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Optimal Variable Acceptance Sampling Plan under Progressive Type-II Censoring for the Mixture of Exponential-Rayleigh Distributions

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Abstract

Mixture distributions are widely utilized in various practical problems, such as clinical experiments and electronic component life testing. Despite this, the literature does not extensively cover acceptance sampling plans associated with these distributions. In this paper, variable acceptance sampling plans are designed for a mixture of exponential-Rayleigh distributions using partially accelerated life tests. Under progressive Type-II censoring scheme, the maximum likelihood estimators of the unknown parameters of the mixture distribution are derived for Arrhenius and linear life-stress relationships. Based on these relationships, optimal variable sampling plans are formulated. The plan parameters are determined by solving corresponding optimization problems. The study presents numerical findings, a comparative analysis, and sensitivity assessments. Finally, the practical applicability and relevance of the proposed acceptance sampling plans are demonstrated using real-world datasets. These datasets include breast cancer patients' records and failure lifetime data from communication transmitter-receivers in a commercial aircraft.

Keywords: Arrhenius life-stress relation, partially accelerated life test, linear life-stress relation, maximum likelihood estimation, optimal test plan.

1. Introduction

In today's global economy, quality has transitioned from being optional to becoming an essential need. An acceptance sampling plan (ASP) is an essential tool in statistical quality control to ensure quality. It enables consumers or manufacturers to determine whether to accept or reject a product lot based on the findings from a random sample. It is economically efficient when conducting a 100% examination of each unit is impractical or testing might cause damage to items. Furthermore, in clinical trials, sampling strategies are of utmost importance in evaluating the efficiency and safety of novel medicines.

Censoring schemes are integral in life testing due to the various constraints like time, cost, availability, etc. The two commonly utilized approaches are Type-I, involving termination at a predefined time, and Type-II, which concludes testing after a specific number of observed

failures (see Aslam, Azam, and Jun 2017; Gui and Aslam 2017). Later, a mixture of Type-I and Type-II called hybrid censoring schemes was introduced to reduce the testing time.

In contrast, progressive censoring schemes allow the removal of surviving items before a test's predetermined time. It is an appropriate censoring scheme in medical survival analysis, where unpredicted and uncontrolled removal of items occurs at intermediate stages of treatment. For example, suppose, for a clinical test, n cancer patients undergo chemotherapy post-surgery. The objective is to track the lifespan of m patients out of n. However, in such studies, patients frequently discontinue participation by missing follow-up appointments or failing to report their time of death. This persists even in hospital settings, where, following the first patient's demise, others might withdraw due to personal reasons of faith in medical staff, dissatisfaction with hospital facilities, or seeking alternate treatments. Subsequent patient withdrawals occur following subsequent deaths, shaping observations until the lifetimes of m (pre-fixed number) patients are recorded. Notably, the random nature of patient dropouts at each stage impedes predictability. Technically, the aforementioned can be restated as follows: Testing begins with a sample of size n. After observing the first failure, $X_{(1,m,n)}$, R_1 $(0 \leq R_1 \leq n-m)$ surviving units are removed. At the second failure, $X_{(2,m,n)}$, R_2 $(0 \leq R_2 \leq n-m-R_1)$ units are removed from surviving units. The testing continues until the occurrence of the m^{th} failure, $X_{(m,m,n)}$, and at this point, the remaining $R_m = n-m-\sum_{i=1}^{m-1} R_i$ units are removed. Note that here R_i is a random variable. By assuming a probability,p, for the removal of each remaining unit at every stage, the number of units removed at the i^{th} failure, R_i, follows a binomial distribution as described by Tse, Yang, and Yuen (2000). This censoring scheme is called the progressive Type-II censoring scheme with binomial removals (PTIICS-BR) and is illustrated in Fig. 1. For $i = 1, 2, \cdots, m$, $R_i = 0$, this scheme reduces to complete sampling, and for $i = 1, 2, \cdots, m-1$,$R_i = 0$ and $r_m = n-m$, progressive Type-II censoring scheme (PTIICS) becomes Type-II censoring.

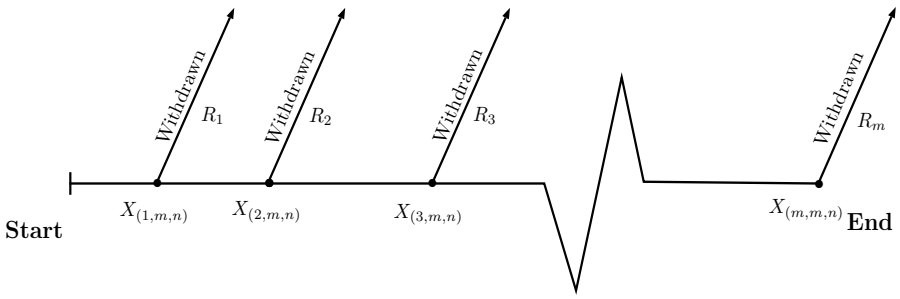

Figure 1: Progressive Type II censoring model

In recent years, the estimation of parameters for different distributions under PTIICS has been studied by researchers like Vishwakarma, Kaushik, Pandey, Singh, and Singh (2018), Wu and Chang (2002), and Pathak, Kumar, Singh, Singh, Tiwari, and Kumar (2023). Under PTIICS-BR, Singh, Singh, and Kumar (2016) proposed the maximum likelihood estimation and Bayes estimation of Poisson-exponential distribution with an application in cancer patients' data. Ahmadi Nadi, Gildeh, Kazempoor, Tran, and Tran (2023) developed a conditional reliability sampling plan for the quantile of the Weibull distribution based on PTIICS-BR with three cost models. Balakrishnan and Cramer (2014) investigated PTIICS in detail, covering both the theoretical principles and practical techniques related to progressive censoring schemes.

Nowadays, products exhibit higher reliability, and acquiring life test data under normal conditions requires a longer duration. This prompted the use of accelerated and partially accelerated life testing, as they help to reduce cost and time than traditional tests. In an accelerated life test (ALT), units undergo testing only at accelerated stress conditions, whereas in a partially accelerated life test (PALT), test units are tested under both accelerated and normal conditions. PALT is mostly used when there is no prior information about the stress-

dependent model. Hassan, Assar, and Zaky (2015) explored constant-stress PALT using multi-censored data. They considered the inverted Weibull distribution for the lifetime of the test unit. Also, obtained the maximum likelihood estimator (MLE) of the parameters and the accelerating factor. On a related note, Ismail (2016) investigated the estimation of Weibull distribution parameters based on hybrid censored data under constant-stress partially accelerated test model.

Srivastava and Mittal (2013) proposed a sampling plan for products having truncated logistic distribution truncated at point zero under Type-I censoring under constant PALT. They focused on determining the optimal allocation of sample proportion to both conditions by minimizing the generalized asymptotic variance of MLE of the acceleration factor and model parameters. Meanwhile, Kumar, Bajeel, and Ramyamol (2020) designed an optimal ASP for Weibull distribution under PALT based on the Type-II censoring scheme. They considered two different life-stress relationships, namely, linear relation and Arrhenius. A detailed comparative study based on testing cost is also presented.

Recently, researchers have been paying attention to the application of mixture distributions for modeling failure data. Failures can be classified into two or more subclasses based on their reasons. Like in the above-mentioned example, for cancer patients, demise (failure) might result from different causes, such as the progression of cancer itself, an accidental event, or another unrelated disease. Statistical analysis addresses such complexities by employing mixture distributions, combining two or more component distributions to create a more comprehensive model for failure events.

Muralidharan and Lathika (2005) introduced a mixture of exponential-Rayleigh distributions and compared the estimators of the parameters obtained via maximum likelihood estimation and the method of moments. Some of the other studies in the parameter inference can be seen in Mendenhall and Hader (1958), Muralidharan (2000), Kharazmi, Kumar, and Dey (2023), and Mathai and Kumar (2023). Despite the significance of mixture distributions in several practical statistical areas, variable ASPs for mixed distributions are not well addressed in the literature. Ramyamol and Kumar (2019) derived single and sequential variable sampling plans for a mixture of two exponential distributions. They have also defined two different sequential sampling plans, and optimal plan parameters are obtained by minimizing total testing cost. However, they have used the primitive censoring schemes, Type-I and Type-II schemes, which are no longer considered in practical applications.

In this work, we propose a variable repetitive group sampling plan for units whose lifetime follows the mixture of exponential-Rayleigh distributions (MERD) under PTIICS using PALT. This is a novel concept in the ASPs for mixture distribution as there are no works available in the literature, using PALT or PTIICS for any mixture distributions. Moreover, the MLE of the distribution parameters serves as the deciding factor by utilizing the lifetime of the units in the lot. Under PALT, two different life stress relationships, namely, Arrhenius and linear relationship are considered for deriving ASPs. The main purpose of this study is to obtain the optimal values of plan parameters in the two different ASPs by minimizing the expected total cost (ETC) and satisfying the Type-I and Type-II error constraints.

The rest of the article is organized as follows in different sections: Section 2 discusses the derivation of the likelihood function for the MERD based on PTIICS. MLE of the mixture distribution parameters and design of an optimal ASP using linear life-stress relation, which minimizes the ETC, are explained in Section 3. In Section 4, mixture distribution parameters are estimated, and an optimal ASP is proposed based on Arrhenius life-stress relation. Analysis of numerical results, comparative study, and sensitivity analysis are presented in Section 5. Also illustrates a real-life case study of two different data sets: cancer patients' lifetime data and failure data of communication transmitter-receivers of a single commercial airline. The concluding remarks are presented in Section 6.

2. Progressive Type-II censoring scheme

Consider a random sample of n independent and identically distributed (i.i.d.) units with lifetime following the MERD in the proportion ρ and $1-\rho$ are put to test. The cumulative distribution function (CDF) of the MERD is

$$F(x \mid \vartheta, \varphi) = \rho \cdot \left(1 - exp\left(-\frac{x}{\vartheta}\right)\right) + (1-\rho) \cdot \left(1 - exp\left(-\frac{x^2}{\varphi}\right)\right), \tag{1}$$

where $\vartheta, \varphi > 0$, $x \geq 0$ and $0 \leq \rho \leq 1$, and the probability density function (PDF) is

$$f(x \mid \vartheta, \varphi) = \rho \cdot \left(\frac{1}{\vartheta} exp\left(-\frac{x}{\vartheta}\right)\right) + (1-\rho) \cdot \left(\frac{2x}{\varphi} exp\left(-\frac{x^2}{\varphi}\right)\right). \tag{2}$$

This is a combination of Weibull distributions with parameters $(\vartheta, 1)$, and $(\varphi, 2)$. The mixture shows a varying hazard rate, as the hazard rate of the exponential distribution remains constant while the hazard rate of the Rayleigh distribution is monotonically increasing.

Under PTIICS, suppose that after the initial failure, $X_{(1,m,n)}$, R_1 items are removed. After the second failure $X_{(2,m,n)}$, R_2 items are removed, and so on, until R_m items are removed, followed by mth failure in random from the test. The sample thus obtained are represented as $\left[X_{(1,m,n)}, R_1\right], \left[X_{(2,m,n)}, R_2\right], \cdots \left[X_{(m,m,n)}, R_m\right]$ (see Fig. 1). As explained in Section 1, the number of items removed at ith stage, R_i is a binomial random variable. But, if the number of items removed at each stage is fixed, i.e, $R_i = r_i$ for $i = 1, 2, \cdots, m$, the conditional likelihood function can be expressed as follows (see Balakrishnan, Cramer, and Kamps 2001)

$$\begin{aligned} L(\vartheta, \varphi; x \mid R = r) &= f_{\left(X_{(1,m,n)}, \cdots, X_{(m,m,n)}\right)}(x_1, \cdots, x_m) \\ &= a \prod_{i=1}^{m} f(x_i) \left[1 - F(x_i)\right]^{r_i}, \quad 0 \leq x_i < \infty, \end{aligned} \tag{3}$$

where x_i is the realization of the random variable $X_{(i,m,n)}, i = 1, \cdots, m$, $a = \prod_{i=1}^{m} \sum_{j=i}^{m} (r_j + 1)$, $n, m \in N$, $r_i \in N_0$, and $n = m + \sum_{i=1}^{m} r_i$.

As previously stated, the number of items removed at each stage is random and independent, with each unit having a probability p. Therefore, the number of units R_i that are eliminated at the ith failure $X_{(i,m,n)}$; $i = 1, 2, \cdots, (m-1)$, follows to a binomial distribution, i.e., $R_i \sim \left(n - m - \sum_{j=0}^{i-1} r_j,\ p\right)$ and $r_0 = 0$. Thus,

$$P(R_1 = r_1; p) = \binom{n-m}{r_1} p^{r_1} (1-p)^{n-m-r_1}, \tag{4}$$

and for $i = 2, 3, \cdots, m-1$,

$$P(R_i = r_i \mid R_{i-1} = r_{i-1}, \cdots R_1 = r_1) = \binom{n - m - \sum_{j=0}^{i-1} r_j}{r_i} p^{r_i} (1-p)^{n-m-\sum_{j=0}^{i} r_j}. \tag{5}$$

Therefore,

$$\begin{aligned} P(R = r; p) = P(R_1 = r_1)\, P(R_2 = r_2 \mid R_1 = r_1)\, P(R_3 = r_3 \mid R_2 = r_2, R_1 = r_1) \\ \cdots P(R_{m-1} = r_{m-1} \mid R_{m-2} = r_{m-2}, \cdots R_1 = r_1). \end{aligned} \tag{6}$$

Substituting equations (4) and (5) in equation (6), we get

$$P(R=r;p)=\frac{(n-m)!p^{\sum_{i=1}^{m-1} r_i}(1-p)^{(m-1)(n-m)-\sum_{i=1}^{m-1}(m-i)r_i}}{\left(n-m-\sum_{i=1}^{m-1} r_i\right)!\prod_{i=1}^{m-1} r_i!}. \tag{7}$$

Then the complete likelihood function is given by

$$\begin{aligned} L(\vartheta,\varphi,p;x) &= L(\vartheta,\varphi;x\mid R=r)\ P(R=r;p) \\ &= \frac{a(n-m)!p^{\sum_{i=1}^{m-1} r_i}(1-p)^{(m-1)(n-m)-\sum_{i=1}^{m-1}(m-i)r_i}}{\left(n-m-\sum_{i=1}^{m-1} r_i\right)!\prod_{i=1}^{m-1} r_i!}\ \prod_{i=1}^{m} f\left(x_i\right)\left[1-F\left(x_i\right)\right]^{r_i}. \end{aligned} \tag{8}$$

Further, the computation of the MLE of the parameters ϑ and φ, uses only the conditional likelihood, $L(\vartheta,\varphi;x\mid R=r)$. This is because $P(R=r;p)$, is independent of ϑ and φ (from equation (7)) and hence remains constant.

3. ASP under progressive Type-II censoring using PALT: Linear life-stress relation

This section presents the design of an ASP for the MERD using PALT. The plan is based on a linear life-stress relation between a unit's lifespan under accelerated stress conditions (represented as Y) and its lifespan under normal stress conditions (represented as X). The relationship is given by $Y=\frac{X}{\zeta}$, where ζ is the acceleration factor and $\zeta>0$.

According to the PALT procedure, initially, a sample of n i.i.d. units is taken. Subsequently, the sample is divided into proportions of δ and $(1-\delta)$ for testing under accelerated and normal stress conditions, respectively. Accordingly, from a set of n units, $n\delta$ units are allocated randomly for accelerated stress testing, while $n(1-\delta)$ units are chosen for normal stress testing. Subsequent testing procedures using PTIICS are conducted separately for samples under both stress conditions.

3.1. Progressive Type-II censoring for normal stress condition

Suppose that random variable X denotes the lifetime of units under normal stress condition following the MERD with unknown parameters ϑ and φ. The PDF of X is given by

$$f_1(x\mid\vartheta,\varphi)=\rho_1\cdot g_1(x\mid\vartheta)+(1-\rho_1)\cdot h_1(x\mid\varphi), \tag{9}$$

where $\vartheta,\varphi>0,\ x\geq 0,\ 0\leq\rho_1\leq 1$,

$$g_1(x\mid\vartheta)=\frac{1}{\vartheta}exp\left(-\frac{x}{\vartheta}\right)\quad\text{and}\quad h_1(x\mid\varphi)=\frac{2x}{\varphi}exp\left(-\frac{x^2}{\varphi}\right). \tag{10}$$

The corresponding CDF of X is

$$F_1(x\mid\vartheta,\varphi)=\rho_1\cdot G_1(x\mid\vartheta)+(1-\rho_1)\cdot H_1(x\mid\varphi), \tag{11}$$

$$\text{where}\quad G_1(x\mid\vartheta)=1-exp\left(-\frac{x}{\vartheta}\right)\quad\text{and}\quad H_1(x\mid\varphi)=1-exp\left(-\frac{x^2}{\varphi}\right). \tag{12}$$

Under PTIICS, $n(1-\delta)$ units are tested in normal stress condition until the η^{th} failure, $X_{(\eta,\eta,n(1-\delta))}$ occurs and $\eta\leq n(1-\delta)$ is fixed in prior. Let $r_i, i=1,2,\cdots,\eta-1$ denotes the number of items removed after i^{th} failure, $X_{(i,\eta,n(1-\delta))}$. Let x_i be the realization of the

random variable $X_{(i,\eta,n(1-\delta))}$. Then, by using equation (3), the conditional likelihood under normal stress condition is given by

$$L_{\mathrm{N}}(\vartheta,\varphi;x) = a_1 \prod_{i=1}^{\eta} f_1(x_i)\left[1-F_1(x_i)\right]^{r_i} \tag{13}$$

where $a_1 = \prod_{i=1}^{\eta}\sum_{j=i}^{\eta}(r_j+1)$ and $n(1-\delta) = \eta + \sum_{i=1}^{\eta} r_i$.

As in (Muralidharan and Lathika 2005; Mathai and Kumar 2023), define,

$U(x) = \begin{cases} 1, & \text{if } X \sim \text{Exponential}(\vartheta) \\ 0, & \text{if } X \sim \text{Rayleigh}(\varphi) \end{cases}$ and among η failures, let η_0 $(\eta_0 < \eta)$ be the observed number of units following exponential life. Hence, $\sum_{i=1}^{\eta} U(x_i) = \eta_0$.

Thus, equation (13) becomes,

$$L_{\mathrm{N}}(\vartheta,\varphi;x) = a_1 \prod_{i=1}^{\eta}\Big(\{\rho_1 g_1(x_i)\left[1-G_1(x_i)\right]^{r_i}\}^{U(x_i)} \{(1-\rho_1)h_1(x)\left[1-H_1(x)\right]^{r_i}\}^{1-U(x_i)}\Big). \tag{14}$$

By substituting equations (10) and (12) in equation (14), we get

$$L_{\mathrm{N}}(\vartheta,\varphi;x) = a_1 \left(\frac{\rho_1}{\vartheta}\right)^{\eta_0}\left(\frac{2(1-\rho_1)}{\varphi}\right)^{\eta-\eta_0} exp\left(-\sum_{i=1}^{\eta_0}\frac{x_i+x_i r_i}{\vartheta} - \sum_{i=\eta_0+1}^{\eta}\frac{x_i^2+x_i^2 r_i}{\varphi}\right)\prod_{i=\eta_0+1}^{\eta} x_i \tag{15}$$

3.2. Progressive Type-II censoring for accelerated stress condition

Let the random variable Y be the lifetime of units tested under accelerated stress condition with acceleration factor ζ. Let $Y = \frac{X}{\zeta}$, based on linear life-stress relation, with CDF given by

$$F_2(y \mid \vartheta,\varphi) = P(Y \le y) = P\left(\frac{X}{\zeta} \le y\right) = P(X \le \zeta y). \tag{16}$$

Therefore,

$$F_2(y \mid \vartheta,\varphi,\zeta) = \rho_2 G_2(X \le \zeta y) + (1-\rho_2)H_2(X \le \zeta y), \tag{17}$$

$$\begin{aligned} \text{where} \quad G_2(y \mid \vartheta,\zeta) &= G_2(X \le \zeta y) = 1 - exp\left(-\frac{\zeta y}{\vartheta}\right) \\ H_2(y \mid \varphi,\zeta), &= H_2(X \le \zeta y) = 1 - exp\left(-\frac{(\zeta y)^2}{\varphi}\right). \end{aligned} \tag{18}$$

The PDF is given by,

$$f_2(y \mid \vartheta,\varphi,\zeta) = \frac{\mathrm{d}}{\mathrm{d}y}F_2(y \mid \vartheta,\varphi,\zeta) = \rho_2 g_2(y \mid \vartheta,\zeta) + (1-\rho_2)h_2(y \mid \varphi,\zeta), \tag{19}$$

$$\text{where} \quad g_2(y \mid \vartheta,\zeta) = \frac{\zeta}{\vartheta}exp\left(-\frac{\zeta y}{\vartheta}\right) \quad \text{and} \quad h_2(y \mid \varphi,\zeta) = \frac{2\zeta^2 y}{\varphi}exp\left(-\frac{(\zeta y)^2}{\varphi}\right). \tag{20}$$

It can be seen that Y follows the MERD with known parameter ζ and unknown parameters ϑ and φ.

Suppose out of n samples, $n\delta$ are randomly chosen and tested with accelerated stress conditions under PTIICS. The test is terminated after the occurrence of μ^{th} failure, $Y_{(\mu,\mu,n\delta)}$.

Let $s_i, i = 1, 2, \cdots, \mu$, be the number of units removed after the i^{th} failure where $s_\mu = n\delta - \mu - \sum_{i=1}^{\mu-1} s_i$ and hence, $n\delta = \mu + \sum_{i=1}^{\mu} s_i$.

As explained in Subsection 3.1, the conditional likelihood under accelerated stress test is derived as follows

$$L_{\mathrm{A}}(\vartheta, \varphi, \zeta; y) = a_2 \prod_{i=1}^{\mu} f_2(y_i) \left[1 - F_2(y_i)\right]^{s_i} \tag{21}$$

where y_i is the realization of the random variable $Y_{(i,\mu,n\delta)}$, $s_i \in N_0, 1 \leq i \leq \mu$, $a_2 = \prod_{i=1}^{\mu} \sum_{j=i}^{\mu} (s_j + 1)$, and $\mu \in N$.

By defining, $V(y) = \begin{cases} 1, & \text{if} \quad Y \sim \text{Exponential}(\vartheta) \\ 0, & \text{if} \quad Y \sim \text{Rayleigh}(\varphi) \end{cases}$, we get $\sum_{i=1}^{\mu} V(y_i) = \mu_0$, where μ_0 ($\mu_0 < \mu$) is the number of failures observed, out of μ following exponential distribution.

Using equations (17), (18), (19), and (20) in equation (21), we get

$$L_{\mathrm{A}}(\vartheta, \varphi, \zeta; y) = a_2 \left(\frac{\rho_2 \zeta}{\vartheta}\right)^{\mu_0} \left(\frac{2\zeta^2(1-\rho_2)}{\varphi}\right)^{\mu-\mu_0} exp\left(-\zeta \sum_{i=1}^{\mu_0} \frac{y_i + y_i s_i}{\vartheta} - \zeta^2 \sum_{i=\mu_0+1}^{\mu} \frac{y_i^2 + y_i^2 s_i}{\varphi}\right) \prod_{i=\mu_0+1}^{\mu} y_i. \tag{22}$$

3.3. Maximum likelihood estimation of parameters

In this section, MLE of the unknown parameters ϑ and φ of the MERD are derived, where units are tested under PALT based on PTIICS. The conditional likelihood function for observed data under PALT is given by (see Kumar *et al.* 2020)

$$\mathcal{L}(\vartheta, \varphi; x, y) = L_N(\vartheta, \varphi; x) L_A(\vartheta, \varphi; y). \tag{23}$$

Using equations (15) and (22), equation (23) becomes

$$\mathcal{L}(\vartheta, \varphi; x, y) = a_1 a_2 \left(\frac{\rho_1}{\vartheta}\right)^{\eta_0} \left(\frac{\rho_2 \zeta}{\vartheta}\right)^{\mu_0} \left(\frac{2(1-\rho_1)}{\varphi}\right)^{\eta-\eta_0} \left(\frac{2\zeta^2(1-\rho_2)}{\varphi}\right)^{\mu-\mu_0} exp\left(-\sum_{i=1}^{\eta_0} \frac{x_i + x_i r_i}{\vartheta} - \sum_{i=\eta_0+1}^{\eta} \frac{x_i^2 + x_i^2 r_i}{\varphi} - \zeta \sum_{i=1}^{\mu_0} \frac{y_i + y_i s_i}{\vartheta} - \zeta^2 \sum_{i=\mu_0+1}^{\mu} \frac{y_i^2 + y_i^2 s_i}{\varphi}\right) \prod_{i=\eta_0+1}^{\eta} x_i \prod_{i=\mu_0+1}^{\mu} y_i \tag{24}$$

Then, the log-likelihood function is given by

$$\begin{aligned} \ln \mathcal{L} &= \ln L_N + \ln L_A. \\ &= \ln a_1 + \ln a_2 + \eta_0 (\ln \rho_1 - \ln \vartheta) + \mu_0 [\ln(\rho_2 \zeta) - \ln \vartheta] + \sum_{i=\eta_0+1}^{\eta} \ln x_i + \sum_{i=\mu_0+1}^{\mu} \ln y_i \\ &\quad (\eta - \eta_0) [\ln(2(1-\rho_1)) - \ln \varphi] + (\mu - \mu_0) \left[\ln\left(2\zeta^2(1-\rho_2)\right) - \ln \varphi\right] \\ &\quad - \sum_{i=1}^{\eta_0} \frac{x_i + x_i r_i}{\vartheta} - \sum_{i=\eta_0+1}^{\eta} \frac{x_i^2 + x_i^2 r_i}{\varphi} - \zeta \sum_{i=1}^{\mu_0} \frac{y_i + y_i s_i}{\vartheta} - \zeta^2 \sum_{i=\mu_0+1}^{\mu} \frac{y_i^2 + y_i^2 s_i}{\varphi} \end{aligned} \tag{25}$$

Differentiating both sides of equation (25),with respect to ϑ and φ and equating to zero, we get

$$\frac{\partial}{\partial \vartheta} \ln \mathcal{L} = 0 \quad \text{and} \quad \frac{\partial}{\partial \varphi} \ln \mathcal{L} = 0.$$

The normal equations are obtained as,

$$-\frac{\eta_0+\mu_0}{\vartheta}+\frac{1}{\vartheta^2}\left[\sum_{i=1}^{\eta_0}(x_i+x_i r_i)+\zeta\sum_{i=1}^{\mu_0}(y_i+y_i s_i)\right]=0 \tag{26}$$

$$-\frac{1}{\varphi}\left[(\eta-\eta_0)+(\mu-\mu_0)\right]+\frac{1}{\varphi^2}\left[\sum_{i=\eta_0+1}^{\eta}\left(x_i^2+x_i^2 r_i\right)+\zeta^2\sum_{i=\mu_0+1}^{\mu}\left(y_i^2+y_i^2 s_i\right)\right]=0. \tag{27}$$

Finally, the expressions for the MLEs of ϑ and φ are obtained, respectively, as

$$\widehat{\vartheta}=\frac{P_1+\zeta\ P_2}{\eta_0+\mu_0}, \tag{28}$$

$$\widehat{\varphi}=\frac{Q_1+\zeta^2\ Q_2}{(\eta-\eta_0)+(\mu-\mu_0)}, \tag{29}$$

where $P_1=\sum_{i=1}^{\eta_0}X_i\,(1+r_i)$, $P_2=\sum_{i=1}^{\mu_0}Y_i\,(1+s_i)$, $Q_1=\sum_{i=\eta_0+1}^{\eta}X_i^2\,(1+r_i)$ and $Q_2=\sum_{i=\mu_0+1}^{\mu}Y_i^2\,(1+s_i)$.

Next, for developing the ASP, the distribution of the estimators is to be derived. Here, $X_{(i,\eta,n(1-\delta))}$, $i=1,2,\cdots,\eta_0$ follows an exponential distribution with mean ϑ and variance ϑ^2, and $Y_{(i,\mu,n\delta)}$, $i=1,2,\cdots,\mu_0$ follows an exponential distribution with mean $\frac{\vartheta}{\zeta}$ and variance $\left(\frac{\vartheta}{\zeta}\right)^2$. Also, $X_{(i,\eta,n(1-\delta))}$, $i=\eta_0+1,\cdots,\eta$ follows a Rayleigh distribution with mean $\frac{\sqrt{\varphi\pi}}{2}$ and variance $\frac{4-\pi}{4}\varphi$, and $Y_{(i,\mu,n\delta)}$, $i=\mu_0+1,\cdots,\mu$ follows a Rayleigh distribution with mean $\frac{\sqrt{\varphi\pi}}{2\zeta}$ and variance $\frac{4-\pi}{4}\frac{\varphi}{\zeta^2}$.

Let $A(X_i)=X_i\,(1+r_i)$, $i=1,2,\cdots,\eta_0$. Then, $A(X_i)$ follows an exponential distribution with mean $\vartheta\,(1+r_i)$ and variance $(1+r_i)^2\,\vartheta^2$. Here, r_i's need not always be distinct, so the sum of the independent exponential random variables involves a combination of Erlang and hypo exponential distributions, whose expression for closed-form density is not available explicitly. Hence, by the Lindeberg central limit theorem, $P_1=\sum_{i=1}^{\eta_0}A(X_i)$ follows a normal distribution with mean $\vartheta\sum_{i=1}^{\eta_0}(1+r_i)$ and variance $\vartheta^2\sum_{i=1}^{\eta_0}(1+r_i)^2$. Similar approximations are considered in the literature (see Kumar and Bajeel 2018; Rajgopal and Mazumdar 1997, 1996). Similarly, we get $P_2=\sum_{i=1}^{\mu_0}Y_i\,(1+s_i)\sim N\left(\frac{\vartheta}{\zeta}\sum_{i=1}^{\mu_0}(1+s_i),\left(\frac{\vartheta}{\zeta}\right)^2\sum_{i=1}^{\mu_0}(1+s_i)^2\right)$.

For $i=\eta_0+1,\cdots,\eta$, X_i^2 follows an exponential distribution with mean φ and variance φ^2. Thus, $X_i^2\,(1+r_i)$ follows an exponential distribution with mean $\varphi\,(1+r_i)$ and variance $(1+r_i)^2\,\varphi^2$. Again, invoking the Lindeberg central limit theorem, we get $Q_1=\sum_{i=\eta_0+1}^{\eta}X_i^2\,(1+r_i)\sim N\left(\varphi\sum_{i=\eta_0+1}^{\eta}(1+r_i),\varphi^2\sum_{i=\eta_0+1}^{\eta}(1+r_i)^2\right)$. Similarly, $Q_2=\sum_{i=\mu_0+1}^{\mu}Y_i^2\,(1+s_i)\sim N\left(\frac{\varphi}{\zeta^2}\sum_{i=\mu_0+1}^{\mu}(1+s_i),\frac{\varphi^2}{\zeta^4}\sum_{i=\mu_0+1}^{\mu}(1+s_i)^2\right)$.

Moreover, from equations (28) and (29), we get that,

$$\widehat{\vartheta}=\frac{P_1+\zeta\ P_2}{\eta_0+\mu_0}\sim\text{Normal distribution with mean, }\frac{\vartheta}{\eta_0+\mu_0}\left[\sum_{i=1}^{\eta_0}(1+r_i)+\sum_{i=1}^{\mu_0}(1+s_i)\right]$$
$$\text{and variance, }\left(\frac{\vartheta}{\eta_0+\mu_0}\right)^2\left[\sum_{i=1}^{\eta_0}(1+r_i)^2+\sum_{i=1}^{\mu_0}(1+s_i)^2\right], \tag{30}$$

and $\widehat{\varphi} = \dfrac{Q_1 + \zeta^2\, Q_2}{(\eta - \eta_0) + (\mu - \mu_0)} \sim$ Normal distribution with

$$\text{mean, } \frac{\varphi}{(\eta - \eta_0) + (\mu - \mu_0)} \left[\sum_{i=\eta_0+1}^{\eta} (1 + r_i) + \sum_{i=\mu_0+1}^{\mu} (1 + s_i) \right]$$

$$\text{and variance, } \frac{\varphi^2}{[(\eta - \eta_0) + (\mu - \mu_0)]^2} \left[\sum_{i=\eta_0+1}^{\eta} (1 + r_i)^2 + \sum_{i=\mu_0+1}^{\mu} (1 + s_i)^2 \right]. \quad (31)$$

3.4. ASP under linear life-stress relation

In this section, an ASP is derived under PTIICS based on linear life-stress relation. For that, consider a lot of units with failure lifetime following the MERD with PDF given by equation (2). Let ϑ_A and, φ_A denote the acceptance quality level (AQL) and ϑ_R and, φ_R denotes the rejection quality level (RQL) of an item in the lot. The lot is accepted or rejected depending on the following probability conditions:

$$\begin{aligned} &\mathrm{P}\left(\text{Reject the lot} \mid \vartheta \geq \vartheta_A \text{ and } \varphi \geq \varphi_A\right) \leq \alpha, \\ &\mathrm{P}\left(\text{Accept the lot} \mid \vartheta \leq \vartheta_R \text{ and } \varphi \leq \varphi_R\right) \leq \beta, \end{aligned} \quad (32)$$

where α is the producer's risk and β is the consumer's risk.

The ASP is outlined as follows:

1. Take a random sample of n units from the lot under consideration. According to the PALT scheme, $n\delta$ units are tested under accelerated stress condition and $n\,(1-\delta)$ under normal stress condition.
2. In both stress conditions under PTIICS, as elaborated in Subsections 3.1 and 3.2, testing is terminated upon attaining a pre-determined number of failures, μ and η out of $n\delta$ and $n\,(1-\delta)$ units, respectively.
3. From the data obtained, calculate the MLEs $\widehat{\vartheta}$ and $\widehat{\varphi}$ of the parameters ϑ and φ, respectively.
4. Let $\widehat{\psi} = \widehat{\vartheta} + \widehat{\varphi}$. Accept the lot, if $\widehat{\psi} \geq t_2$. Reject the lot, if $\widehat{\psi} < t_1$.
5. If $t_1 \leq \widehat{\psi} < t_2$, continue the process by repeating Steps 1 to 3.

Let p_a, p_r, and p_c denote the probability of accepting the lot, rejecting the lot, and continuing the process, respectively, while observing a sample from the lot. The probabilities of long-run acceptance and rejection are obtained as $P_A = \frac{p_a}{1-p_c}$ and $P_R = \frac{p_r}{1-p_c}$, respectively, and the expected number of samples involved in testing is $\frac{1}{1-p_c}$ (see Sherman 1965).

According to the plan, for deriving the probabilities p_a, p_r, and p_c, the distribution of $\widehat{\psi}$ is required and from equations (30) and (31), we get that,

$$\widehat{\psi} \sim N\left[E\left(\widehat{\vartheta}\right) + E\left(\widehat{\varphi}\right), V\left(\widehat{\vartheta}\right) + V\left(\widehat{\varphi}\right)\right]. \quad (33)$$

Then the required probabilities are given by,

$$p_a = P\left(\widehat{\psi} \geq t_2\right) = P\left[Z \geq \frac{t_2 - E\left(\widehat{\psi}\right)}{\sqrt{V\left(\widehat{\psi}\right)}}\right], \quad (34)$$

$$p_r = P\left(\widehat{\psi} < t_1\right) = P\left[Z < \frac{t_1 - E\left(\widehat{\psi}\right)}{\sqrt{V\left(\widehat{\psi}\right)}}\right], \quad (35)$$

$$p_c = P\left(t_1 \le \widehat{\psi} < t_2\right) = P\left[\frac{t_1 - E\left(\widehat{\psi}\right)}{\sqrt{V\left(\widehat{\psi}\right)}} \le Z < \frac{t_2 - E\left(\widehat{\psi}\right)}{\sqrt{V\left(\widehat{\psi}\right)}}\right], \tag{36}$$

where Z denotes the standard normal random variable.

Next, the optimal values of the unknown parameters of the sampling plan, namely, t_1, t_2, η, μ, r_i, $i = 1, 2, \cdots, \eta$ and $s_i, i = 1, 2, \cdots, \mu$, are determined by solving a nonlinear optimization problem. The objective of this problem is to minimize the ETC of testing while satisfying the conditions in equation (32). Note that here the total cost is the sum of the cost associated with testing and the cost of the items that have failed. The cost of testing is determined by multiplying the cost of testing a sample per unit of time by the overall testing duration. Due to the random nature of both the decision time $\widehat{\psi}$ and the number of samples, the total testing time becomes a random variable, with expectation $\frac{E\left(\widehat{\psi}\right)}{1-p_c}$. Also, the total expected number of failed items is $\frac{\eta+\mu}{1-p_c}$. As the total testing time and number of failed items are random quantities, here we have to consider the ETC. Thus, the ETC for conducting this sampling plan is

$$\begin{aligned} \text{ETC} &= \text{Expected cost of testing} + \text{Expected cost of failed items} \\ &= C_t \frac{E\left(\widehat{\psi}\right)}{1-p_c} + C_f \frac{\eta+\mu}{1-p_c}, \end{aligned} \tag{37}$$

where C_t is the cost of testing a unit for unit time and C_f is the cost of a failed item.

The required optimization problem, which minimizes the ETC given by equation (37) at AQL (i.e., $\vartheta = \vartheta_A$ and $\varphi = \varphi_A$) subjected to conditions in equation (32) is formulated as:

$$\begin{aligned} \underset{(t_1,t_2,\eta,\mu,r_i,s_i)}{\text{Min}} \quad & C_t \frac{E\left(\widehat{\psi}\right)}{1-p_c} + C_f \frac{\eta+\mu}{1-p_c} \text{ at } \vartheta = \vartheta_A, \varphi = \varphi_A \\ \text{subject to} \quad & \left(\frac{p_r}{1-p_c} \mid \vartheta \ge \vartheta_A \text{ and } \varphi \ge \varphi_A\right) \le \alpha, \\ & \left(\frac{p_a}{1-p_c} \mid \vartheta \le \vartheta_R \text{ and } \varphi \le \varphi_R\right) \le \beta, \end{aligned} \tag{38}$$

where $t_2 > t_1 > 0$, $0 < \eta \le n(1-\delta)$, $0 < \mu \le n\delta$, $0 \le r_i \le n(1-\delta) - \eta - \sum_{j=1}^{i-1} r_i$, for $i = 1, 2, \cdots, \eta - 1$, $r_0 = s_0 = 0$, $0 \le s_i \le n\delta - \mu - \sum_{j=1}^{i-1} s_i$, for $i = 1, 2, \cdots, \mu - 1$, $r_\eta = n(1-\delta) - \eta - \sum_{i=1}^{\eta-1} r_i$, and $s_\mu = n\delta - \mu - \sum_{i=1}^{\mu-1} s_i$.

Substituting the expressions for probabilities from equations (34), (35), and (36) in equation (38), we get

$$\begin{aligned} \underset{(t_1,t_2,\eta,\mu,r_i,s_i)}{\text{Min}} \quad & \frac{C_t\, E\left(\widehat{\psi}\right) + C_f\,(\eta+\mu)}{1 - P\left[\frac{t_1-E\left(\widehat{\psi}\right)}{\sqrt{V\left(\widehat{\psi}\right)}} \le Z < \frac{t_2-E\left(\widehat{\psi}\right)}{\sqrt{V\left(\widehat{\psi}\right)}}\right]} \text{ at } \vartheta = \vartheta_A, \varphi = \varphi_A \\ \text{subject to} \quad & \left(\frac{P\left[Z < \frac{t_1-E\left(\widehat{\psi}\right)}{\sqrt{V\left(\widehat{\psi}\right)}}\right]}{1 - P\left[\frac{t_1-E\left(\widehat{\psi}\right)}{\sqrt{V\left(\widehat{\psi}\right)}} \le Z < \frac{t_2-E\left(\widehat{\psi}\right)}{\sqrt{V\left(\widehat{\psi}\right)}}\right]} \middle| \vartheta \ge \vartheta_A \text{ and } \varphi \ge \varphi_A\right) \le \alpha, \end{aligned} \tag{39}$$

$$\left(\frac{P\left[Z \geq \frac{t_2 - E\left(\widehat{\psi}\right)}{\sqrt{V\left(\widehat{\psi}\right)}} \right]}{1 - P\left[\frac{t_1 - E\left(\widehat{\psi}\right)}{\sqrt{V\left(\widehat{\psi}\right)}} \leq Z < \frac{t_2 - E\left(\widehat{\psi}\right)}{\sqrt{V\left(\widehat{\psi}\right)}} \right]} \middle| \vartheta \leq \vartheta_R \text{ and } \varphi \leq \varphi_R \right) \leq \beta.$$

Observing equations (30), (31), and (33), we get that the expressions for $E\left(\widehat{\psi}\right)$ and $V\left(\widehat{\psi}\right)$ has terms involving ϑ and φ. But they are unknown and hence, the optimization problem is re-written as,

$$\begin{aligned}
\underset{(t_1,t_2,\eta,\mu,r_i,s_i)}{\text{Min}} \quad & \frac{C_t\ E\left(\widehat{\psi}\right) + C_f\left(\eta + \mu\right)}{1 - P\left[\frac{t_1 - E\left(\widehat{\psi}\right)}{\sqrt{V\left(\widehat{\psi}\right)}} \leq Z < \frac{t_2 - E\left(\widehat{\psi}\right)}{\sqrt{V\left(\widehat{\psi}\right)}} \right]} \text{ at } \vartheta = \vartheta_A, \varphi = \varphi_A \\
\text{subject to} \quad & \underset{\vartheta \geq \vartheta_A \text{ and } \varphi \geq \varphi_A,}{max} \left(\frac{P\left[Z < \frac{t_1 - E\left(\widehat{\psi}\right)}{\sqrt{V\left(\widehat{\psi}\right)}} \right]}{1 - P\left[\frac{t_1 - E\left(\widehat{\psi}\right)}{\sqrt{V\left(\widehat{\psi}\right)}} \leq Z < \frac{t_2 - E\left(\widehat{\psi}\right)}{\sqrt{V\left(\widehat{\psi}\right)}} \right]} \right) \leq \alpha, \qquad (40) \\
& \underset{\vartheta \leq \vartheta_R \text{ and } \varphi \leq \varphi_R,}{max} \left(\frac{P\left[Z \geq \frac{t_2 - E\left(\widehat{\psi}\right)}{\sqrt{V\left(\widehat{\psi}\right)}} \right]}{1 - P\left[\frac{t_1 - E\left(\widehat{\psi}\right)}{\sqrt{V\left(\widehat{\psi}\right)}} \leq Z < \frac{t_2 - E\left(\widehat{\psi}\right)}{\sqrt{V\left(\widehat{\psi}\right)}} \right]} \right) \leq \beta. \qquad (41)
\end{aligned}$$

From equations (30), (31), and (33), it is clear that, in equation (40), P_R decreases with respect to ϑ and φ and in equation (41), P_A increases as both ϑ and φ increases. Hence, maximum value of P_R is obtained at $\vartheta = \vartheta_A$ and $\varphi = \varphi_A$ and maximum of P_A is obtained at $\vartheta = \vartheta_R$ and $\varphi = \varphi_R$. Thus, the required nonlinear optimization problem, P_I is derived as:

$$\begin{aligned}
\underset{(t_1,t_2,\eta,\mu,r_i,s_i)}{\text{Min}} \quad & \frac{C_t\ E\left(\widehat{\psi}\right) + C_f\left(\eta + \mu\right)}{1 - P\left[\frac{t_1 - E\left(\widehat{\psi}\right)}{\sqrt{V\left(\widehat{\psi}\right)}} \leq Z < \frac{t_2 - E\left(\widehat{\psi}\right)}{\sqrt{V\left(\widehat{\psi}\right)}} \right]} \text{ at } \vartheta = \vartheta_A, \varphi = \varphi_A \\
\text{subject to} \quad & \frac{P\left[Z < \frac{t_1 - E\left(\widehat{\psi}\right)}{\sqrt{V\left(\widehat{\psi}\right)}} \right]}{1 - P\left[\frac{t_1 - E\left(\widehat{\psi}\right)}{\sqrt{V\left(\widehat{\psi}\right)}} \leq Z < \frac{t_2 - E\left(\widehat{\psi}\right)}{\sqrt{V\left(\widehat{\psi}\right)}} \right]} \leq \alpha, \text{ at } \vartheta = \vartheta_A \text{ and } \varphi = \varphi_A, \qquad (42) \\
& \frac{P\left[Z \geq \frac{t_2 - E\left(\widehat{\psi}\right)}{\sqrt{V\left(\widehat{\psi}\right)}} \right]}{1 - P\left[\frac{t_1 - E\left(\widehat{\psi}\right)}{\sqrt{V\left(\widehat{\psi}\right)}} \leq Z < \frac{t_2 - E\left(\widehat{\psi}\right)}{\sqrt{V\left(\widehat{\psi}\right)}} \right]} \leq \beta, \text{ at } \vartheta = \vartheta_R \text{ and } \varphi = \varphi_R, \qquad (43)
\end{aligned}$$

where $t_2 > t_1 > 0$, $0 < \eta \leq n(1-\delta)$, $0 < \mu \leq n\delta$, $0 \leq r_i \leq n(1-\delta) - \eta - \sum_{j=1}^{i-1} r_i$, for $i = 1, 2, \cdots, \eta - 1$, $r_0 = s_0 = 0$, $0 \leq s_i \leq n\delta - \mu - \sum_{j=1}^{i-1} s_i$, for $i = 1, 2, \cdots, \mu - 1$,

$r_\eta = n(1-\delta) - \eta - \sum_{i=1}^{\eta-1} r_i$, and $s_\mu = n\delta - \mu - \sum_{i=1}^{\mu-1} s_i$.

The nonlinear optimization problem, P_I is solved using the genetic algorithm (GA) solver in MATLAB. The algorithm used in the GA solver for solving the optimization problem is presented in the Appendix. The optimum values of the plan parameters and their corresponding ETC are illustrated in Table 1 in Section 5.

4. ASP under progressive Type-II censoring using PALT: Arrhenius life-stress relation

Under PALT, again consider a random sample of size n, consisting of i.i.d. random units following the MERD from the lot. The random sample is then allocated for testing under accelerated and normal stress conditions in a ratio of $\delta : (1-\delta)$. Specifically, $n\delta$ units from the sample undergo testing under accelerated stress condition while $n(1-\delta)$ units are subjected to testing under normal stress condition. As explained in subsections 3.1 and 3.2, under each stress condition units are tested using the PTIICS. In this section, the Arrhenius life-stress relationship is applied in PALT and the relationship is given by,

$$\mathcal{A}(\omega) = b_0 \; exp\left(\frac{b_1}{\omega}\right), \tag{44}$$

where ω is the stress level, $\mathcal{A}$ is a quantifiable life measure, and $b_0, b_1 > 0$ are the parameters in this model to be determined.

Based on the PTIICS, testing under accelerated stress condition, ceases upon observing the μ failures, where $0 < \mu \leq n\delta$ is prefixed. After each failure $(i = 1, 2, \cdots, \mu)$, $S_i = s_i$ surviving units are removed after each i^{th} failure. Similarly, under normal stress condition, $n(1-\delta)$ items are tested until the occurrence of η failures, where $0 < \eta \leq n(1-\delta)$. Post each failure $(i = 1, 2, \cdots, \eta)$, $R_i = r_i$ surviving units are removed.

Let the random variable X denote the lifetime under normal life-stress condition following the MERD with parameters ϑ_1, and φ_1 and the PDF is given by,

$$f_3\left(x \mid \vartheta_1, \varphi_1\right) = \rho_3 \cdot g_3\left(x \mid \vartheta_1\right) + (1-\rho_3) \cdot h_3\left(x \mid \varphi_1\right) \tag{45}$$

where $\vartheta_1, \varphi_1 > 0,\ x \geq 0,\ 0 \leq \rho_3 \leq 1,$

$$g_3(x \mid \vartheta_1) = \frac{1}{\vartheta_1} exp\left(-\frac{x}{\vartheta_1}\right) \quad \text{and} \quad h_3(x \mid \varphi_1) = \frac{2x}{\varphi_1} exp\left(-\frac{x^2}{\varphi_1}\right). \tag{46}$$

and the CDF of X is of the form

$$F_3(x \mid \vartheta_1, \varphi_1) = \rho_3 \cdot G_3(x \mid \vartheta_1) + (1-\rho_3) \cdot H_3(x \mid \varphi_1), \tag{47}$$

$$\text{where} \quad G_3(x \mid \vartheta_1) = 1 - exp\left(-\frac{x}{\vartheta_1}\right) \quad \text{and} \quad H_3(x \mid \varphi_1) = 1 - exp\left(-\frac{x^2}{\varphi_1}\right). \tag{48}$$

Let the random variable Y represent the lifetime of a unit under accelerated life-stress condition following the MERD with parameters ϑ_2 and φ_2. The PDF of Y is given by,

$$f_4\left(y \mid \vartheta_2, \varphi_2\right) = \rho_4 \cdot g_4\left(y \mid \vartheta_2\right) + (1-\rho_4) \cdot h_4\left(y \mid \varphi_2\right) \tag{49}$$

where $\vartheta_2, \varphi_2 > 0,\ y \geq 0,\ 0 \leq \rho_4 \leq 1,$

$$g_4(y \mid \vartheta_2) = \frac{1}{\vartheta_2} exp\left(-\frac{y}{\vartheta_2}\right) \quad \text{and} \quad h_4(y \mid \varphi_2) = \frac{2y}{\varphi_1} exp\left(-\frac{y^2}{\varphi_2}\right). \tag{50}$$

and the CDF of Y is

$$F_4(y \mid \vartheta_2, \varphi_2) = \rho_4 \cdot G_4(y \mid \vartheta_2) + (1 - \rho_4) \cdot H_4(y \mid \varphi_2), \tag{51}$$

$$\text{where} \quad G_4(y \mid \vartheta_2) = 1 - exp\left(-\frac{y}{\vartheta_2}\right) \quad \text{and} \quad H_4(y \mid \varphi_2) = 1 - exp\left(-\frac{y^2}{\varphi_2}\right). \tag{52}$$

4.1. Maximum likelihood estimation of parameters

Let ω_1 and ω_2 represent the normal and accelerated stress levels, respectively. By considering Arrhenius life-stress model, we assume that

$$\vartheta_1 + \varphi_1 = b_0 exp\left(\frac{b_1}{\omega_1}\right), \text{ and } \qquad \vartheta_2 + \varphi_2 = b_0 exp\left(\frac{b_1}{\omega_2}\right). \tag{53}$$

The MLE of the parameters ϑ_1, ϑ_2, φ_1, and φ_2 are to be determined as in Section 3. The conditional likelihood function for observed data under normal stress level ω_1, is derived as in Subsection 3.1 and from equation (15), we get

$$L_N(\vartheta_1, \varphi_1; x) = a_1 \left(\frac{\rho_3}{\vartheta_1}\right)^{\eta_0} \left(\frac{2(1-\rho_3)}{\varphi_1}\right)^{\eta-\eta_0} exp\left(-\sum_{i=1}^{\eta_0} \frac{x_i + x_i r_i}{\vartheta_1} - \sum_{i=\eta_0+1}^{\eta} \frac{x_i^2 + x_i^2 r_i}{\varphi_1}\right) \prod_{i=\eta_0+1}^{\eta} x_i, \tag{54}$$

where η_0 is the number of failed items following exponential (ϑ_1), $\eta - \eta_0$ items follow Rayleigh (φ_1), $a_1 = \prod_{i=1}^{\eta} \sum_{j=i}^{\eta} (r_j + 1)$, and $n(1-\delta) = \eta + \sum_{i=1}^{\eta} r_i$.

Similarly, the conditional likelihood function for observed data obtained under accelerated stress level ω_2 is obtained as in Subsection 3.2 and from equation (22), we get

$$L_A(\vartheta_2, \varphi_2; y) = a_2 \left(\frac{\rho_4}{\vartheta_2}\right)^{\mu_0} \left(\frac{2(1-\rho_4)}{\varphi_2}\right)^{\mu-\mu_0} exp\left(-\sum_{i=1}^{\mu_0} \frac{y_i + y_i s_i}{\vartheta_2} - \sum_{i=\mu_0+1}^{\mu} \frac{y_i^2 + y_i^2 s_i}{\varphi_2}\right) \prod_{i=\mu_0+1}^{\mu} y_i, \tag{55}$$

where μ_0 denotes the number of failed items having exponential life with parameter ϑ_2, while $\mu - \mu_0$ items follow Rayleigh distribution with parameter φ_2, $a_2 = \prod_{i=1}^{\mu} \sum_{j=i}^{\mu} (s_j + 1)$, and $n\delta = \mu + \sum_{i=1}^{\mu} s_i$.

Under PALT, the joint likelihood function is derived using the normal and accelerated stress conditions from equations (54) and (55), we get

$$\begin{aligned} \mathcal{L}(\vartheta_1, \vartheta_2, \varphi_1, \varphi_2; x, y) &= L_N(\vartheta_1, \varphi_1; x) \cdot L_A(\vartheta_2, \varphi_2; y), \\ &= a_1 a_2 \left(\frac{\rho_3}{\vartheta_1}\right)^{\eta_0} \left(\frac{\rho_4}{\vartheta_2}\right)^{\mu_0} \left(\frac{2(1-\rho_3)}{\varphi_1}\right)^{\eta-\eta_0} \left(\frac{2(1-\rho_4)}{\varphi_2}\right)^{\mu-\mu_0} \\ &\quad exp\left(-\sum_{i=1}^{\eta_0} \frac{x_i + x_i r_i}{\vartheta_1} - \sum_{i=\eta_0+1}^{\eta} \frac{x_i^2 + x_i^2 r_i}{\varphi_1}\right) \prod_{i=\eta_0+1}^{\eta} x_i \\ &\quad exp\left(-\sum_{i=1}^{\mu_0} \frac{y_i + y_i s_i}{\vartheta_2} - \sum_{i=\mu_0+1}^{\mu} \frac{y_i^2 + y_i^2 s_i}{\varphi_2}\right) \prod_{i=\mu_0+1}^{\mu} y_i. \end{aligned} \tag{56}$$

Next, by partially differentiating $\ln \mathcal{L}$ with respect to ϑ_1, ϑ_2, φ_1 and φ_2 and equating them to zero, we get the MLEs of the parameters ϑ_1, ϑ_2, φ_1, and φ_2, respectively as

$$\widehat{\vartheta}_1 = \frac{1}{\eta_0} \sum_{i=1}^{\eta_0} X_i \left(1 + r_i\right), \tag{57}$$

$$\widehat{\vartheta}_2 = \frac{1}{\mu_0} \sum_{i=1}^{\mu_0} Y_i \left(1 + s_i\right), \tag{58}$$

$$\widehat{\varphi}_1 = \frac{1}{(\eta - \eta_0)} \sum_{i=\eta_0+1}^{\eta} X_i^2 \left(1 + r_i\right), \tag{59}$$

$$\widehat{\varphi}_2 = \frac{1}{(\mu - \mu_0)} \sum_{i=\mu_0+1}^{\mu} Y_i^2 \left(1 + s_i\right). \tag{60}$$

To develop an ASP using the parameters, ϑ_1, ϑ_2, φ_1, and φ_2, the information about their estimator's distribution is needed. Here under normal stress condition, $X_{(i,\eta,n(1-\delta))}, i = 1, 2, \cdots, \eta_0$ follows an exponential distribution with mean, θ_1 and variance, θ_1^2 while for $i = \eta_0 + 1, \cdots, \eta$, $X_{(i,\eta,n(1-\delta))}$ follows Rayleigh with mean, $\frac{\sqrt{\varphi_1 \pi}}{2}$ and variance, $\varphi_1 \frac{(4-\pi)}{4}$. In accelerated stress condition, $Y_{(i,\mu,n\delta)}, i = 1, 2, \cdots, \mu_0$ follows an exponential distribution with mean, θ_2 and variance, θ_2^2 and for $i = \mu_0 + 1, \cdots, \mu$, $Y_{(i,\mu,n\delta)}$ follows Rayleigh with mean, $\frac{\sqrt{\varphi_2 \pi}}{2}$ and variance, $\varphi_2 \frac{(4-\pi)}{4}$.

Hence, for $i = 1, 2, \cdots, \eta_0$, $A\left(X_i\right) = X_i \left(1 + r_i\right)$ follows exponential distribution with mean, $\vartheta_1 \left(1 + r_i\right)$ and variance, $\vartheta_1^2 \left(1 + r_i\right)^2$. By invoking Lindeberg central limit theorem as in Subsection 3.3, we get, $\sum_{i=1}^{\eta_0} A\left(X_i\right)$ follows normal distribution with mean, $\vartheta_1 \sum_{i=1}^{\eta_0} \left(1 + r_i\right)$ and variance, $\vartheta_1^2 \sum_{i=1}^{\eta_0} \left(1 + r_i\right)^2$. Thus, from equation (57), we get

$$\widehat{\vartheta}_1 = \frac{1}{\eta_0} \sum_{i=1}^{\eta_0} X_i \left(1 + r_i\right) \sim N \left[\frac{\vartheta_1}{\eta_0} \sum_{i=1}^{\eta_0} \left(1 + r_i\right), \ \frac{\vartheta_1^2}{\eta_0^2} \sum_{i=1}^{\eta_0} \left(1 + r_i\right)^2 \right]. \tag{61}$$

In a similar manner, applying the explanation provided in Subsection 3.3 and using Lindeberg central limit theorem in equations (58), (59), and (60), we get

$$\widehat{\vartheta}_2 \sim N \left[\frac{\vartheta_2}{\mu_0} \sum_{i=1}^{\mu_0} \left(1 + s_i\right), \ \frac{\vartheta_2^2}{\mu_0^2} \sum_{i=1}^{\mu_0} \left(1 + s_i\right)^2 \right], \tag{62}$$

$$\widehat{\varphi}_1 \sim N \left[\frac{\varphi_1}{(\eta - \eta_0)} \sum_{i=\eta_0+1}^{\eta} \left(1 + r_i\right), \ \frac{\varphi_1^2}{(\eta - \eta_0)^2} \sum_{i=\eta_0+1}^{\eta} \left(1 + r_i\right)^2 \right], \tag{63}$$

$$\widehat{\varphi}_2 \sim N \left[\frac{\varphi_2}{(\mu - \mu_0)} \sum_{i=\mu_0+1}^{\mu} \left(1 + s_i\right), \ \frac{\varphi_2^2}{(\mu - \mu_0)^2} \sum_{i=\mu_0+1}^{\mu} \left(1 + s_i\right)^2 \right]. \tag{64}$$

Applying the equations (57), (58), (59), and (60), in the Arrhenius life relation, given by equation (53), we get

$$\widehat{\vartheta}_1 + \widehat{\varphi}_1 = \widehat{b}_0 exp\left(\frac{\widehat{b}_1}{\omega_1}\right), \quad \text{and} \quad \widehat{\vartheta}_2 + \widehat{\varphi}_2 = \widehat{b}_0 exp\left(\frac{\widehat{b}_1}{\omega_2}\right).$$

Taking logarithm on both sides gives

$$\ln\left(\widehat{\vartheta}_1 + \widehat{\varphi}_1\right) = \ln \widehat{b}_0 + \frac{\widehat{b}_1}{\omega_1}, \tag{65}$$

$$\ln\left(\widehat{\vartheta}_2 + \widehat{\varphi}_2\right) = \ln \widehat{b}_0 + \frac{\widehat{b}_1}{\omega_2}. \tag{66}$$

Solving the simultaneous equations (65) and (66) for $\widehat{b}_0$ and $\widehat{b}_1$, we get estimators for b_0 and b_1 respectively, as (see Kumar *et al.* 2020)

$$\widehat{b}_0 = \frac{\left(\widehat{\vartheta}_1 + \widehat{\varphi}_1\right)^{\frac{\omega_1}{\omega_1-\omega_2}}}{\left(\widehat{\vartheta}_2 + \widehat{\varphi}_2\right)^{\frac{\omega_2}{\omega_1-\omega_2}}}, \tag{67}$$

$$\widehat{b}_1 = \frac{\omega_1\omega_2}{\omega_2-\omega_1} \ln\left(\frac{\widehat{\vartheta}_1 + \widehat{\varphi}_1}{\widehat{\vartheta}_2 + \widehat{\varphi}_2}\right). \tag{68}$$

4.2. ASP under Arrhenius life-stress relation

In this section, an ASP is designed based on the Arrhenius life-stress relation under PTIICS. The plan is derived using the MLE of the parameters ϑ_1, ϑ_2, φ_1, and φ_2. As in Subsection 3.4, assume that the lifetime of the unit follows the MERD given by equation (1). Based on PALT, suppose ϑ_{1A}, ϑ_{2A}, φ_{1A}, and φ_{2A} denotes the AQL and ϑ_{1R}, ϑ_{2R}, φ_{1R}, and φ_{2R} denote the RQL of an item in the lot. The acceptance or rejection of the lot is contingent upon meeting the following probability conditions:

$$\begin{aligned} &\mathrm{P}\left(\text{Reject the lot} \mid \vartheta_1 \geq \vartheta_{1A},\ \vartheta_2 \geq \vartheta_{2A},\ \varphi_1 \geq \varphi_{1A},\ \text{ and } \varphi_2 \geq \varphi_{2A}\right) \leq \alpha, \\ &\mathrm{P}\left(\text{Accept the lot} \mid \vartheta_1 \leq \vartheta_{1R},\ \vartheta_2 \leq \vartheta_{2R},\ \varphi_1 \leq \varphi_{1R},\ \text{ and } \varphi_2 \leq \varphi_{2R}\right) \leq \beta, \end{aligned} \tag{69}$$

where α is the producer's risk and β is the consumer's risk.

The ASP comprises the following steps:

1. Take a random sample of n units from the lot. Using the PALT method, $n\delta$ units are tested under accelerated stress condition and $n(1-\delta)$ under normal stress condition.
2. Applying PTIICS in both the stress conditions, (subsections 3.1 and 3.2) the testing concludes upon reaching predetermined failure thresholds, denoted as μ and η, out of $n\delta$ and $n(1-\delta)$ units, respectively.
3. Using the observed data, calculate the MLEs $\widehat{\vartheta}_1$, $\widehat{\vartheta}_2$, $\widehat{\varphi}_1$ and $\widehat{\varphi}_2$ of the parameters ϑ_1, ϑ_2, φ_1 and φ_2, respectively.
4. Define, $\widehat{\psi} = \widehat{\vartheta_1} + \widehat{\vartheta_2} + \widehat{\varphi_1} + \widehat{\varphi_2}$. Accept the lot, if $\widehat{\psi} \geq t_2$. Reject the lot, if $\widehat{\psi} < t_1$.
5. If $t_1 \leq \widehat{\psi} < t_2$, continue the process by repeating Steps 1 to 3.

Let p_a, p_r, and p_c denote the probability of acceptance, rejection of the lot, and continuation of the process, respectively. From equations (61), (62), (63), and (64), we get

$$\widehat{\psi} \sim N\left[E\left(\widehat{\vartheta}_1\right) + E\left(\widehat{\vartheta}_2\right) + E\left(\widehat{\varphi}_1\right) + E\left(\widehat{\varphi}_2\right), V\left(\widehat{\vartheta}_1\right) + V\left(\widehat{\vartheta}_2\right) + V\left(\widehat{\varphi}_1\right) + V\left(\widehat{\varphi}_2\right)\right]. \tag{70}$$

Thus, from equation (70), we get the required probabilities are obtained as,

$$p_a = P\left(\widehat{\psi} \geq t_2\right) = P\left[Z \geq \frac{t_2 - E\left(\widehat{\psi}\right)}{\sqrt{V\left(\widehat{\psi}\right)}}\right], \tag{71}$$

$$p_r = P\left(\widehat{\psi} < t_1\right) = P\left[Z < \frac{t_1 - E\left(\widehat{\psi}\right)}{\sqrt{V\left(\widehat{\psi}\right)}}\right], \tag{72}$$

$$p_c = P\left(t_1 \leq \widehat{\psi} < t_2\right) = P\left[\frac{t_1 - E\left(\widehat{\psi}\right)}{\sqrt{V\left(\widehat{\psi}\right)}} \leq Z < \frac{t_2 - E\left(\widehat{\psi}\right)}{\sqrt{V\left(\widehat{\psi}\right)}}\right], \tag{73}$$

As in Subsection 3.4, the main objective of this section is to obtain the optimal values of t_1, t_2, η, μ, $r_i, i = 1, 2, \cdots, \eta$ and $s_i, i = 1, 2, \cdots, \mu$ by minimizing the ETC subject to probability constraints given by equation (69). Here, the total cost is a sum of the cost of testing and the cost of failure items. The cost of testing is obtained as a product of the cost of testing a sample in unit time and the total testing time. But, the total testing time is a random variable with expectation, $\frac{E\left(\widehat{\psi}\right)}{1-p_c}$. This is because, the time to make a decision while testing a sample, $\widehat{\psi}$, and the total number of samples are both random variables with expectation, $E\left(\widehat{\psi}\right)$ and $\frac{1}{1-p_c}$, respectively. Hence, the total number of failed items is again a random variable with expectation $\frac{\eta+\mu}{1-p_c}$. Now, since the total cost is random, we consider the ETC. Thus, from equation (37), we get,

$$\text{ETC} = C_t \frac{E\left(\widehat{\psi}\right)}{1-p_c} + C_f \frac{\eta+\mu}{1-p_c}, \tag{74}$$

where C_t is the cost of testing a unit for unit time and C_f is the cost of a failed item.

The required optimization problem thus minimizes the ETC given by equation (74) at AQL subjected to the probability conditions in equation (69) and is formulated as

$$\begin{aligned} \underset{(t_1,t_2,\eta,\mu,r_i,s_i)}{\text{Min}} \quad & C_t \frac{E\left(\widehat{\psi}\right)}{1-p_c} + C_f \frac{\eta+\mu}{1-p_c} \text{ at AQL} \\ \text{such that} \quad & \left(\frac{p_r}{1-p_c} \middle| \begin{array}{ll} \vartheta_1 \geq \vartheta_{1A}, & \vartheta_2 \geq \vartheta_{2A} \\ \varphi_1 \geq \varphi_{1A}, & \varphi_2 \geq \varphi_{2A} \end{array} \right) \leq \alpha, \\ & \left(\frac{p_a}{1-p_c} \middle| \begin{array}{ll} \vartheta_1 \leq \vartheta_{1R}, & \vartheta_2 \leq \vartheta_{2R} \\ \varphi_1 \leq \varphi_{1R}, & \varphi_2 \leq \varphi_{2R} \end{array} \right) \leq \beta, \end{aligned} \tag{75}$$

where $t_2 > t_1 > 0$, $0 < \eta \leq n(1-\delta)$, $0 < \mu \leq n\delta$, $0 \leq r_i \leq n(1-\delta) - \eta - \sum_{j=1}^{i-1} r_i$, for $i = 1, 2, \cdots, \eta - 1$, $r_0 = s_0 = 0$, $0 \leq s_i \leq n\delta - \mu - \sum_{j=1}^{i-1} s_i$, for $i = 1, 2, \cdots, \mu - 1$, $r_\eta = n(1-\delta) - \eta - \sum_{i=1}^{\eta-1} r_i$, and $s_\mu = n\delta - \mu - \sum_{i=1}^{\mu-1} s_i$.

Substituting the probabilities given in equations (71), (72), and (73) in equation (75) and from equations (61), (62), (63), (64) and (70), one can observe that $E\left(\widehat{\psi}\right)$ and $V\left(\widehat{\psi}\right)$ have terms involving ϑ_1, ϑ_2, φ_1 and φ_2. Given that they are unknown, the optimization problem

is re-formulated as,

$$\underset{(t_1,t_2,\eta,\mu,r_i,s_i)}{\text{Min}} \quad \frac{C_t\ E\left(\widehat{\psi}\right) + C_f\left(\eta+\mu\right)}{1-P\left[\frac{t_1-E\left(\widehat{\psi}\right)}{\sqrt{V\left(\widehat{\psi}\right)}} \le Z < \frac{t_2-E\left(\widehat{\psi}\right)}{\sqrt{V\left(\widehat{\psi}\right)}}\right]} \text{ at AQL}$$

$$\text{such that} \quad \underset{\left\{\begin{array}{ll}\vartheta_1 \ge \vartheta_{1A}, & \vartheta_2 \ge \vartheta_{2A}\\ \varphi_1 \ge \varphi_{1A}, & \varphi_2 \ge \varphi_{2A}\end{array}\right\}}{max} \left(\frac{P\left[Z < \frac{t_1-E\left(\widehat{\psi}\right)}{\sqrt{V\left(\widehat{\psi}\right)}}\right]}{1-P\left[\frac{t_1-E\left(\widehat{\psi}\right)}{\sqrt{V\left(\widehat{\psi}\right)}} \le Z < \frac{t_2-E\left(\widehat{\psi}\right)}{\sqrt{V\left(\widehat{\psi}\right)}}\right]}\right) \le \alpha, \quad (76)$$

$$\underset{\left\{\begin{array}{ll}\vartheta_1 \le \vartheta_{1R}, & \vartheta_2 \le \vartheta_{2R}\\ \varphi_1 \le \varphi_{1R}, & \varphi_2 \le \varphi_{2R}\end{array}\right\}}{max} \left(\frac{P\left[Z \ge \frac{t_2-E\left(\widehat{\psi}\right)}{\sqrt{V\left(\widehat{\psi}\right)}}\right]}{1-P\left[\frac{t_1-E\left(\widehat{\psi}\right)}{\sqrt{V\left(\widehat{\psi}\right)}} \le Z < \frac{t_2-E\left(\widehat{\psi}\right)}{\sqrt{V\left(\widehat{\psi}\right)}}\right]}\right) \le \beta. \quad (77)$$

From equations (57), (58), (59), (60), and (70), one can clearly say that, P_R in equation (76) decreases with respect to ϑ_i and φ_i, $i = 1, 2$ and, in equation (77), P_A increases with respect to ϑ_i and φ_i, $i = 1, 2$. Therefore, P_R attains its maximum at $\vartheta_i = \vartheta_{iA}$, and $\varphi_i = \varphi_{iA}$, $i = 1, 2$. Similarly, maximum of equation (77) is obtained at $\vartheta_i = \vartheta_{iR}$ and $\varphi = \varphi_{iR}$, $i = 1, 2$. Thus, we get the required nonlinear optimization problem, P_{II} as:

$$\underset{(t_1,t_2,\eta,\mu,r_i,s_i)}{\text{Min}} \quad \frac{C_t\ E\left(\widehat{\psi}\right) + C_f\left(\eta+\mu\right)}{1-P\left[\frac{t_1-E\left(\widehat{\psi}\right)}{\sqrt{V\left(\widehat{\psi}\right)}} \le Z < \frac{t_2-E\left(\widehat{\psi}\right)}{\sqrt{V\left(\widehat{\psi}\right)}}\right]} \text{ at AQL}$$

$$\text{subject to} \quad \frac{P\left[Z < \frac{t_1-E\left(\widehat{\psi}\right)}{\sqrt{V\left(\widehat{\psi}\right)}}\right]}{1-P\left[\frac{t_1-E\left(\widehat{\psi}\right)}{\sqrt{V\left(\widehat{\psi}\right)}} \le Z < \frac{t_2-E\left(\widehat{\psi}\right)}{\sqrt{V\left(\widehat{\psi}\right)}}\right]} \le \alpha, \text{ at AQL}, \quad (78)$$

$$\frac{P\left[Z \ge \frac{t_2-E\left(\widehat{\psi}\right)}{\sqrt{V\left(\widehat{\psi}\right)}}\right]}{1-P\left[\frac{t_1-E\left(\widehat{\psi}\right)}{\sqrt{V\left(\widehat{\psi}\right)}} \le Z < \frac{t_2-E\left(\widehat{\psi}\right)}{\sqrt{V\left(\widehat{\psi}\right)}}\right]} \le \beta, \text{ at RQL}, \quad (79)$$

where $t_2 > t_1 > 0$, $0 < \eta \le n(1-\delta)$, $0 < \mu \le n\delta$, $0 \le r_i \le n(1-\delta) - \eta - \sum_{j=1}^{i-1} r_i$,

for $i = 1, 2, \cdots, \eta - 1$, $r_0 = s_0 = 0$, $0 \le s_i \le n\delta - \mu - \sum_{j=1}^{i-1} s_i$, for $i = 1, 2, \cdots, \mu - 1$,

$r_\eta = n(1-\delta) - \eta - \sum_{i=1}^{\eta-1} r_i$, and $s_\mu = n\delta - \mu - \sum_{i=1}^{\mu-1} s_i$.

The GA solver in MATLAB is employed to solve the nonlinear optimization problem, P_{II}. Details of the algorithm used in the GA solver are provided in the Appendix. The obtained optimal values for the plan parameters and their associated ETC are depicted in Table 2 in Section 5.

5. Numerical results and real data application: Comparisons and sensitivity analysis

In this section, some numerical results of the proposed optimal acceptance sampling plans in Sections 3 and 4 are presented in Tables 1 and 2. Specifically, Section 3 focuses on ASP for the MERD under PTIICS-BR utilizing a linear life-stress relation, with the corresponding numerical results for some examples provided in Table 1. For example, from Table 1, consider the following set of parameter values: $C_t = C_f = 1$, $n = 50$, $\vartheta_A = 300$, $\varphi_A = 200$, $\vartheta_R = 60$, $\varphi_R = 60$, $\alpha = \beta = 0.01$, we get the optimum values of parameters as $\eta = 10$, $\mu = 10$, $\{r_i : i = 1, 2, \cdots, \eta\} = \{1, 1, 1, 1, 1, 4, 1, 1, 1, 3\}$, $\{s_i : i = 1, 2, \cdots, \mu\} = \{1, 1, 1, 1, 1, 1, 1, 1, 1, 6\}$, $t_1 = 421.4393$ and $t_2 = 443.7485$ with ETC = 1155.9383 units. Thus, according to the plan, we accept the lot if $\widehat{\psi}$ exceeds 443.7485, and reject the lot if $\widehat{\psi} < 421.4393$.

Table 2 illustrates the ASP explained in Section 4, using Arrhenius life-stress relation for the MERD under PTIICS-BR. ETC is obtained for different values of plan parameters. For a set of values: $C_t = C_f = 1$, $n = 40$, $\vartheta_{1A} = 600$,$\vartheta_{2A} = 600$, $\varphi_{1A} = 500$,$\varphi_{2A} = 800$, $\vartheta_{1R} = 150$,$\vartheta_{2R} = 150$, $\varphi_{1R} = 100$,$\varphi_{2R} = 200$,$\alpha = \beta = 0.01$, we get the optimum values of parameters as $\eta = 9$, $\mu = 7$, $\{r_i : i = 1, 2, \cdots, \eta\} = \{1, 1, 1, 1, 1, 1, 1, 1, 3\}$, $\{s_i : i = 1, 2, \cdots, \mu\} = \{1, 1, 3, 1, 1, 3, 3\}$, $t_1 = 2040.2095$ and $t_2 = 2609.1731$ with ETC = 6437.0946 units. Hence, if $\widehat{\psi} \geq 2609.1731$, we accept the lot and reject the lot, if $\widehat{\psi} < 2040.2095$.

5.1. Comparitative study and sensitivity analysis for linear and Arrhenius life-stress relationship models

A comparative study is done to analyze the ASPs under linear and Arrhenius life-stress models based on the ETC obtained. Some numerical examples are presented in Table 3 by varying α and β and for a fixed set of plan parameters: $C_t = C_f = 1$, $n = 50$, $\vartheta_A = \vartheta_{1A} = \vartheta_{2A} = 1500$, $\varphi_A = \varphi_{1A} = \varphi_{2A} = 1200$, $\vartheta_R = \vartheta_{1R} = \vartheta_{2R} = 300$, $\varphi_R = \varphi_{1R} = \varphi_{2R} = 200$. According to the findings in Table 3, the ETC obtained from the linear life-stress model is lower than that of the Arrhenius model. The term $Red_{\%}$ represents the percentage of cost reduction achieved in the linear model compared to the Arrhenius life-stress model.

Table 3: Comparison of expected total costs obtained in the ASPs based on Arrhenius (Column A) and linear (Column B) life-stress models by varying the values of α and β

α	β	A	B	$Red_{\%}$
0.01	0.01	15727.2376	7463.6589	52.56
0.01	0.05	13475.1429	6328.1726	52.99
0.01	0.1	12020.9999	6125.8824	49.08
0.05	0.05	11671.9999	5926.2632	49.17
0.05	0.1	11356.3333	5651.5714	50.25
0.1	0.1	11077.3333	5533.0909	50.09

Next, we conduct a sensitivity analysis to demonstrate the significance of decision-making. The choice of sample size, denoted as n, holds particular importance when conducting the ASPs using linear and Arrhenius life-stress models. The influence of n on the resulting ETC from both models is depicted in Fig. 2, revealing a consistent increase in ETC as n increases. Fig. 3 illustrates the effect of the producer's risk, α, on ETC by fixing the consumer's risk, β, and varying the values of α. From the graph, we can see that the ETC increases as the value of α decreases. Similarly, Fig. 4 shows the same impact of the consumer's risk, β, on ETC while keeping α constant and varying the values of β. Furthermore, Fig. 5 and Table 3 depicts the combined effect of simultaneously changing α and β on the ETC. Notably, a consistent trend emerges where the ETC rises as both α and β decrease.

Table 1: ASP based on linear life-stress model for $C_t = C_f = 1$, and $\delta = 0.5$

n	$(\vartheta_A, \vartheta_R)$	(φ_A, φ_R)	(α, β)	η	μ	r	s	t_1	t_2	ETC
50	(300,60)	(200,60)	(0.01,0.01)	10	10	1,1,1,1,1,4,1,1,1,3	1,1,1,1,1,1,1,1,1,6	421.4393	443.7485	1155.9383
			(0.01,0.05)	10	10	1,1,1,1,1,3,1,1,1,4	1,1,1,1,1,1,1,1,1,6	357.2334	403.5479	1150.3789
			(0.05,0.05)	10	10	1,1,1,1,1,1,1,1,1,6	1,1,1,1,1,1,1,1,1,6	427.4462	427.8395	1137.6471
40	(1500,500)	(1000,300)	(0.01,0.01)	10	9	1,1,1,1,1,1,1,1,1	1,1,1,1,1,1,1,1,3	2291.2316	2597.4215	5249.3140
			(0.01,0.05)	9	10	1,1,1,1,1,1,1,1,3	1,1,1,1,1,1,1,1,1	2295.3323	2323.6404	5204.2897
			(0.05,0.05)	10	9	1,1,1,1,1,1,1,1,1	1,1,1,1,1,1,1,1,3	2504.1427	2504.7812	5152.3334
30	(500,100)	(1500,400)	(0.01,0.01)	6	7	1,1,1,1,4	1,1,1,1,1,2	1463.4334	1936.2090	4628.4299
			(0.01,0.05)	6	7	4,1,1,1,1	1,1,1,1,1,2	1361.79	1709.3758	4522.1334
			(0.05,0.05)	6	7	1,1,1,1,4	1,1,1,1,1,2	1947.0427	1947.9105	4477.2857

Table 2: ASP based on Arrhenius life-stress model for $C_t = C_f = 1$, and $\delta = 0.5$

n	$(\vartheta_{1A}, \vartheta_{2A}, \vartheta_{1R}, \vartheta_{2R})$	$(\varphi_{1A}, \varphi_{2A}, \varphi_{1R}, \varphi_{2R})$	(α, β)	η	μ	r	s	t_1	t_2	ETC
50	(1200,1000,200,300)	(1500,1500,200,250)	(0.01,0.01)	8	10	1,1,1,1,1,1,1,10	1,6,1,1,1,1,1,1,1,1	3681.5574	3965.8805	13320.8181
			(0.01,0.05)	12	10	1,1,1,1,1,1,1,1,1,1,1,2	1,1,1,1,1,6,1,1,1,1	2841.5450	3164.3081	11176.1766
			(0.05,0.05)	12	12	1,1,1,1,1,1,1,1,1,1,1,2	1,1,1,1,2,1,1,1,1,1,1,1	4189.3906	4189.6367	10664.9091
40	(600,600,150,150)	(500,800,100,200)	(0.01,0.01)	9	7	1,1,1,1,1,1,1,1,3	1,1,3,1,1,3,3	2040.2095	2609.1731	6437.0946
			(0.01,0.05)	8	9	1,1,1,1,1,1,1,5	1,3,1,1,1,1,1,1,1	1741.7991	1988.8539	5550.1557
			(0.05,0.05)	10	9	1,1,1,1,1,1,1,1,1,1	1,1,1,3,1,1,1,1,1	2175.3446	2175.6032	5168.9999
30	(500,800,100,200)	(300,400,50,50)	(0.01,0.01)	6	7	1,1,1,1,1,4	1,1,1,1,1,1,2	1912.9239	2455.4271	4678.2740
			(0.01,0.05)	7	7	1,1,1,1,1,1,2	1,1,1,1,1,1,2	1249.2252	2087.9867	4346.5583
			(0.05,0.05)	7	7	1,1,1,1,1,1,2	1,1,1,1,1,1,2	1780.3044	1870.8123	4239.2577

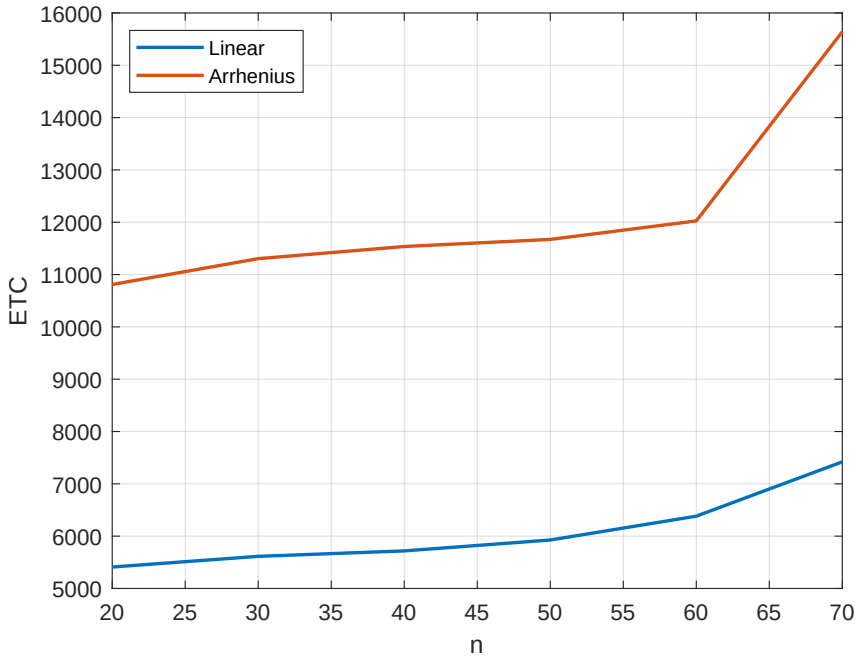

Figure 2: Changes in ETC obtained for linear and Arrhenius model for varying sample size with $C_t = C_f = 1$,$\vartheta_A = \vartheta_{1A} = \vartheta_{2A} = 1500$, $\varphi_A = \varphi_{1A} = \varphi_{2A} = 1200$, $\vartheta_R = \vartheta_{1R} = \vartheta_{2R} = 300$, $\varphi_R = \varphi_{1R} = \varphi_{2R} = 200$, $\alpha = 0.05$, and $\beta = 0.05$

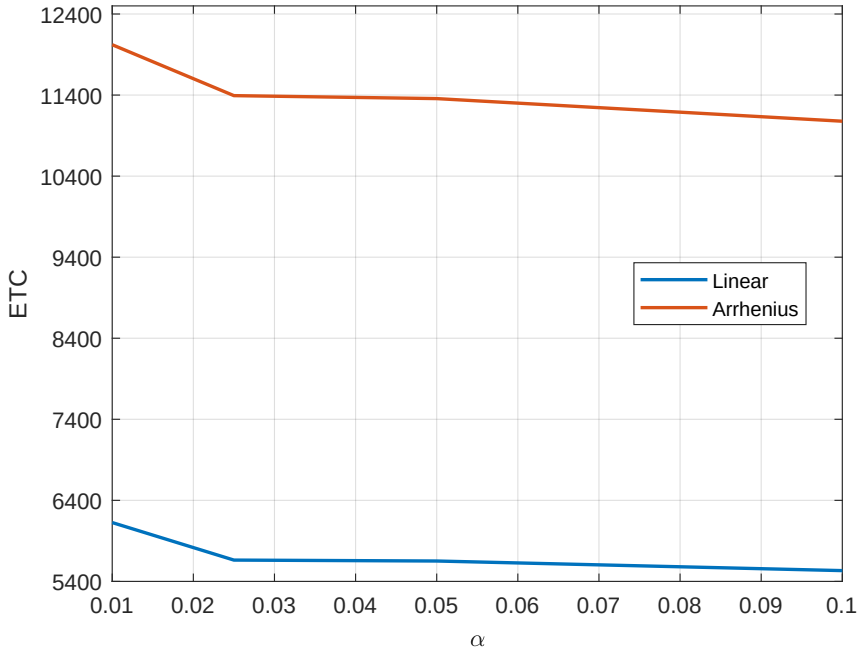

Figure 3: Changes in ETC obtained for linear and Arrhenius model for different values of α with $C_t = C_f = 1$, $n = 50$, $\vartheta_A = \vartheta_{1A} = \vartheta_{2A} = 1500$, $\varphi_A = \varphi_{1A} = \varphi_{2A} = 1200$, $\vartheta_R = \vartheta_{1R} = \vartheta_{2R} = 300$, $\varphi_R = \varphi_{1R} = \varphi_{2R} = 200$, and $\beta = 0.1$

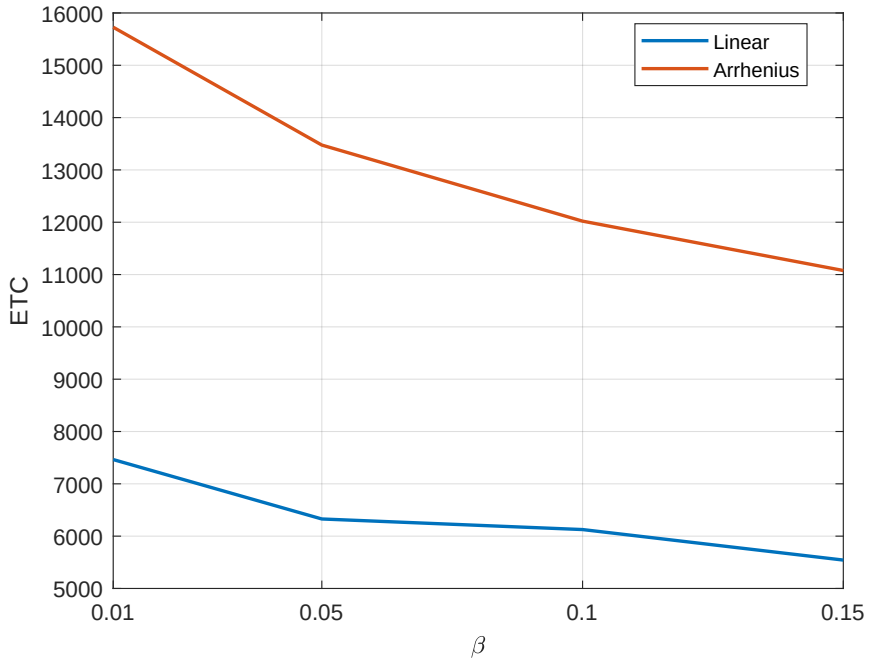

Figure 4: Changes in ETC obtained for linear and Arrhenius model for different values of β with $C_t = C_f = 1$, $n = 50$, $\vartheta_A = \vartheta_{1A} = \vartheta_{2A} = 1500$, $\varphi_A = \varphi_{1A} = \varphi_{2A} = 1200$, $\vartheta_R = \vartheta_{1R} = \vartheta_{2R} = 300$, $\varphi_R = \varphi_{1R} = \varphi_{2R} = 200$, and $\alpha = 0.01$

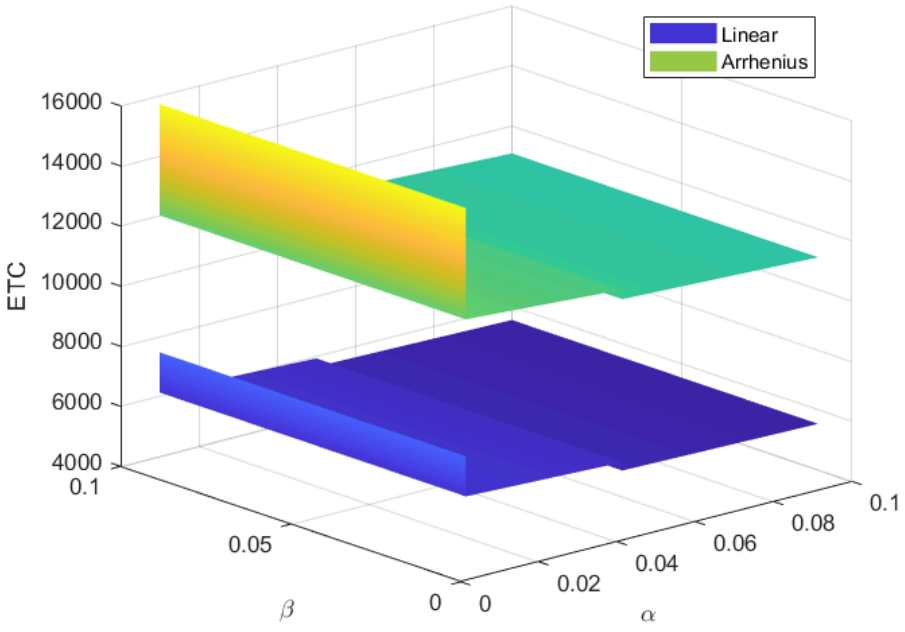

Figure 5: Changes in ETC obtained for linear and Arrhenius model by varying the values of α and β with $C_t = C_f = 1$, $n = 50$, $\vartheta_A = \vartheta_{1A} = \vartheta_{2A} = 1500$, $\varphi_A = \varphi_{1A} = \varphi_{2A} = 1200$, $\vartheta_R = \vartheta_{1R} = \vartheta_{2R} = 300$, and $\varphi_R = \varphi_{1R} = \varphi_{2R} = 200$

5.2. Comparative analysis using real-life data

The proposed MERD can be effectively applied in the testing of electronic components, such as capacitors, where ensuring reliability is crucial for device performance. Capacitors typically fail due to multiple types of mechanisms. For example, two types of failure mechanisms for capacitors may be, electrical overload (modeled by an exponential distribution) and physical degradation caused by long-term wear (modeled by a Rayleigh distribution). To expedite the testing process, an ASP based on PTIICS under PALT can be used, allowing the test to end early if a predetermined number of failures are detected, reducing time and cost. By combining these models, the ASP can determine whether a batch of capacitors meets quality standards based on observed failure data, ensuring high-quality electronic components.

In this section, the proposed ASPs are demonstrated using a real-life case, and a comparative study is done with an existing model in the literature. For that, we consider a dataset used in Ramyamol and Kumar (2019). Their study revolves around the duration until the failure of ARC-1 VHF communication transmitter-receivers in a particular commercial airline. Failed units underwent removal for maintenance, yet in some instances, initially deemed failures functioned adequately upon reaching the maintenance center, their failure status was unverified. To practically address this, it's essential to integrate the information from unconfirmed failures within the overall failure dataset when formulating ASPs for variables. Consequently, it's justifiable to categorize the sample of failures into confirmed and unconfirmed subsets. They considered the two subpopulations to follow exponential distributions. Here we consider the subpopulation of confirmed failures as exponential and unconfirmed failures as Rayleigh data ($\mathring{y}$) derived from the transformation $\mathring{y} = \sqrt{2\mathring{x}}$ of the exponential data ($\mathring{x}$), as explained in Tahir, Aslam, Hussain, and Abbas (2017). This approach is justified since both are lifetime distributions. Furthermore, the Kolmogorov-Smirnov (KS) test is used to confirm the MERD's fit to the considered dataset. For the exponential subpopulation data, the test statistic is 0.1232, which is less than the critical value, $KS_{(33,0.05)} = 0.23$ from the KS table. Similarly, for the Rayleigh data, the test statistic is $0.2 < 0.338$ ($KS_{(15,0.05)}$ table value). Thus, the MERD provides a good fit for the data, with a proportion coefficient of $\rho = 0.3098$. Based on the dataset of size $n = 48$, the maximum likelihood estimates for ϑ and φ are found to be 232.42 and 407.20, respectively. In Table 4, the lifetime of unconfirmed failures is marked with the "$*$" symbol.

Table 4: Time to failure dataset of ARC-1 VHF communication transmitter-receivers of a commercial airline

16	392	368*	408	304	208	344*	256	560*	488	60	32*	360
72	168	200*	144*	96	120	328	80	64*	112*	104*	120*	552
272	304*	152*	112	184	136*	152	224	576	384	16	246*	168
194	216	168*	616	80	208	232	72	56				

A sample of size $n = 48$ with $\rho = 0.3098$ is taken as in Ramyamol and Kumar (2019). ASP under PTIICS using the linear life-stress model is applied to this exponential-Rayleigh data, as its ETC is lower than the Arrhenius model (from Table 3). For the demonstration of the ASP, choose $C_t = C_f = 1$, $\vartheta_A = 336$, $\varphi_A = 234$, $\vartheta_R = \varphi_R = 100$, $\zeta = 2$, $\delta = 0.5$, $\alpha = 0.05$, and $\beta = 0.1$. By solving the nonlinear optimization problem P_I, we get $\eta = 11$, $\mu = 11$, $\{r_i : i = 1, 2, \cdots, \eta\} = \{1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 3\}$, $\{s_i : i = 1, 2, \cdots, \mu\} = \{1, 1, 1, 1, 1, 1, 2, 1, 1, 1, 2\}$, $t_1 = 578.4570$ and $t_2 = 578.8613$.

Thus, among 48 samples in Table 4, the first 24 samples are distributed to accelerated testing, and the other 24 samples are tested in normal conditions. Next, in each condition, testing terminates after observing 11 failures. Next, according to PTIICS-BR surviving items are removed after each failure. Then from the observed data, we get $\eta_0 = \mu_0 = 7$, $\widehat{\vartheta} = 500$, and

$\widehat{\varphi} = 715.9963$. Hence, we accept the lot with ETC = 1214 units since $\widehat{\vartheta}+\widehat{\varphi} > t_2$. Meanwhile, in Ramyamol and Kumar (2019), they accept the lot with a lower ETC for exponential mixture under Type-I censoring, even though testing is stopped at time, $T = 2000$ units. Their ETC hinges primarily on the cost of failed items alone. Consequently, our approach appears more realistic and practically applicable when considering mixture distributions than the existing plans.

5.3. Application to breast cancer data

In this subsection, we demonstrate the practical application of our proposed model through the analysis of survival times (in months) among 45 breast cancer patients who underwent simple or radical mastectomy for breast cancer tumor treatment. The dataset is sourced from Collett (2023) and comprises survival times of women who died because of cancer and due to diseases other than breast cancer. Thus, it is a mixture data and we consider the lifetime subpopulation of women who died from cancer as exponentially distributed and died from other diseases as Rayleigh distributed. Moreover, after the surgery, a section of the tumor is treated with a marker called Helix pomatia agglutinin (HPA), and each tumor undergoes subsequent classification based on staining, where a positive stain indicates the presence of potential metastasis within the tumor. For our model application, we consider the patients with positively stained tumors as samples in accelerated condition. Table 5 presents the dataset of women who underwent surgical treatment with categorization based on staining and uses "*" to represent the survival time of patients who died due to diseases other than breast cancer.

Table 5: Survival lifetime of breast cancer patients

	5	8	10	13	18	24	26	26	31
Positive staining	35	40	41	48	50	59	61	68	71
(*Accelerated condition*)	76*	105*	107*	109*	113	116*	118	143	154*
	162*	188*	212*	217*	225*				
Negative staining	23	47	69	70*	71*	100*	101*	148	181
(*Normal condition*)	198*	208*	212*	224*					

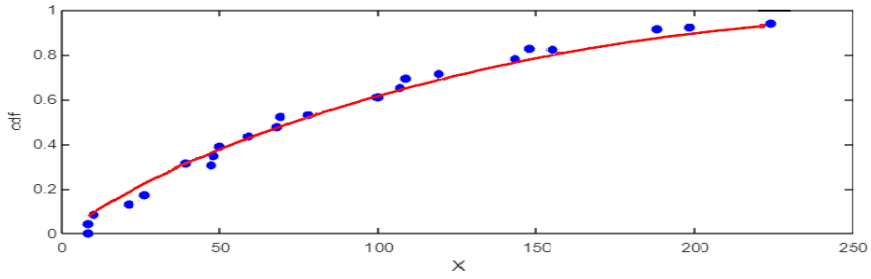

Figure 6: CDF plots for the breast cancer data

The fitting of the MERD to the given data set is verified using the KS test and the CDF plot given in Fig. 6. In the case of exponential subpopulation data, the value of test statistics is $0.1408 < 0.27$ ($KS_{(26,0.05)}$ table value), while for Rayleigh data, the test statistics value is computed as $0.1617 < 0.301$ ($KS_{(19,0.05)}$ table value). Therefore, the MERD offers a satisfactory fit for the considered data, with a proportion coefficient of $\rho = 0.5778$. From the dataset of size $n = 45$, the maximum likelihood estimates of ϑ and φ are obtained as 56.7692 and 25519.95, respectively.

ASP under PTIICS using linear life-stress relation with $\zeta = 2$, and $\delta = 0.711$ is considered

here as it offers a lower ETC than the Arrhenius model. To illustrate the proposed model, choose $C_t = C_f = 1$, $\vartheta_A = 57$, $\varphi_A = 25520$, $\vartheta_R = 30$, $\varphi_R = 5000$, $\alpha = \beta = 0.05$. Note that these values are user-defined and their choice plays a vital role in ASP. By solving the nonlinear optimization problem P_I, we get the optimal values as $\eta = 6$, $\mu = 7$, $\{r_i : i = 1, 2, \cdots, \eta\} = \{2, 1, 1, 1, 1, 1\}$, $\{s_i : i = 1, 2, \cdots, \mu\} = \{1, 1, 1, 1, 1, 1, 19\}$, $t_1 = 8470.7544$, and $t_2 = 10653.7755$. Based on the ASP, 32 out of 45 samples are tested under accelerated conditions. In normal testing conditions, the test was concluded after the occurrence of the 6th failure. While under accelerated conditions, testing ceased after observing 7 failures. Following the PTIICS methodology, surviving items were removed after each failure, and from the observed data, we get $\eta_0 = 2$, $\mu_0 = 7$, $\widehat{\vartheta} = 176.1111$, and $\widehat{\varphi} = 49524$.Therefore, the decision is to accept the lot with an ETC of 68696.3462 units. This choice is made based on the ASP condition $\widehat{\vartheta} + \widehat{\varphi} > t_2$. Thus, we can conclude that surgical treatment can be accepted for breast cancer patients.

6. Conclusion

While mixture distributions have wide application in everyday life, the literature lacks extensive application regarding acceptance sampling plans associated with these distributions. In this work, ASPs for the MERD under PTIICS-BR are designed under PALT. Two different life-stress models, namely, linear and Arrhenius, are considered. For each life-stress model, the MLE of the unknown parameters of the mixture distribution is obtained and is used for deriving the ASPs. In both cases, optimal values of the plan parameters are obtained by solving a nonlinear optimization problem of minimizing the ETC at given producer's and consumer's risks. Here we have considered ETC since the total testing cost is random. Also, note that the actual cost may be less than the ETC obtained here. Tables 1 and 2 present the optimal ASP using these models for various examples. A comparative analysis of these models is done based on the ETC and is illustrated in Table 3. The ETC derived from the linear life-stress model proves to be lower than that of the Arrhenius model. This observation emphasizes the cost-effectiveness of the linear model in comparison to the Arrhenius model. The sensitivity analysis conducted on both models demonstrates that as the sample size increases, the ETC also increases. Additionally, it is observed that decreasing values of both producer's and consumer's risk result in an increased ETC. In addition, to demonstrate the practical application, the ASP under the linear life-stress relation is applied to two different real-life data sets. The first set includes cancer patients' lifetime data, while the second set contains failure data from communication transmitter-receivers of a single commercial airline. To further improve the efficiency of reliability testing, future research could extend the proposed ASP by considering step-stress accelerated life testing to obtain data. Secondly, an adaptive progressive Type-II censoring scheme may be considered as a future scope for research in modeling clinical research data.

Appendix

The following steps present the algorithm used in the GA solver available in a widely used public domain software, MATLAB.

Step 1. Initialization:

-Randomly generate an initial population of candidate solutions.

- Evaluate the fitness of each individual based on the objective function.

Step 2. Selection:

-Select individuals from the population based on their fitness (e.g., tournament selection, roulette wheel).

-Favor individuals with better fitness for reproduction.

Step 3. Crossover (Recombination):

-Create new offspring by combining pairs of selected individuals (parents).

-Apply crossover with a certain probability to exchange information between parents.

Step 4. Mutation:

-Introduce random mutations in the offspring with a certain probability.

-This ensures diversity and prevents premature convergence to local optima.

Step 5. Fitness Evaluation:

-Evaluate the fitness of the new offspring.

-Replace the least fit individuals in the population with the new offspring.

Step 6. Termination:

- Repeat the process (selection, crossover, mutation) for a fixed number of generations or until convergence criteria (e.g., no significant change in fitness) are met.

Solution Existence:
The GA solver provides a solution when the population converges, and the best individual satisfies the problem's constraints.

This algorithm summarizes how GA works in MATLAB, addressing the existence of a solution through convergence and fitness evaluation. The GA solver used to solve the nonlinear optimization problems in this paper provides the best solution satisfying all the constraints mentioned in the problems.

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