017),

$$\bar{X}_N(t) = \frac{1}{c\sqrt{N}}\left(\tilde{X}_{[N\cdot t]} - [N \cdot t]\right) \overset{\text{Skd}}{\Longrightarrow} W_t,\, t \in [0,1]\,, \text{ as } N \to \infty, \tag{12}$$

where $\overset{\text{Skd}}{\Longrightarrow}$ means the weak convergence in the Skorokhod space, $\{W_t : t \geq 0\}$ is the standard Brownian motion, and c is a normalizing constant given by $c^2 = \mathbb{V}ar\,[\tau_1]\,\frac{e^{\lambda}+1}{e^{\lambda}-1}$. Hence, the process $\tilde{X}_t - t$ is asymptotically self-similar (see, for details in Kerss *et al.* (2014), Heyde and Leonenko (2005) and Leonenko *et al.* (2011)) such that

$$\tilde{X}_t - t \overset{\text{d}}{\approxeq} \sqrt{t}\left(\tilde{X}_1 - 1\right),$$

where $\overset{\text{d}}{\approxeq}$ denotes asymptotic equality in distribution.

Remark 3.1. Let $\tau_t = X_t, t = 0, 1, 2, ...$ be a discrete time stochastic process with $R\Gamma(\frac{\nu}{2}, \frac{\delta^2}{2})$ marginal distribution, and $\tilde{Y}_N = \sum_0^{N-1} \tau_t$. For

$$\bar{Y}_N(t) = \frac{1}{\bar{c}\sqrt{N}}\left(\tilde{Y}_{[N\cdot t]} - \mathbb{E}\left[\tilde{Y}_{[N\cdot t]}\right]\right) = \frac{1}{\bar{c}\sqrt{N}}\left(\tilde{Y}_{[N\cdot t]} - [N \cdot t]\right),\, t \in [0,1]\,,$$

similar to Billingsley (1968, p.178–179), one can prove

$$\delta_N = \sup_t \left|\bar{Y}_N(t) - \bar{X}_N(t)\right| \xrightarrow{\text{P}} 0,\, N \to \infty.$$

It means that limit distributions of mixing stochastic processes with discrete time $\tau_t, t = 0, 1, 2, ...$ and continuous time $X_t, t \geq 0$ coincide.

Model III: Following Barndorff-Nielsen and Shepard (2001), Barndorff-Nielsen and Leonenko (2005) and Grahovac, Leonenko, and Taqqu (2019), we have a strictly stationary superpositions of OU processes (supOU):

$$X_t = \int\int_{(0,\infty)\times(-\infty,t]} e^{-\xi(t-s)}\Lambda(d\xi, ds),\ t\in\mathbb{R} \tag{13}$$

where Λ is a homogeneous infinitely divisible random measure (Lévy basis) on $\mathbb{R}_+\times\mathbb{R}$ with comulant function $\log\mathbb{E}e^{i\zeta\Lambda(A)} = (\pi\times Leb)(A)\kappa_L(\zeta)$, for $A\in\mathcal{B}(\mathbb{R}_+\times\mathbb{R})$ and involves π such that $\int_{(0,\infty)}\frac{\pi(dx)}{x} < \infty$, Leb and κ_L which we now define. The probability measure π on $\mathbb{R}_+$ "randomizes" the rate parameter ξ, Leb is the Lebesgue measure on $\mathbb{R}$, and κ_L is the cumulant function of some infinitely divisible random variable $L(1)$ with Lévy-Khintchine triplet $\left(\tilde{a}, \tilde{b}, \mu_L\right)$:

$$\begin{aligned}&\kappa_L(\zeta) = \log\mathbb{E}e^{i\zeta L(1)} = i\zeta\tilde{a} - \frac{\zeta^2}{2}\tilde{b} + \int_{\mathbb{R}}\left(e^{i\zeta x} - 1 - i\zeta x\mathbf{1}_{[-1,1]}(x)\right)\mu_L(dx),\\ &\int_{\mathbb{R}}\min(1,x^2)\mu_L(dx) < \infty,\ \int_{|x|>1}\log|x|\mu_L(dx) < \infty,\ \tilde{a}\in\mathbb{R},\ \tilde{b}\geq 0.\end{aligned} \tag{14}$$

As $R\Gamma(a,b)$ is a self-decomposable marginal distribution, we can find corresponding BDLP Lévy process such that the strictly stationary supOU process (13) has the chosen marginal distribution (see Barndorff-Nielsen and Shepard (2001) and Grahovac *et al.* (2019) for more details). Heyde and Leonenko (2005), Barndorff-Nielsen and Shepard (2001) and Massing (2018) also show that the Lévy triplet should be $(a_0, 0, \mu_L)$ in which μ_L is a σ-finite measure in $(0,\infty)$ satisfying $\int_0^\infty (u\wedge 1)\,\mu_L(du) < \infty$. More precisely,

$$\begin{aligned}&a_0 = a - \int_{|u|\leq 1} u\mu_L(du),\ \mu_L(du) = \left(\frac{1}{u}\int_0^\infty e^{-su}2bg_a(4bs)ds\right)du,\\ &g_a(x) = \frac{2}{\pi^2 x}\left(J_a^2(\sqrt{x}) + Y_a^2(\sqrt{x})\right)^{-1},\ x>0.\end{aligned}$$

where J_a and Y_a are the Bessel functions of the first and the second kind respectively. We also know that the Blumenthal-Getoor index of this process $\beta_{BG} = \inf\left\{s\geq 0 : \int_{|x|\leq 1}|x|^s\,\mu_L(dx)\right\}$ satisfies $0\leq\beta_{BG}\leq 1$ for a subordinator (see Section 6 in Blumenthal and Getoor (1961)).
The correlation function of supOU process X_t, under the condition that $\mathbb{E}[X_t^2] < \infty$, is of the form

$$\rho_X(s) = \int_{\mathbb{R}_+} e^{-\xi s}\pi(d\xi),\ s\geq 0. \tag{15}$$

Using the Tauberian theorem (see, e.g., Fasen and Klüppelberg (2007)), one can obtain that if for some $\alpha > 0$ and some slowly varying at infinity function $\ell(\cdot)$, the density $\pi((0,x])\sim p(x)$ of a measure π satisfies $p(x)\sim\ell(x^{-1})x^\alpha$, $x\to 0$, then the correlation function (15) becomes $\rho_X(s) = \Gamma(1+\alpha)\ell(s)s^{-\alpha}$, $s\to\infty$, and, in particular, $\alpha\in(0,1)$ yields the LRD. It is also worth noting that this model can be downgraded to Model I, a SRD case, when $\xi\equiv\lambda$. We use some elementary properties of slowly varying functions below (see (Bingham, Goldie, and Teugels 1987, Proposition 1.3.6)).

Proposition 3.1. *If $\ell(x)$ is a slowly varying function, then*

1. *$\ell(x)^k$ is slowly varying for each $k\in\mathbb{R}$;*
2. *for each $\varepsilon > 0$, $\lim_{x\to\infty} x^\varepsilon\ell(x) = \infty$ and $\lim_{x\to\infty} x^{-\varepsilon}\ell(x) = 0$.*

Using Theorem 3.2 in Grahovac *et al.* (2019), as $\tilde{b} = 0$ and $\beta_{BG} < 1+\alpha$, $\alpha\in(0,1)$, we have

$$\frac{1}{N^{1/(\alpha+1)}\ell^{\#}(N)^{1/(\alpha+1)}}\tilde{X}_{N\cdot t} \overset{\text{fdd}}{\Longrightarrow} L_{1+\alpha}(t),$$

where $\overset{\text{fdd}}{\Longrightarrow}$ means the convergence to a finite-dimensional distribution, $\ell^{\#}(x)$ is *de Bruijn conjugate* of $\frac{1}{\ell(x^{1/(\alpha+1)})}$, and $L_{1+\alpha}$ is $(1+\alpha)$-stable Lévy process. The *de Bruijn conjugate* of some slowly varying function ℓ is the unique slowly varying function $\ell^{\#}$ such that $\ell(x)\ell^{\#}(x\ell(x)) \to 1$ and $\ell^{\#}(x)\ell(x\ell^{\#}(x)) \to 1$ (Grahovac *et al.* 2019). We then use Proposition 3.1 above. Let $H = \frac{1}{1+\alpha}$, we get

$$\frac{1}{N^H \ell^{\#}(N)^H} \left(\tilde{X}_{N\cdot t} - N \cdot t\right) \overset{\text{fdd}}{\Longrightarrow} L_{1+\alpha}(t),\, t \in [0,1]\,, \text{ as } N \to \infty. \tag{16}$$

In other words, through a well-defined normalizing factor, $\tilde{X}_t - t$ is asymptotically H-self-similar. As $H = \frac{1}{1+\alpha}$ and $0 < \alpha < 1$, we also have $\frac{1}{2} < H < 1$.

Model IV: Let $\eta_1(t)$, $\eta_2(t)$, ..., $\eta_\nu(t)$, $\nu > 4$, $t \geq 0$ be independent copies of zero mean, mean-square continuous Gaussian stationary stochastic process with a correlation function $\rho_\eta(s) \geq 0$, $s \in \mathbb{R}$. The chi-squared process $\chi^2_\nu(t)$ is defined by

$$\chi^2_\nu(t) = \frac{\eta_1^2(t) + \eta_2^2(t) + ... + \eta_\nu^2(t)}{2},\, t \geq 0.$$

Note that

$$\mathbb{E}[\chi^2_\nu(t)] = \frac{\nu}{2},\, \mathbb{V}ar[\chi^2_\nu(t)] = \frac{\nu}{2},\, \varrho_{\chi^2}(s) = \mathbb{C}ov[\chi^2_\nu(t), \chi^2_\nu(t+s)] = \frac{\nu}{2}\rho^2_\eta(s),\, s \in \mathbb{R}.$$

Let $X_t = G(\chi^2_\nu(t))$, $G(u) = \frac{\delta^2}{2u}$, then X_t is a strictly stationary process with $R\Gamma(\frac{\nu}{2}, \frac{\delta^2}{2})$ marginal distribution. Note that $G(u) \in L_2((0,\infty), p(u)du)$ if $\nu > 4$, where $f_{\Gamma(\frac{\nu}{2},1)} = p(u) = e^{-u}u^{\frac{\nu}{2}-1}/\Gamma(\frac{\nu}{2})$, $u > 0$. A more recent study about mathematical properties of this model can be found in Wang (2013).

It is worth noting that this model can construct both long- and short-memory processes. If the correlation function $\varrho_{\chi^2}(s)$ is summable, then $\chi^2_\nu(t)$ and X_t have short-memory, otherwise, it has long-memory. For example, using the correlation function $\rho_\eta(s) = (1+s^2)^{-\frac{\alpha}{2}}$, then $\mathbb{C}orr[\chi^2_\nu(t), \chi^2_\nu(t+s)] = \rho^2_\eta(s)$ and the stationary process $\chi^2_\nu(t)$ has LRD if $0 < \alpha < \frac{1}{2}$ and SRD if $\frac{1}{2} < \alpha < 1$. Similarly, when $0 < \alpha < \frac{1}{2}$, the process X_t have long-memory; while $\frac{1}{2} < \alpha < 1$ gives short-memory (see Leonenko *et al.* (2011) for more details). Similar results hold for class of correlation function $\rho_\eta(s) = \frac{\mathcal{L}(s)}{s^\alpha}$, where $\mathcal{L}(s)$ is a slowly varying function. If the SRD condition holds, then

$$\frac{1}{\sigma\sqrt{N}} \left(\tilde{X}_{[N\cdot t]} - [N \cdot t]\right) \overset{\text{fdd}}{\Longrightarrow} W_t, \text{ as } N \to \infty, \tag{17}$$

where W_t is a standard Brownian motion and σ^2 is variance of X_t. If X_t has LRD, then

$$\frac{1}{N^{1-\alpha}} \left(\tilde{X}_{[N\cdot t]} - [N \cdot t]\right) \overset{\text{fdd}}{\Longrightarrow} -\frac{1}{\nu}\sum_{i=1}^{\nu} R_i(t), \text{ as } N \to \infty, \tag{18}$$

where $R_i(t)$, $i = 1, 2, ..., \nu$ are independent Rosenblatt processes. For detailed proofs, see Section 7.2 in Leonenko *et al.* (2011) (similarly Section 5.1 in Heyde and Leonenko (2005)) and Section 6 in Taqqu (1975). Hence, $\tilde{X}_t$ is H-self-similar for $H = 1 - \alpha$ and $\alpha \in (0, \frac{1}{2})$. Here we confirm that $H \in \left(\frac{1}{2}, 1\right)$ for long-memory.

Remark 3.2. Similar to Lemma 5.2 in Berman (1979), see also Lemma 1 in Leonenko and Taufer (2006), one can prove that the limit distribution of a normalized sum of the stochastic process LRD under discrete time $\tau_t = X_t, t = 0, 1, 2, ...,$ where $X_t, t \geq 0$ is stochastic process with continuous time as in Proposition 3.1, and that in (16) and (18) are the same. See also Remark 3.1 for similar fact for (17).

3.2. Construction of fractal activity time

Let $\tau_t = X_t$ and $T_t = \tilde{X}_t$, $t = 1, 2, ...$ We consider a typical construction of T_t below:

$$T_t = \sum_{i=1}^{[t]} \tau_i + \tau_{[t]+1} \left(t - [t]\right), \, T_0 = 0, \tag{19}$$

where $\{\tau_t : t \geq 0\}$ is a strict stationary process with finite second moment, also recall the assumption $\mathbb{E}[\tau_i] = 1$, $i = 1, 2, ...$ This construction creates a discrete model of T_t. Limit distributions of normalized sum of this stochastic process with discrete time are the same as limit distributions of stochastic processes with continuous time (see Remarks 3.1 and 3.2) and limit theorems in (12), (16), (17), (18), and Proposition 3.1. As a kind of adjustment to Lambert's theorem, we know that if covariance functions are similar asymptotically at infinity and second moments exist, then asymptotic distributions of sums and integrals of stationary processes are the same. Hence, limit theorems for sums and integrals are the same, including both SRD and LRD cases (see Lemma 1 in Leonenko and Taufer (2006) for proof). The time unit in this construction model can be any natural time period, e.g. 5 minutes, 2 hours, 1 day. Ideally, we should pick a time unit as small as possible in practice.

As T_t is H-self-similar, we have $T_{ct} - ct \overset{\mathrm{d}}{\approxeq} c^H \left(T_t - t\right)$, where $\overset{\mathrm{d}}{\approxeq}$ denotes asymptotic equality in distributions, H is the self-similar parameter, and $H = 1/2$ and $1/2 < H < 1$ for SRD and LRD respectively. Hence, with $c = Y$, $t = 1$, as well as the assumption $T_0 = 0$, we have $T_Y \overset{\mathrm{d}}{\approxeq} Y + Y^H \left(T_1 - 1\right)$ and

$$f_{T_Y}(s) \approxeq \frac{1}{Y^H} f_\tau\left(\frac{s}{Y^H} - \left(Y^{1-H} - 1\right)\right), \tag{20}$$

where $f_\tau(s)$ is the marginal density of τ_1.

3.3. European option pricing

The derivative pricing typically requires the discounted underlying asset price $\{\mathrm{e}^{-rt} S_t, t \geq 0\}$ to be a martingale, in which r is the risk-free rate. The simplest solution is a skew-correcting martingale construction that constrains $\mu = r$ and $\theta = -\frac{1}{2}\sigma^2$ (see details in, e.g., Finlay and Seneta (2006); Finlay *et al.* (2009))). We consider the σ-algebras $\mathcal{F}_s = \sigma\{\{W(u), u \leq T_s\}, \{T_u, u \leq s\}\}$ and $\mathcal{F}^\star_{s,t} = \sigma\{\{W(u), u \leq T_s\}, \{T_u, u \leq s\}, T_t\}$, $0 \leq s \leq t$, then $\mathcal{F}_s \subseteq \mathcal{F}^\star_{s,t}$. Then

$$\left.\begin{aligned}
\mathbb{E}[\mathrm{e}^{-rt} S_t \mid \mathcal{F}_s] &= S_0 \mathbb{E}[\mathrm{e}^{(\mu-r)t+\theta(T_t-T_s+T_s)+\sigma\left(W_{T_t}-W_{T_s}+W_{T_s}\right)} \mid \mathcal{F}_s] \\
&= S_s \mathrm{e}^{(\mu-r)t-\mu s} \mathbb{E}[\mathbb{E}[\mathrm{e}^{\theta(T_t-T_s)+\sigma\left(W_{T_t}-W_{T_s}\right)} \mid \mathcal{F}^\star_{s,t}] \mid \mathcal{F}_s] \\
&= \mathrm{e}^{-rs} S_s \mathrm{e}^{(\mu-r)(t-s)} \mathbb{E}[\mathrm{e}^{\left(\theta+\frac{1}{2}\sigma^2\right)(T_t-T_s)} \mid \mathcal{F}_s] \\
&= \mathrm{e}^{-rs} S_s
\end{aligned}\right., \; 0 \leq s \leq t. \tag{21}$$

where we used the moment generating function of a normal variable. In this case, we take $\mu = r$ and $\theta = -\frac{1}{2}\sigma^2$ which renders $\mathrm{e}^{-rt} S_t$ a Q-martingale under the risk-neutral probability measure Q, with four free parameters r, σ, ν and H, which can be estimated from real data. With such a setup, one can use this model to price options.

Hence, we say

$$S^\star_t = S_0 \mathrm{e}^{rt - \frac{1}{2}\sigma^2 T_t + \sigma W_{T_t}},$$

is the underlying pricing process under the risk-neutral measure Q.

As a summary, for options expire at time Y, the call price $C(\cdot)$ is and put price $P(\cdot)$ are formulated below (see Kerss *et al.* (2014), Heyde and Leonenko (2005), Leonenko *et al.* (2011), Leonenko *et al.* (2012) and Castelli *et al.* (2017)):

$$\begin{aligned} C(S,K,r,Y) &= \mathbb{E}_{T_Y}[S\Phi(d_1(T_Y)) - Ke^{-rY}\Phi(d_2(T_Y))], \\ P(S,K,r,Y) &= \mathbb{E}_{T_Y}[Ke^{-rY}\Phi(-d_2(T_Y)) - S\Phi(-d_1(T_Y))], \\ d_1(T_Y) &= \frac{\log\frac{S}{K} + rY + \frac{1}{2}\sigma^2 T_Y}{\sigma\sqrt{T_Y}},\ d_2(T_Y) = \frac{\log\frac{S}{K} + rY - \frac{1}{2}\sigma^2 T_Y}{\sigma\sqrt{T_Y}}, \end{aligned} \tag{22}$$

where S is the spot underlying price, K is the strike price, r is the risk-free rate, $\Phi(\cdot)$ is the standard normal cumulative distribution function. $\mathbb{E}[g(T_Y)]$ means the expectation of r.v. $g(T_Y)$ with respect to T_Y, where $g(\cdot)$ is Borel function.

3.4. Market incompleteness

A market is complete if there exists a predictable strategy that replicates claims of the option at every time point. In other words, a perfectly hedge can be formulated. The processes we consider in this paper imply incomplete markets. Hence, it is impossible to hedge the randomness of W_t and T_t at the same time. While Section 3.6 below shows that, on average, the delta hedging strategy produces a risk-free return in the long-run. We do not aim at a comprehensive study of how one should optimize hedging under such incomplete market condition, but focus on comparing different option risk measures and delta hedging results for SRD and LRD.

3.5. European option Greeks

Option Greeks are typical risk measures of European options. We derive formulas of DELTA, GAMMA, VEGA, and THETA for the FATGBM model. Details can be found in B. For simplicity, we use $d_1 = d_1(T_Y)$ and $d_2 = d_2(T_Y)$ throughout this paper.

Definition 3.3 (DELTA)**.** DELTA measures the change in an option price caused by a change in the underlying price:

$$\text{DELTA}_{\text{call}} = \frac{\partial C}{\partial S} = \mathbb{E}_{T_Y}[\Phi(d_1)],\ \text{DELTA}_{\text{put}} = \frac{\partial P}{\partial S} = \mathbb{E}_{T_Y}[\Phi(d_1)] - 1. \tag{23}$$

Definition 3.4 (GAMMA)**.** GAMMA measures the change in an option's DELTA caused by a change in the underlying price:

$$\left\{ \begin{array}{ll} \text{GAMMA}_{\text{call}} & = \frac{\partial^2 C}{\partial S^2} \\ \text{GAMMA}_{\text{put}} & = \frac{\partial^2 P}{\partial S^2} \end{array} \right\} = \frac{1}{\sigma S}\mathbb{E}_{T_Y}\left[\frac{\phi(d_1)}{\sqrt{T_Y}}\right]. \tag{24}$$

Definition 3.5 (VEGA)**.** VEGA measures the change in an option price caused by a change in volatility:

$$\left\{ \begin{array}{ll} \text{VEGA}_{\text{call}} & = \frac{\partial C}{\partial \sigma} \\ \text{VEGA}_{\text{put}} & = \frac{\partial P}{\partial \sigma} \end{array} \right\} = S\mathbb{E}_{T_Y}\left[\sqrt{T_Y}\phi(d_1)\right]. \tag{25}$$

Definition 3.6 (THETA)**.** THETA measures the change in an option price resulting from reducing in time to maturity:

$$\begin{aligned} \text{THETA}_{\text{call}} &= -\frac{\partial C}{\partial Y} = -rKe^{-rY}\mathbb{E}_{T_Y}[\Phi(d_2)] - \frac{\sigma S}{2}\mathcal{I}, \\ \text{THETA}_{\text{put}} &= -\frac{\partial P}{\partial Y} = rKe^{-rY}\mathbb{E}_{T_Y}[\Phi(-d_2)] - \frac{\sigma S}{2}\mathcal{I}, \\ \mathcal{I} &= \frac{H}{Y}\mathbb{E}_{T_Y}[\sqrt{T_Y}\phi(d_1)] + (1-H)\,\mathbb{E}_{T_Y}[\frac{\phi(d_1)}{\sqrt{T_Y}}]. \end{aligned} \tag{26}$$

3.6. Delta hedging of European options

We construct a portfolio V_t with the risky asset S_t and its European call option C_t: $V_t = C_t + \gamma_t S_t$, in which γ_t is the hedging ratio defined as $\gamma_t = -\frac{\partial C_t}{\partial S_t}$. Given the skew-correcting martingale and (2), we have

$$dS_t = \mu S_t dt + \sigma S_t dW_{T_t}, \tag{27}$$

Using Equations (22) and (23), we have

$$dC_t = \left(\frac{\partial C_t}{\partial t} + \mu \frac{\partial C_t}{\partial S_t} S_t\right) dt + \frac{\sigma^2}{2} \frac{\partial^2 V_t}{\partial S_t^2} S_t^2 dT_t + \sigma \frac{\partial C_t}{\partial S_t} S_t dW_{T_t}. \tag{28}$$

We get the delta hedging result:

$$\begin{aligned} dV_t =& \gamma_t dS_t + dC_t = \frac{\partial C_t}{\partial t} dt + \frac{\sigma^2}{2} \frac{\partial^2 C_t}{\partial S_t^2} S_t^2 dT_t + \mu \left(\gamma_t + \frac{\partial C_t}{\partial S_t}\right) S_t dt + \sigma \left(\gamma_t + \frac{\partial C_t}{\partial S_t}\right) S_t dW_{T_t} \\ =& \frac{\partial C_t}{\partial t} dt + \frac{\sigma^2}{2} \frac{\partial^2 C_t}{\partial S_t^2} S_t^2 dT_t. \end{aligned} \tag{29}$$

If we take expectation, it is

$$\mathbb{E}_{T_t}[dV_t] = \mathbb{E}\left[\frac{\partial C_t}{\partial t} dt + \frac{\sigma^2}{2} \frac{\partial^2 C_t}{\partial S_t^2} S_t^2 dT_t\right] = \left(\frac{\partial C_t}{\partial t} + \frac{\sigma^2}{2} \frac{\partial^2 C_t}{\partial S_t^2} S_t^2\right) dt.$$

This is in the same format as the GBM model: $d\tilde{V}_t = \left(\frac{\partial \tilde{C}_t}{\partial t} + \frac{\sigma^2}{2} \frac{\partial^2 \tilde{C}_t}{\partial S_t^2} S_t^2\right) dt$, in which $\tilde{C}_t$ is the Black-Scholes call option price. Using the same method, we get same results for delta hedging of put options.

4. Simulation of option pricing and hedging

We analyse option pricing and Greeks of underlying prices driven by SRD and LRD Student FATGBM through simulations. Recall the Equation (4) and consider the skew-correcting martingale in Kerss *et al.* (2014) (and the reference therein), we have the following stochastic representation for underlying return process:

$$R_t \stackrel{\mathrm{d}}{=} r + \sigma \sqrt{\tau_t} W_1,$$

where $W_1 \sim N(0,1)$ and $\tau_t \sim R\Gamma(\frac{\nu}{2}, \frac{\nu-2}{2})$ with certain defined dependence structure. The model should satisfy $\nu > 4$. The mean and variance of τ_t are 1 and $\frac{2}{\nu-4}$ respectively.

The parameter setting is listed in Table 1. We use Model I to simulate SRD processes with parameter λ and Model IV for LRD processes with paramter α. To focus on impact of fat-tail and dependencies, we use parameters that give the same mean and variance of R_t. We also compare results with independent Student FATGBM and GBM.

4.1. Generating return processes with SRD

The simulation algorithm for Model I is introduced in Taufer and Leonenko (2009). In short, we have

$$\tau_t = \mathrm{e}^{-\lambda} \tau_{t-1} + \varepsilon_t,\ t = 1, 2, ...,$$

in which ε_t is generated from distribution corresponding to the characteristic function (ch.f.):

$$\kappa_{\mathcal{L}(1)}(\zeta) = \exp(\kappa(\zeta) - \kappa(\zeta \mathrm{e}^{-\lambda})) = \frac{\mathrm{i}\zeta}{\left(-\mathrm{i}\zeta \frac{2}{\nu-2}\right)^{1/2}} \frac{K_{\nu/2-1}\left((\nu-2)\left(-\mathrm{i}\zeta \frac{2}{\nu-2}\right)^{1/2}\right)}{K_{\nu/2}\left((\nu-2)\left(-\mathrm{i}\zeta \frac{2}{\nu-2}\right)^{1/2}\right)},\ \zeta \in \mathbb{R},\ \zeta \neq 0,$$

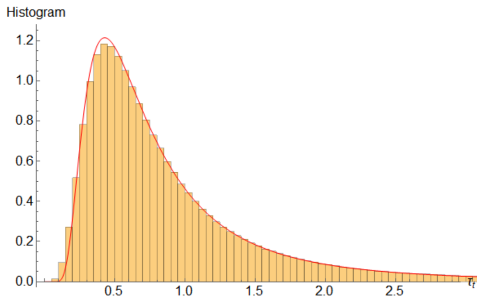
Table 1: Simulation parameter setting.

	Student FATGBM					GBM
	Model A1 (SRD)	Model A2 (LRD)	Model B (LRD)	Model A' (IID)	Model B' (IID)	(IID)
μ	0.0005	0.0005	0.0005	0.0005	0.0005	0.0005
σ	0.0200	0.0200	0.0200	0.0200	0.0200	0.0200
ν	5.0000	5.0000	6.0000	5.0000	6.0000	–
λ	0.7000	–	–	–	–	–
H	0.5000	0.8000	0.6000	–	–	–

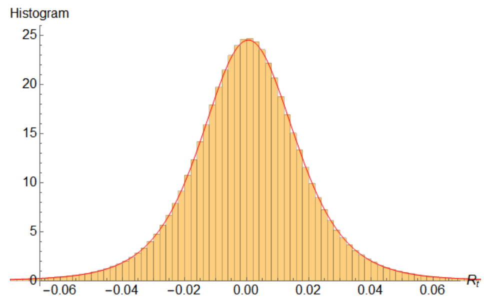
and $\kappa_{\mathcal{L}(1)}(0) = 0$, $\kappa(\zeta)$ is the ch.f. of $R\Gamma\left(\frac{\nu}{2}, \frac{\nu-2}{2}\right)$, and $K_\lambda(x)$ is the modified Bessel functions of the third kind (see Heyde and Leonenko (2005) for more details). Taufer and Leonenko (2009) proves that τ_t has stationary $R\Gamma(\frac{\nu}{2}, \frac{\nu-2}{2})$ distribution and autocorrelation

$$\mathbb{C}orr[\tau_t, \tau_{t+s}] = \mathrm{e}^{-\lambda s}, \; s = 1, 2, \ldots. \tag{30}$$

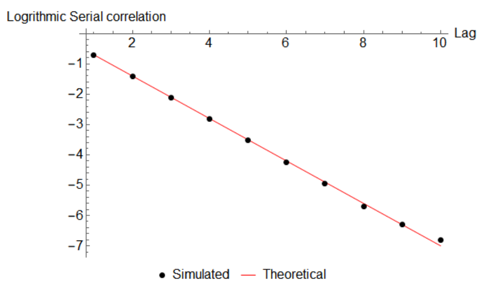
Because there is no explicit solution of probability density function with ch.f. $\kappa_{\mathcal{L}(1)}(\zeta)$, we apply the numerical method in Taufer and Leonenko (2009). We generate $1,000$ paths of $50,000$ fractal activity time intervals and returns for Model A1. We examine the goodness-of-fit for marginal distributions of τ_t and R_t in Figures 4a and 4b respectively. According to (30), logarithm-scale autocorrelations of both τ_t and R_t^2 are linear of lagged time (see Figures 4c and 4d).



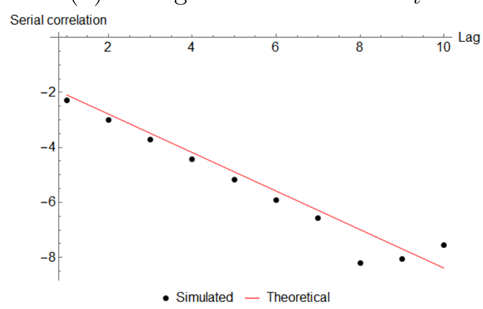
(a) Histogram of simulated τ_t



(b) Histogram of simulated $\tilde{R}_t$



(c) Serial correlation of simulated τ_t



(d) Serial correlation of simulated $\tilde{R}_t^2$

Figure 4: Simulation results of fractal activity time with SRD

4.2. Generating return processes with LRD

Recall the fractal time constructed by Model IV:

$$\tau_t = \left(\frac{2}{\nu-2}\chi^2_\nu(t)\right)^{-1} = \left(\frac{\eta_1^2(t) + \eta_2^2(t) + \ldots + \eta_\nu^2(t)}{\nu - 2}\right)^{-1}, \; \nu > 4, \; t = 1, 2, \ldots.$$

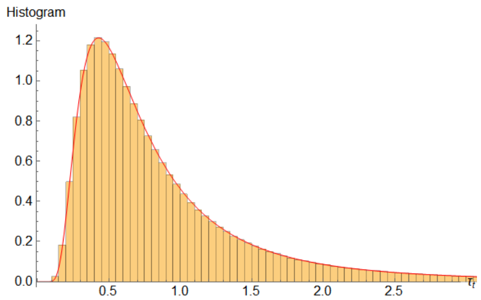
τ_t is a stationary process whose marginal distribution is $R\Gamma(\frac{\nu}{2}, \frac{\nu-2}{2})$ with many elegant properties (see Heyde and Leonenko (2005) for more details). To obtain LRD, we use the correlation function $\rho_\eta(s) = (1+s^2)^{-\frac{1-H}{2}}$, $s = 1, 2, ...$, which requires $1/2 < H < 1$. The same holds for class of correlations $\rho_\eta(s) = \frac{\mathcal{L}(s)}{s^{\frac{1-H}{2}}}$, where $\mathcal{L}(s)$ is slowly varying at infinity function. Leonenko *et al.* (2011) gives the autocorrelation function of τ_t:

$$\mathbb{C}orr[\tau_t, \tau_{t+s}] = \frac{\nu-4}{2} \sum_{k=1}^{\infty} C_k^2(\nu)\rho_\eta^{2k}(s),\ s = 1, 2, ...,$$

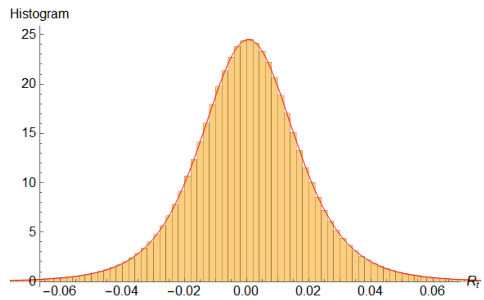
where $C_k^2(\nu)$ is given by

$$C_k(\nu) = \left(\frac{\nu}{2} - 1\right) \int_0^\infty \frac{f_{\Gamma(\nu/2,1)}(x) L_k^{\nu/2-1}(u) \left\{k! \frac{\Gamma(\nu/2)}{\Gamma(\nu/2+k)}\right\}^{1/2} dx}{x},$$

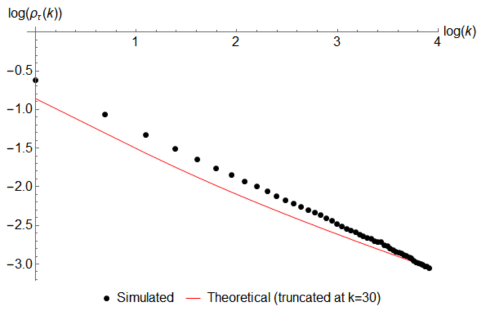
where $L_k^\beta(u) = \frac{1}{k!} u^{-\beta} e^u \frac{d^k}{du^k}(u^{\beta+k}e^{-u})$ is the generalized Laguerre polynomials.



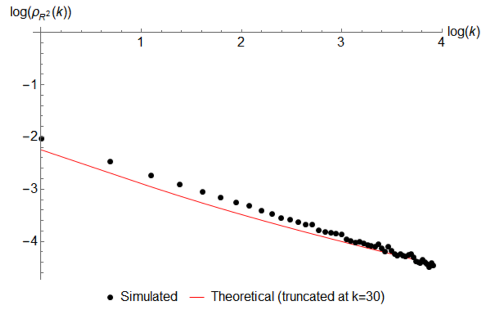
(a) Histogram of simulated τ_t



(b) Histogram of simulated $\tilde{R}_t$



(c) Serial correlation of simulated τ_t



(d) Serial correlation of simulated $\tilde{R}_t^2$

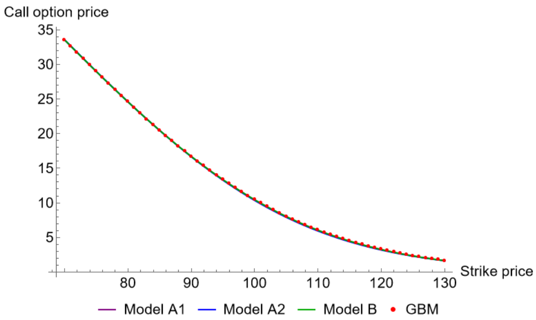
Figure 5: Statistical features of fractal activity time with LRD

Similar to the simulation in Section 4.1, we generate 1,000 paths of 5,000 fractal active time intervals τ_t and returns R_t for Model A2. The probability density and the LRD structure of our simulated data are presented in Figure 5.

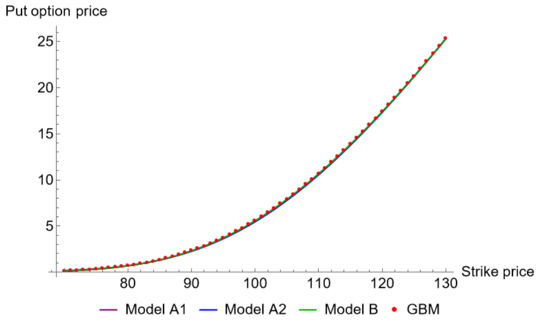
4.3. Student FATGBM vs. GBM

We compare option prices and Greeks of Models A1, A2, B and the benchmark GBM (i.e. the Black-Scholes) in Table 1. In Figure 6, we observe that option pricing based on Student FATGBM is not different much from the Black-Scholes pricing.

Comparison of Greeks are presented in Figures 7, 8, 9 and 10. Here we show a few key observations. We find trivial differences among at-the-money DELTAs (see Figures 7a and 7b). While DELTA of Model A2 for in-the-money and out-of-money options clearly deviate from those of other models (see Figures 7c, 7d, 7e and 7f). In Figure 10, same findings also observed for THETA. For options expiring soon, Student models produce higher GAMMA and lower VEGA than the Black-Scholes model, especially the at-the-money option under strong LRD,



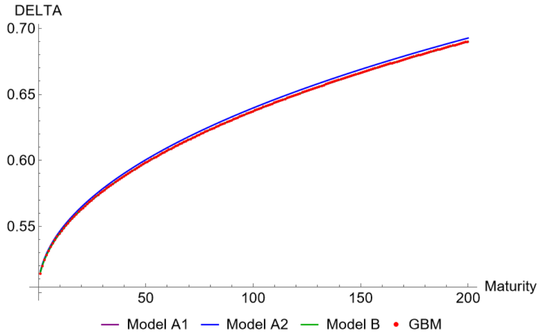
(a) Call option prices



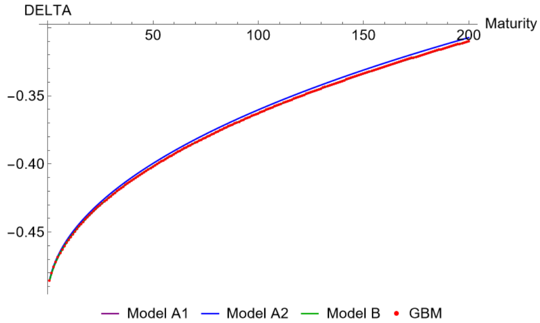
(b) Put option prices

Figure 6: Option prices of Student FATGBM and GBM ($S_0 = 100$, $Y = 100$)

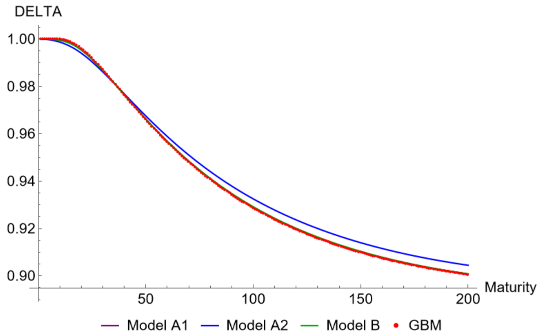
i.e., the Model A2 (see Figures 8a and 9a). But for the options with long time-to-maturity, only GAMMA and VEGA of Model A2 are clearly different from others (see Figures 8b and 9b). According to these observations, we conclude that dependence structure is a key factor that makes the option prices and risks under FATGBM models different from the classic Black-Scholes model.



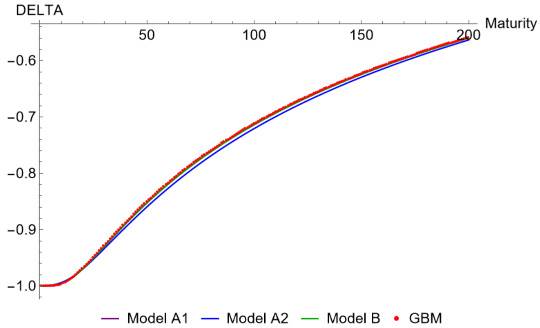
(a) At-the-money call option



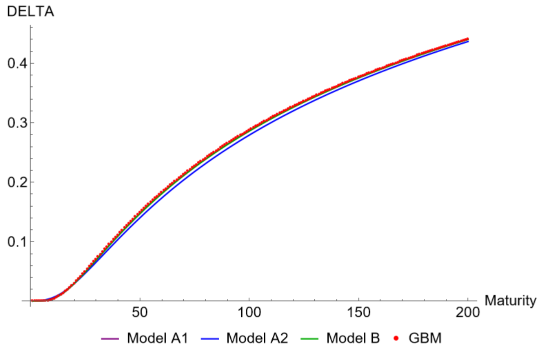
(b) At-the-money put option



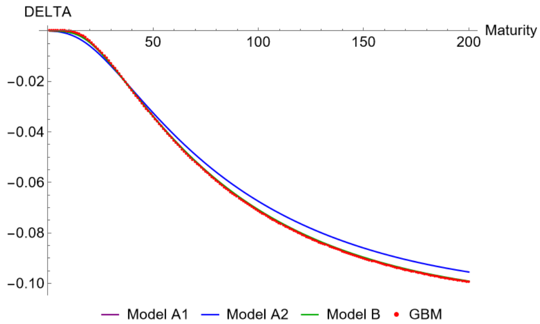
(c) In-the-money call option



(d) In-the-money put option

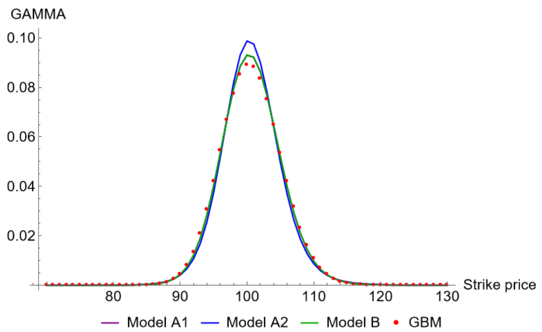


(e) Out-of-money call option

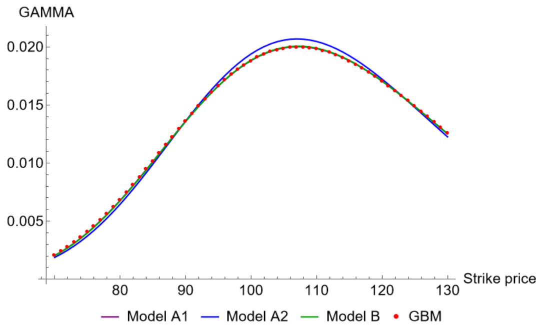


(f) Out-of-money put option

Figure 7: DELTA of Student FATGBM and GBM ($Y = 100$)

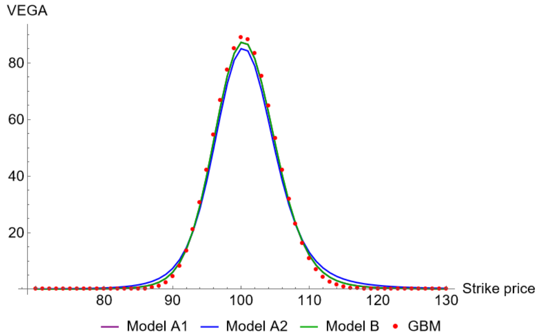


(a) Short maturity: $Y = 5$

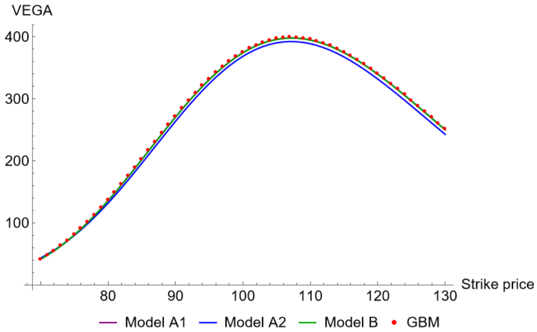


(b) Long maturity: $Y = 100$

Figure 8: GAMMA of FATGBM and GBM

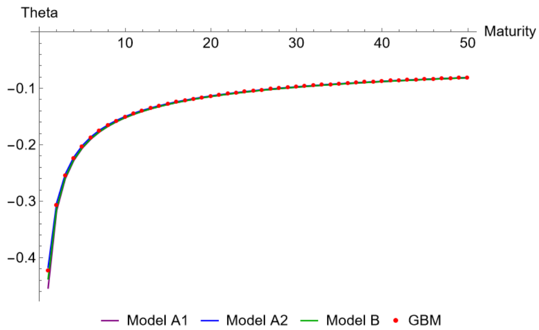


(a) Short maturity: $Y = 5$

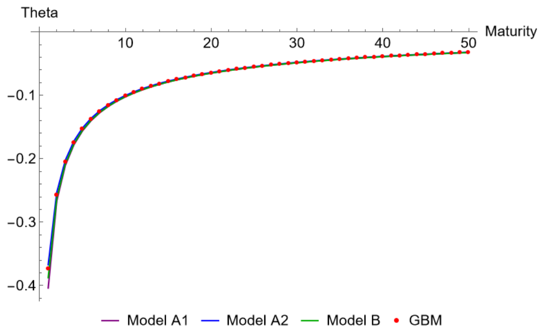


(b) Long maturity: $Y = 100$

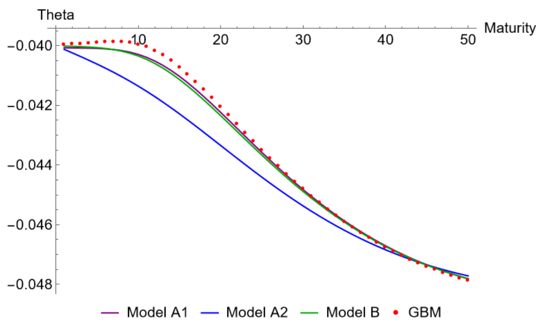
Figure 9: VEGA of FATGBM and GBM



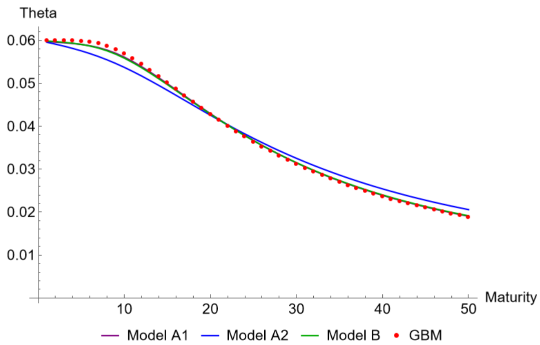
(a) At-the-money call option



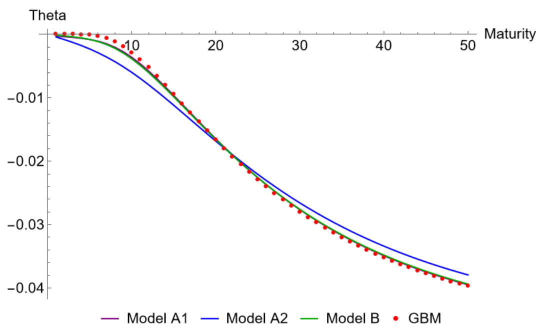
(b) At-the-money put option



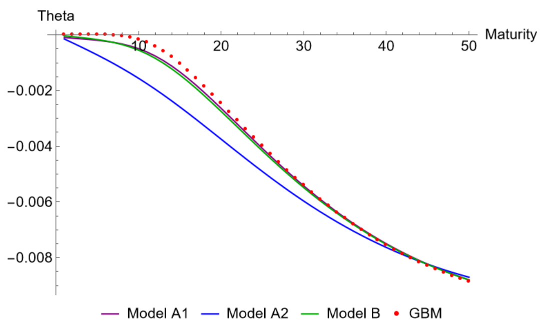
(c) In-the-money call option



(d) In-the-money put option



(e) Out-of-money call option



(f) Out-of-money put option

Figure 10: THETA of Student FATGBM and GBM

5. Delta hedging analysis using simulations

Before applying the delta hedging strategy to real data, we provide some findings using simulated data. This is because mathematical properties of simulated data match better with the theoretical model than real data, which allows us to examine and observe theoretical results better. In addition, financial data is limited and lacks stochastic paths, which may lead to insufficient observations of complex statistical inference.

According to Equations (22) and (23), dependence structures are not considered in the pricing formulas nor the delta hedging ratios. However, they should affect the hedging results. This is because long- or short-memory of T_t should result in conditional probability distributions of τ_t given previous $\{\tau_0, \tau_1, ..., \tau_{t-1}\}$. However, in practice, we cannot "unpack" such information from two random variables T_t and W_t. Hence, when using this model for pricing and hedging at time t, we assume it is the "beginning" of time, i.e., $T_t = t$. We observe hedging errors due to missing information through hedging experiments of simulated prices, in which we create hedging portfolios:

$$V_t = C_t + \gamma_t S_t,\ \gamma_t = -\mathbb{E}_{T_Y}\left[\Phi(d_1)\right], \text{ for call options, and}$$
$$V_t = P_t + \gamma_t S_t,\ \gamma_t = 1 - \mathbb{E}_{T_Y}\left[\Phi(d_1)\right], \text{ for put options.}$$

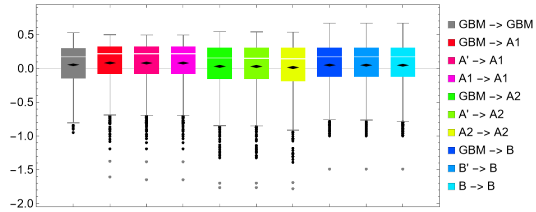
We examine cumulative hedging errors of underlying processes based on Models A1, A2 and B. For the first two models, we compare hedging results of the underlying model, the Model A' and the GBM. Similarly, for the last model, we also compare the underlying model results with the Model B' and the GBM. Note that in Figures 11 and 12, $X \to Y$ means using Model X to hedge option risks of the underlying process follows Model Y. We would like to highlight two points of this experiment design. First, $GBM \to GBM$ creates a benchmark case in which the risk is fully hedged. Second, $H = 0.5$ holds for both SRD and non-dependence models. This is because, for T_t in non-dependence models, we have approximation by standard functional central limit theorem

$$\frac{1}{\sigma\sqrt{N}}\left(T_{[N\cdot t]} - [N\cdot t]\right) \overset{\text{Skd}}{\Longrightarrow} W_t,\ t > 0.$$

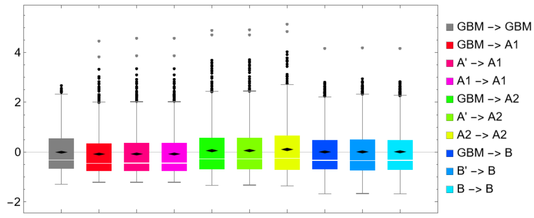
Hence, $A' \to A1$ and $A1 \to A1$ have the same hedging ratios and hedging errors.

We set the initial underlying price $S_0 = 100$ and strike prices $K = 90, 100, 110, 120$. The initial time-to-maturity is 150 time steps. We compare hedging results of 20 and 150 and time steps. The theoretically ideal hedging result should be gaining the risk-free return. We calculate hedging errors as differences from the ideal case in percentage.

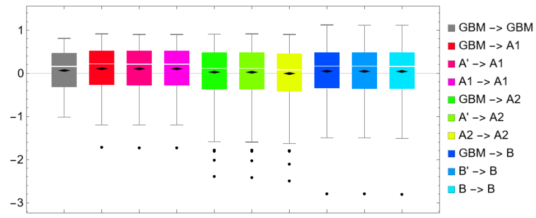
We have four key observations regarding hedging errors. First, it is hard to hedge prices following FATGBM processes in short-term as there are many extreme outliers in Figure 12, especially for Model A2. Also note that GBM does not produce such outliers. This confirms the argument above about the impact of short- and long-memory on hedging errors. The second observation is also related to dependence structures. We find that hedging SRD processes generates much smaller variance of hedging errors than that of hedging LRD price movements. This is true for hedging both short- and long-term. Then, in Figure 12, we get our third finding that we tend to obtain negative outliers when hedging call options and positive outliers when hedging put options. We lastly focus on Model B, which is closer to GBM due to larger ν and has long-memory. Hedging Model B gives more chances of positive errors for call options and negative errors for put options (see, for example, Figures 11a and 11b), which is opposite to the general situation described previously.



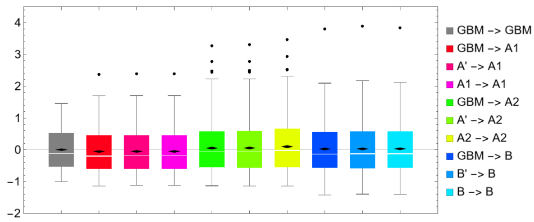
(a) Call options (K=90)



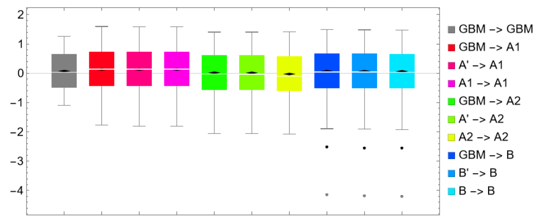
(b) Put options (K=90)



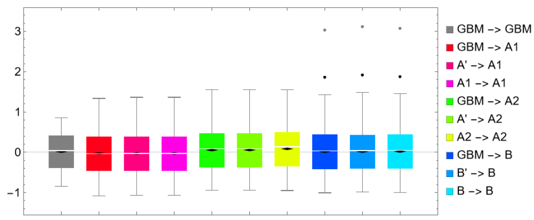
(c) Call options (K=100)



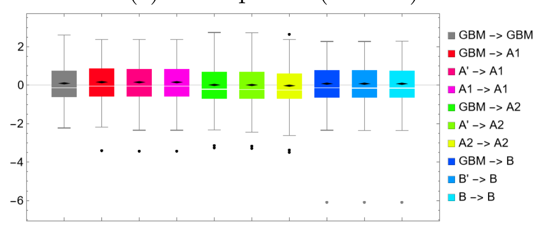
(d) Put options (K=100)



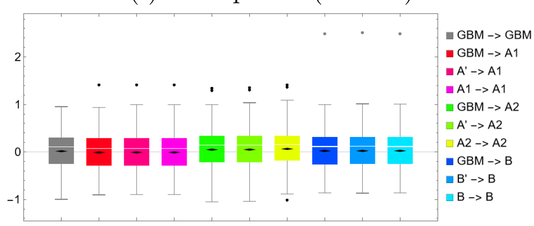
(e) Call options (K=110)



(f) Put options (K=110)

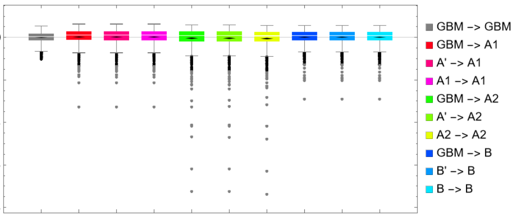


(g) Call options (K=120)

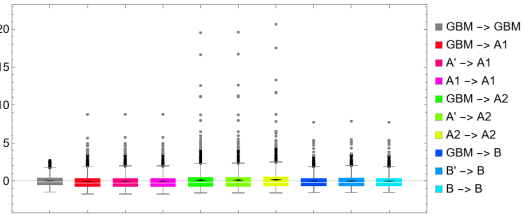


(h) Put options (K=120)

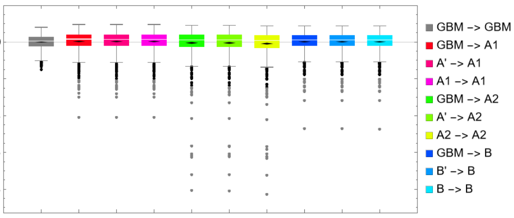
Figure 11: Cumulative delta hedging errors (150 time steps)



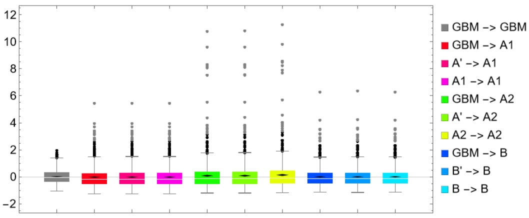
(a) Call options (K=90)



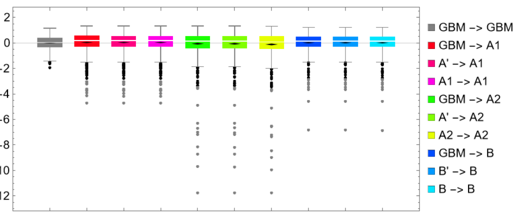
(b) Put options (K=90)



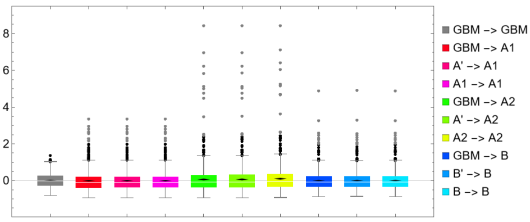
(c) Call options (K=100)



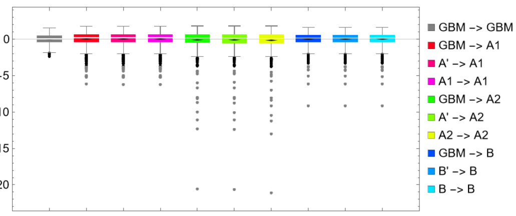
(d) Put options (K=100)



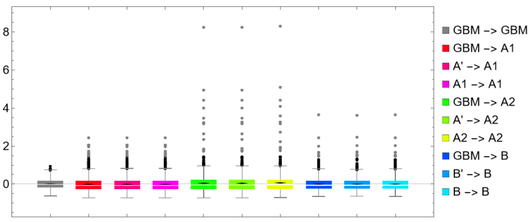
(e) Call options (K=110)



(f) Put options (K=110)



(g) Call options (K=120)



(h) Put options (K=120)

Figure 12: Cumulative delta hedging errors (20 time steps)

6. Application: European options pricing and delta hedging

We apply delta hedging on S&P 500 options using Student FATGBM. We use S&P 500 Index daily returns from 2010-01-04 to 2020-07-31 for model calibration. Then we consider the option S&P 500 expired on 17th December 2021. The hedging ratio is computed by Equation (23).

6.1. Student FATGBM calibration

The advantage of FATGBM is that the density distribution and dependence structure parameters can be estimated separately. As the density functions and moment generating functions of Student distribution both yield explicit formulas, we have a variety ways to fit return data to this model. The results of maximum likelihood estimation (MLE) and generalized method of moment (GMM) are presented in Table 2. Since the daily returns are very small, we use a $\times 100$ multiplier on all return data to prevent precision issues in calibration. In Sections 6.2 and 6.3, we use the GMM result as $\nu > 4$ is required for pricing and hedging.

Table 2: Parameter estimation

	S&P 500 index			U.S./Euro		
	Student dist. (GMM)	Student dist. (MLE)	Empirical data	Student dist. (GMM)	Student dist. (MLE)	Empirical data
$\hat{\mu}$	0.046	0.0847	–	−0.007	−0.003	–
$\hat{\sigma}$	1.103	1.490	–	0.551	0.560	–
$\hat{\nu}$	4.385	2.349	–	7.470	5.003	–
$\hat{\lambda}$	0.064		–	–		–
$\hat{H}$	0.500		–	0.822		–
Mean	0.0459	0.0847	0.0459	−0.007	−0.003	−0.007
Variance	1.216	2.220	1.216	0.303	0.314	0.303
Skewness	0.000	Indeterminate	−0.568	0.000	0.000	−0.004
Kurtosis	18.574	Indeterminate	18.668	4.729	8.981	4.729

To calculate the dependence structure parameter, we first observe the autocorrelation of R_t^2 to determine whether the model exhibits SRD or LRD. For all SRD cases, $\hat{H} = 0.5$ and we calibrate λ in (10) and (11). Given the theoretical autocorrelation equation (31), we can run the linear regression (32) in which $\{y = \mathbb{C}orr\left[R_t^2, R_{t+x}^2\right] : x = 1, 2, 3, ...\}$.

$$\mathbb{C}orr\left[R_t^2, R_{t+s}^2\right] = \sigma^4 \mathbb{C}orr\left[\tau_t, \tau_{t+s}\right] \frac{\mathbb{V}ar\left[\tau_t\right]}{\mathbb{V}ar\left[R_t^2\right]} = \frac{\sigma^2 \mathrm{e}^{-\lambda s}}{2\mu^2(\nu-4) + \sigma^2(\nu-1)}, \; s > 0. \tag{31}$$

$$\begin{aligned} \log(y) &= -\lambda x + \beta_0 \\ \beta_0 &= 2\log(\sigma) - \log\left(2\mu^2(\nu-4) + \sigma^2(\nu-1)\right) \end{aligned} \tag{32}$$

In our data, squared returns of the S&P 500 index shows SRD structure and we get $\hat{\lambda} = 0.064$ (see Figure 13). For LRD cases, we will calibration the self-similarity H parameter. We assume the LRD autocorrelation is a slowly varying function (33). We regress $\log(s)$ on logarithmic aurocorrelation to find the $\hat{\alpha} = 0.216$ and calibrate $\hat{H} = \frac{1}{1+\hat{\alpha}} = 0.822$ (see Figure 14).

$$\mathbb{C}orr\left[R_t^2, R_{t+s}^2\right] \approxeq \alpha_0 \cdot s^{-\alpha}, \; s > 0. \tag{33}$$

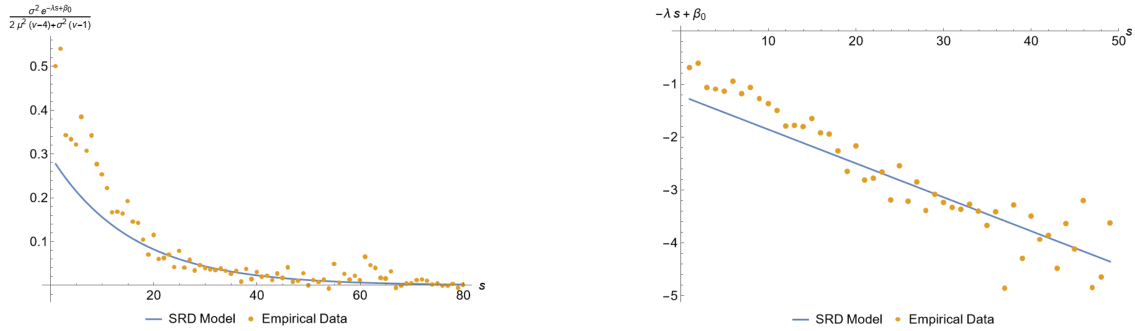


Figure 13: Model fitting of SRD

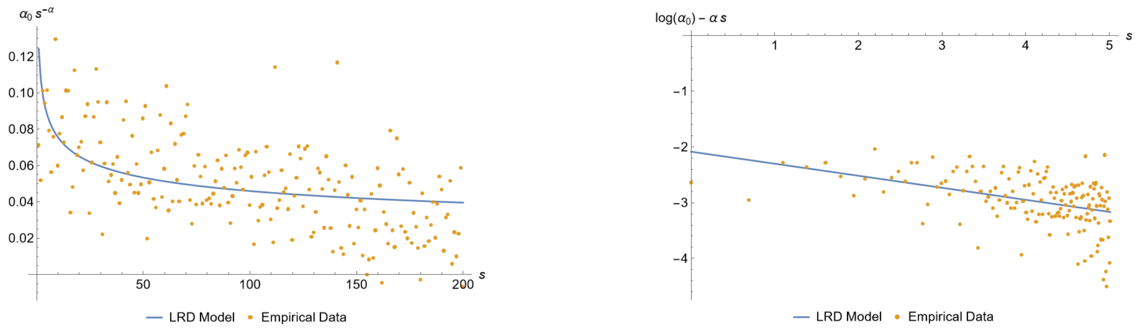
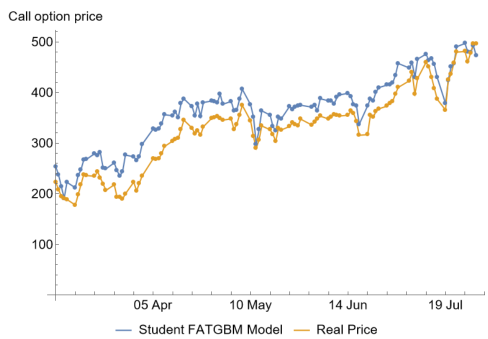


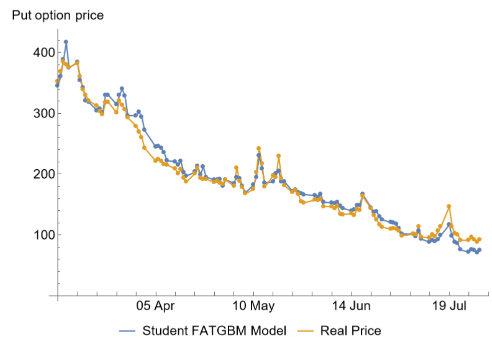
Figure 14: Model fitting of SRD

6.2. Comparison to real S&P 500 option prices

We cannot get FX options data as most of them are OTC traded. Hence, we only compare pricing and hedging for S&P 500 options. We compute call and put option prices using the Student FATGBM model. Although the risk-free rate changes overtime, technically, it does not make many differences if the rate is very low. Hence, we use a flat rate 0.1%, which is the average interbank lending rate in our evaluation. We confirm that option prices given by the models are close to the real market prices, especially for the call options (see Figure 15a and 15b).



(a) Comparison of call option prices

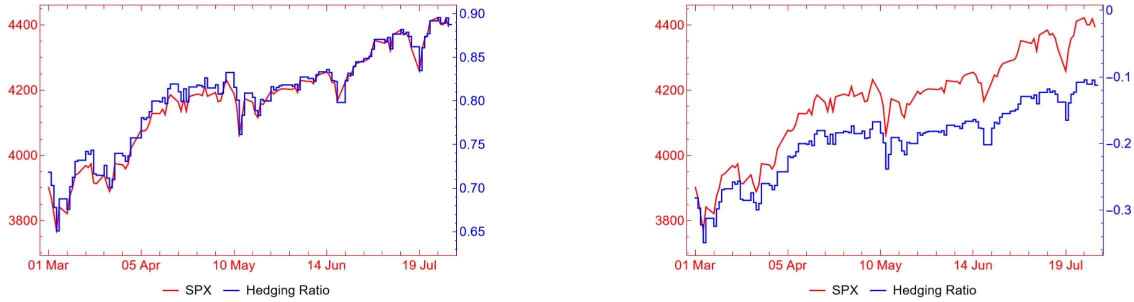


(b) Comparison of put option prices

Figure 15: Comparison of option prices

6.3. Delta hedging of S&P 500 options

We finally compute the delta hedging ratios for the call and put options (see Figure 16). As this is just a single case of hedging experiment, we cannot guarantee a low hedging error. While according to the numerical results in the previous section, we know that, as an SRD example, the SPX option trades will be well hedged in the long-term.



(a) Delta hedging of SPX call option (b) Delta hedging of SPX put option

Figure 16: Delta hedging of SPX options

Acknowledgments

Nikolai Leonenko (NL) would like to thank for support and hospitality during the programme "Fractional Differential Equations" and the programmes "Uncertainly Quantification and Modelling of Material" and "Stochastic systems for anomalous diffusion" in Isaac Newton Institute for Mathematical Sciences, Cambridge. Also NL was partially supported under the ARC Discovery Grant DP220101680 (Australia), LMS grant 42997 (UK), grant FAPESP 22/09201-8 (Brazil) and Croatian Science Foundation (HRZZ) grant Scaling in Stochastic Models (IP-2022-10-8081). Nataliya Shchestyuk (NS) would like to thank for support provided by Knowledge Foundation.

A. Properties of the generalized hyperbilic skew student's t-distribution

The r.v. ST with density (9) has the following properties:

$$\mathbb{E}[ST] = \mu + \theta \tag{34}$$

$$\mathbb{V}ar[ST] = \frac{2\theta^2}{\nu - 4} + \sigma^2 \tag{35}$$

$$\mathbb{S}kew[ST] = 2\theta\sqrt{(\nu-4)(\nu-2)}\left(\frac{\sigma}{\sqrt{2\theta^2 + (\nu-4)\sigma^2}}\right)^3\left(3 + \frac{8\theta^2}{(\nu-6)\sigma^2}\right) \tag{36}$$

$$\mathbb{K}urt[ST] = \frac{6\sigma^4}{\left(2\theta^2 + (\nu-4)\sigma^2\right)^2}\left((\nu-4) + \frac{16\theta^4(\nu-4)}{(\nu-6)\sigma^6} + \frac{8\theta^4(5\nu-22)}{(\nu-6)(\nu-8)\sigma^4}\right) \tag{37}$$

B. Deriving option Greeks of Student FATGBM models

B.1. To derive the formula of DELTA:

$$\text{DELTA}_{\text{call}} = \frac{\partial C}{\partial S} = \mathbb{E}_{T_Y}[\Phi(d_1)], \, \text{DELTA}_{\text{put}} = \frac{\partial P}{\partial S} = \mathbb{E}_{T_Y}[\Phi(d_1)] - 1.$$

We have

$$\frac{\partial C}{\partial S} = \int_0^\infty \frac{\partial}{\partial S}\left(S\Phi(d_1) - Ke^{-rY}\Phi(d_2)\right) \cdot f_{T_Y}(s)ds.$$

We compute the partial derivative $\frac{\partial}{\partial S}\left(S\Phi(d_1) - Ke^{-rY}\Phi(d_2)\right)$:

$$\begin{aligned}\frac{\partial}{\partial S}\left(S\Phi(d_1) - Ke^{-rY}\Phi(d_2)\right) &= \Phi(d_1) + S\frac{\partial\Phi(d_1)}{\partial S} - Ke^{-rY}\frac{\partial\Phi(d_2)}{\partial S} \\ &= \Phi(d_1) + \frac{1}{\sigma\sqrt{s}}\left(\phi(d_1) - \frac{K}{S}e^{-rY}\phi(d_2)\right).\end{aligned}$$

As $d_2 = d_1 - \sigma\sqrt{s}$, we have $\phi(d_2) = \frac{1}{\sqrt{2\pi}} e^{\frac{(d_1-\sigma\sqrt{s})^2}{2}} = \frac{S}{K} e^{rY} \phi(d_1)$. Therefore,

$$\frac{\partial}{\partial S}\left(S\Phi(d_1) - Ke^{-rY}\Phi(d_2)\right) = \Phi(d_1) + \frac{1}{\sigma\sqrt{s}} \cdot 0 = \Phi\left(d_1\right).$$

Taking expectation respect to r.v. T_Y, we get $\text{DELTA}_{\text{call}} = \frac{\partial C}{\partial S} = \mathbb{E}_{T_Y}[\Phi(d_1)]$. Similarly, DELTA of put options is found: $\text{DELTA}_{\text{put}} = \frac{\partial P}{\partial S} = \mathbb{E}_{T_Y}[\Phi(d_1)] - 1$.

B.2. To derive the formula of GAMMA:

$$\left\{\begin{array}{ll} \text{GAMMA}_{\text{call}} & = \frac{\partial^2 C}{\partial S^2} \\ \text{GAMMA}_{\text{put}} & = \frac{\partial^2 P}{\partial S^2} \end{array}\right\} = \frac{1}{\sigma S}\mathbb{E}_{T_Y}\left[\frac{\phi(d_1)}{\sqrt{T_Y}}\right].$$

We have

$$\frac{\partial^2 C}{\partial S^2} = \int_0^\infty \frac{\partial \Phi(d_1)}{\partial S} \cdot f_{T_Y}(s)ds = \int_0^\infty \phi(d_1)\frac{\partial d_1}{\partial S} \cdot f_{T_Y}(s)ds = \int_0^\infty \frac{\phi(d_1)}{S\sigma\sqrt{s}} \cdot f_{T_Y}(s)ds$$

Hence, $\text{GAMMA}_{\text{call}} = \frac{1}{\sigma S}\mathbb{E}_{T_Y}\left[\frac{\phi(d_1)}{\sqrt{T_Y}}\right]$. Doing the same, we find GAMMA of put options has the same formula.

B.3. To derive the formula of VEGA:

$$\left\{\begin{array}{ll} \text{VEGA}_{\text{call}} & = \frac{\partial C}{\partial \sigma} \\ \text{VEGA}_{\text{put}} & = \frac{\partial P}{\partial \sigma} \end{array}\right\} = S\mathbb{E}_{T_Y}\left[\sqrt{T_Y}\phi(d_1)\right].$$

We have

$$\begin{aligned} \frac{\partial C}{\partial \sigma} &= \int_0^\infty \left(S\frac{\partial}{\partial \sigma}\Phi(d_1) - Ke^{-rY}\frac{\partial}{\partial \sigma}\Phi(d_2)\right) \cdot f_{T_Y}(s)ds \\ &= \int_0^\infty \left(S\frac{\sqrt{s}}{2}\phi(d_1) + Ke^{-rY}\frac{\sqrt{s}}{2}\phi(d_2)\right) \cdot f_{T_Y}(s)ds. \end{aligned}$$

We know that $\phi(d_2) = \frac{1}{\sqrt{2\pi}} e^{\frac{(d_1-\sigma\sqrt{s})^2}{2}} = \frac{S}{K} e^{rY} \phi(d_1)$. Hence,

$$\text{VEGA}_{\text{call}} = \frac{\partial C}{\partial \sigma} = \int_0^\infty S\sqrt{s}\phi(d_1) \cdot f_{T_Y}(s)ds = S\mathbb{E}_{T_Y}\left[\sqrt{T_Y}\phi(d_1)\right].$$

We do the same using put option price formula and get the same VEGA formula.

B.4. To derive the formula of THETA:

$$\text{THETA}_{\text{call}} = -\frac{\partial C}{\partial Y} = -rKe^{-rY}\mathbb{E}_{T_Y}[\Phi(d_2)] - \frac{\sigma S}{2}\mathcal{I},\ \text{THETA}_{\text{put}} = -\frac{\partial P}{\partial Y} = rKe^{-rY}\mathbb{E}_{T_Y}[\Phi(-d_2)] - \frac{\sigma S}{2}\mathcal{I},$$
$$\mathcal{I} = \frac{H}{Y}\mathbb{E}_{T_Y}[\sqrt{T_Y}\phi(d_1)] + (1-H)\,\mathbb{E}_{T_Y}[\frac{\phi(d_1)}{\sqrt{T_Y}}].$$

We know that T_Y is a random variable with probability density f_{T_Y}. The case $T_Y \equiv Y$ (i.e. the Black-Scholes model) is not discussed here. We have

$$\begin{aligned} \frac{\partial C}{\partial Y} &= \int_0^\infty \frac{\partial}{\partial Y}\left(\left(S\Phi(d_1) - Ke^{-rY}\Phi(d_2)\right) \cdot f_{T_Y}(s)\right)ds \\ &= \int_0^\infty \frac{\partial}{\partial Y}\left(S\Phi(d_1) - Ke^{-rY}\Phi(d_2)\right) \cdot f_{T_Y}(s)ds + \int_0^\infty \left(S\Phi(d_1) - Ke^{-rY}\Phi(d_2)\right) \cdot \frac{\partial}{\partial Y}f_{T_Y}(s)ds. \end{aligned}$$

The first term $\frac{\partial}{\partial Y}\left(S\Phi(d_1) - Ke^{-rY}\Phi(d_2)\right)$ is

$$\frac{\partial}{\partial Y}\left(S\Phi(d_1) - Ke^{-rY}\Phi(d_2)\right) = rKe^{-rY}\Phi(d_2) + \frac{Sr}{\sigma\sqrt{s}}\left(\phi(d_1) - \frac{K}{S}e^{-rY}\phi(d_2)\right) = rKe^{-rY}\Phi(d_2).$$

The second term $\frac{\partial}{\partial Y} f_{T_Y}(s)$ is

$$\begin{aligned}\frac{\partial}{\partial Y} f_{T_Y}(s) =& \frac{\partial}{\partial Y}\left(\frac{1}{Y^H} f_\tau\left(\frac{s}{Y^H} - \left(Y^{1-H} - 1\right)\right)\right) \\ =& -\frac{H}{Y} f_{T_Y}(s) - \frac{1}{Y^{2H}}\left(\frac{H}{Y}s + (1-H)\right) f'_\tau\left(\frac{s}{Y^H} - \left(Y^{1-H} - 1\right)\right).\end{aligned}$$

Hence,

$$\begin{aligned}&\int_0^\infty \left(S\Phi(d_1) - Ke^{-rY}\Phi(d_2)\right) \cdot \frac{\partial}{\partial Y} f_{T_Y}(s)ds \\ =& -\frac{H}{Y}\int_0^\infty \left(S\Phi(d_1) - Ke^{-rY}\Phi(d_2)\right) \cdot f_{T_Y}(s)ds \\ &- \frac{H}{Y^{H+1}}\int_0^\infty \left(S\Phi(d_1) - Ke^{-rY}\Phi(d_2)\right) \frac{s}{Y^H} f'_\tau\left(\frac{s}{Y^H} - \left(Y^{1-H} - 1\right)\right)ds \\ &- \frac{1-H}{Y^H}\int_0^\infty \left(S\Phi(d_1) - Ke^{-rY}\Phi(d_2)\right) \frac{1}{Y^H} f'_\tau\left(\frac{s}{Y^H} - \left(Y^{1-H} - 1\right)\right)ds \\ =& -\frac{H}{Y}C - \frac{H}{Y^{H+1}}I_1 - \frac{1-H}{Y^H}I_2,\end{aligned}$$

in which

$$\begin{aligned}I_1 &= \int_0^\infty \left(S\Phi(d_1) - Ke^{-rY}\Phi(d_2)\right) \frac{s}{Y^H} f'_\tau\left(\frac{s}{Y^H} - \left(Y^{1-H} - 1\right)\right)ds, \\ I_2 &= \int_0^\infty \left(S\Phi(d_1) - Ke^{-rY}\Phi(d_2)\right) \frac{1}{Y^H} f'_\tau\left(\frac{s}{Y^H} - \left(Y^{1-H} - 1\right)\right)ds.\end{aligned}$$

Define

$$F(s) = \left(S\Phi(d_1) - Ke^{-rY}\Phi(d_2)\right) s f_\tau\left(\frac{s}{Y^H} - \left(Y^{1-H} - 1\right)\right),$$

then we have $\lim_{s\to\infty}\left(S\Phi(d_1) - Ke^{-rY}\Phi(d_2)\right) = S\cdot 1 - Ke^{-rY}\cdot 0 = S$ and $\lim_{s\to\infty} s f_\tau\left(\frac{s}{Y^H} - \left(Y^{1-H} - 1\right)\right) = 0$. Hence, $\lim_{s\to\infty} F(s) = 0$. We also have

$$\lim_{s\to 0}\left(S\Phi(d_1) - Ke^{-rY}\Phi(d_2)\right) = \begin{cases} 0, & \log\frac{S}{K} + \mu Y < 0 \\ \frac{1}{2}S - \frac{1}{2}Ke^{-rY}, & \log\frac{S}{K} + \mu Y = 0 \\ S - Ke^{-rY}, & \log\frac{S}{K} + \mu Y > 0 \end{cases}.$$

and $\lim_{s\to 0} s f_\tau\left(\frac{s}{Y^H} - \left(Y^{1-H} - 1\right)\right) = 0$. Hence,

$$\lim_{s\to 0} F(s) = \left\{\begin{array}{ll} 0, & \log\frac{S}{K} + \mu Y < 0 \\ \frac{1}{2}S - \frac{1}{2}Ke^{-rY}, & \log\frac{S}{K} + \mu Y = 0 \\ S - Ke^{-rY}, & \log\frac{S}{K} + \mu Y > 0 \end{array}\right\} \cdot 0 \cdot f_\tau\left(1 - Y^{1-H}\right) = 0.$$

We then differentiate $F(s)$:

$$\begin{aligned}\frac{d}{ds}F(s) =& \left(S\left(\frac{\sigma}{4s^{1/2}} - \frac{\log\frac{S}{K} + rY}{2\sigma s^{3/2}}\right)\phi(d_1) - Ke^{-rY}\left(-\frac{\sigma}{4s^{1/2}} - \frac{\log\frac{S}{K} + rY}{2\sigma s^{3/2}}\right)\phi(d_2)\right) s f_\tau(\cdot) \\ &+ \left(S\Phi(d_1) - Ke^{-rY}\Phi(d_2)\right) f_\tau(\cdot) + \left(S\Phi(d_1) - Ke^{-rY}\Phi(d_2)\right) \frac{s}{Y^H} f'_\tau(\cdot) \\ =& \frac{\sigma S}{2}\sqrt{s}\phi(d_1) f_\tau(\cdot) + \left(S\Phi(d_1) - Ke^{-rY}\Phi(d_2)\right) f_\tau(\cdot) + \left(S\Phi(d_1) - Ke^{-rY}\Phi(d_2)\right) \frac{s}{Y^H} f'_\tau(\cdot)\end{aligned}$$

Hence,

$$\begin{aligned}I_1 =& \int_0^\infty F'(s)ds - \frac{\sigma S}{2}\int_0^\infty \sqrt{s}\phi(d_1)f_\tau(\cdot)ds - \int_0^\infty \left(S\Phi(d_1) - Ke^{-rY}\Phi(d_2)\right) f_\tau(\cdot)ds \\ =&0-0-\frac{\sigma S}{2}Y^H\mathbb{E}_{T_Y}[\sqrt{s}\phi(d_1)] - Y^H\cdot C \\ =&-\frac{\sigma S}{2}Y^H\mathbb{E}_{T_Y}[\sqrt{T_Y}\phi(d_1)] - Y^H\cdot C\end{aligned} .$$

Define

$$G(s) = \left(S\Phi(d_1) - Ke^{-rY}\Phi(d_2)\right) f_\tau\left(\frac{s}{Y^H} - \left(Y^{1-H}-1\right)\right).$$

then we have $\lim_{s\to\infty} G(s) = S\cdot 0 = 0$. Hence, $\forall Y \geq 1$ or $1 - Y^{1-H} \leq 0$, we get

$$\lim_{s\to 0} G(s) = \left\{\begin{array}{ll} 0, & \log\frac{S}{K}+\mu Y<0 \\ \frac{S-Ke^{-rY}}{2}, & \log\frac{S}{K}+\mu Y=0 \\ \left(S-Ke^{-rY}\right), & \log\frac{S}{K}+\mu Y>0 \end{array}\right\}\cdot f_\tau(1-Y^{1-H}) = 0.$$

We differentiate $G(s)$:

$$\begin{aligned}\frac{d}{ds}G(s) =& \left(S\left(\frac{\sigma}{4s^{1/2}} - \frac{\log\frac{S}{K}+rY}{2\sigma s^{3/2}}\right)\phi(d_1) - Ke^{-rY}\left(-\frac{\sigma}{4s^{1/2}} - \frac{\log\frac{S}{K}+rY}{2\sigma s^{3/2}}\right)\phi(d_2)\right)f_\tau(\cdot) \\ &+ \left(S\Phi(d_1) - Ke^{-rY}\Phi(d_2)\right)\frac{1}{Y^H}f'_\tau(\cdot) \\ =&\frac{\sigma S}{2\sqrt{s}}\phi(d_1)f_\tau(\cdot) + \left(S\Phi(d_1) - Ke^{-rY}\Phi(d_2)\right)\frac{1}{Y^H}f'_\tau(\cdot)\end{aligned} .$$

Hence,

$$I_2 = \int_0^\infty G'(s)ds - \frac{\sigma S}{2}\int_0^\infty \frac{\phi(d_1)}{\sqrt{s}}f_\tau(\cdot)ds - \frac{\sigma S}{2}Y^H\mathbb{E}_{T_Y}[\frac{\phi(d_1)}{\sqrt{T_Y}}] .$$

By replacing I_1 and I_2 in the original formula, we get

$$\begin{aligned}\frac{\partial C}{\partial Y} =&rKe^{-rY}\mathbb{E}_{T_Y}\left[\Phi(d_2)\right] - \frac{H}{Y}C - \frac{H}{Y^{H+1}}I_1 - \frac{1-H}{Y^H}I_2 \\ =&rKe^{-rY}\mathbb{E}_{T_Y}[\Phi(d_2)] - \frac{H}{Y}C - \frac{H}{Y^{H+1}}\left(-\frac{\sigma S}{2}Y^H\mathbb{E}_{T_Y}[\sqrt{T_Y}\phi(d_1)] - Y^H\cdot C\right) \\ &-\frac{1-H}{Y^H}\cdot -\frac{\sigma S}{2}Y^H\mathbb{E}_{T_Y}[\frac{\phi(d_1)}{\sqrt{T_Y}}] \\ =&rKe^{-rY}\mathbb{E}_{T_Y}[\Phi(d_2)] + \frac{\sigma S}{2}\left(\frac{H}{Y}\mathbb{E}_{T_Y}[\sqrt{T_Y}\phi(d_1)] + (1-H)\,\mathbb{E}_{T_Y}[\frac{\phi(d_1)}{\sqrt{T_Y}}]\right)\end{aligned} .$$

To conclude, we have

$$\text{THETA}_{\text{call}} = -rKe^{-rY}\mathbb{E}_{T_Y}[\Phi(d_2)] - \frac{\sigma S}{2}\left(\frac{H}{Y}\mathbb{E}_{T_Y}[\sqrt{T_Y}\phi(d_1)] + (1-H)\,\mathbb{E}_{T_Y}[\frac{\phi(d_1)}{\sqrt{T_Y}}]\right).$$

For put options, we calculate

$$\begin{aligned}\frac{\partial P}{\partial Y} =& \int_0^\infty \frac{\partial}{\partial Y}\left(\left(Ke^{-rY}\Phi(-d_2) - S\Phi(-d_1)\right)\cdot f_{T_Y}(s)\right)ds \\ =& \int_0^\infty \frac{\partial}{\partial Y}\left(Ke^{-rY}\Phi(-d_2) - S\Phi(-d_1)\right)\cdot f_{T_Y}(s)ds + \int_0^\infty \left(Ke^{-rY}\Phi(-d_2) - S\Phi(-d_1)\right)\cdot\frac{\partial}{\partial Y}f_{T_Y}(s)ds. \\ =&-rKe^{-rY}\mathbb{E}_{T_Y}[\Phi(-d_2)] + \frac{\sigma S}{2}\left(\frac{H}{Y}\mathbb{E}_{T_Y}[\sqrt{T_Y}\phi(d_1)] + (1-H)\,\mathbb{E}_{T_Y}[\frac{\phi(d_1)}{\sqrt{T_Y}}]\right)\end{aligned}$$

Using the same method, we find

$$\text{THETA}_{\text{put}} = -rKe^{-rY}\mathbb{E}_{T_Y}[\Phi(-d_2)] + \frac{\sigma S}{2}\left(\frac{H}{Y}\mathbb{E}_{T_Y}[\sqrt{T_Y}\phi(d_1)] + (1-H)\,\mathbb{E}_{T_Y}[\frac{\phi(d_1)}{\sqrt{T_Y}}]\right).$$

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