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Review on Distributions Generated Using Trigonometric Functions

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Abstract

In the present scenario, researchers are interested in developing new distributions based on trigonometric functions to model data. Previously, data modelling was commonly done with standard and existing distribution functions. Recent research has demonstrated that trigonometrically transformed functions can be used to model a variety of data types, including data from the medical, psychiatric, industrial, and economic sectors as well as studies on life time data, observations on seasonal patterns, and data from other fields. This review article discusses some trigonometric distributions and their applications across a range of fields. Using an actual dataset, a comparison analysis of a few selected models is conducted. Maximum likelihood estimates of the model parameters and various information criteria measures are obtained, and goodness of fit tests are performed for comparison.

Keywords: distributions, trigonometric transformations, modelling.

1. Introduction

Modern world is highly concerned with data and modelling. Researchers from every field of study come up with some data and find some distribution models that perfectly fit the data. If such a suitable model is not obtained from the existing distributions, then it becomes necessary to create new distribution models. Majority of the new distributions are produced by generalizing or extending the standard baseline distribution. Sometimes one additional parameter is added to make the model suitable for the data under study. This will make the function more complicated and estimation of parameters will become more challenging. Recently, it has been noticed that transformations based on trigonometric functions simplifies such problems as it is a flexible approach to develop new distributions. There were only very few, which were derived before the twenty-first century for analyzing circular and directional data. Subsequently many such families of distributions emerged after the introduction of SS transformation by Kumar, Singh, and Singh (2015). Some among those families and their particular members are discussed in Tomy and Satish (2021) and are as follows: sine square distribution by Al-Faris and Khan (2008), sine generated (Sin-G) family by Kumar *et al.* (2015) and Souza, Junior, De Brito, Chesneau, Ferreira, and Soares (2019a), SS transformed

Lindley distribution ($\mathrm{SS_L}(\theta)$ by Kumar, Singh, Singh, and Chaurasia (2018), new exponential with trigonometric function (NET) by Bakouch, Chesneau, and Leao (2018), tan generated (Tan-G) family by Souza, O Júnior, Brito, Chesneau, Fernandes, and Ferreira (2021), cosine generated (Cos-G) family by Souza, Junior, de Brito, Ferreira, Soares *et al.* (2019b), a new family of distributions using cosine-sine (CS) transformation by Chesneau, Bakouch, and Hussain (2018), polyno-expo-trigonometric family by Jamal and Chesneau (2019), sine Topp-Leone generated (STL-G) family by Al-Babtain, Elbatal, Chesneau, and Elgarhy (2020), cosine geometric distribution (CGD) by Chesneau, Bakouch, Hussain, and Para (2020), sinh inverted exponential by Hemeda, Abdallah *et al.* (2020), sin power Lomax (SPL) family by Nagarjuna, Vardhan, and Chesneau (2021), sine Kumaraswamy generated family by Chesneau and Jamal (2021), transformed sin generated (TS-G) family by Jamal, Chesneau, Bouali, and Ul Hassan (2021b). The authors showed that, this approach can be utilised to model a variety of data types, including those related to medicine, engineering, economics, life span, psychiatry, survival etc. Also compared to some of the more well-known distributions, these new distributions offered a superior fit to the relevant datasets. One another benefit of this innovative strategy is that it produces new, effective distributions that are flexible enough to fit the data under study. This review article will describe some of the most recent members of the trigonometric families of distributions that are known to exist in the statistical literature.

In Section 2 titled 'Trigonometric distributions in former times' we will discuss the initial developments based on trigonometric functions. In Section 3, 'Recent advancements in trigonometrically transformed distributions', we will see some of the latest developments in this field by the emerging researchers. The authors used statistical test criterions such as Akaike information criterion (AIC), Bayesian information criterion(BIC), consistent Akaike information criterion($CAIC$), Harley-Quinn information criterion ($HQIC$), Cramer-von Mises (A^*), Anderson-Darling (W^*), Kolmogorov-Smirnov (KS) and log-likelihood ($\hat{l}$) statistics to assess the capability of the generated distributions with respect to the selected alternatives. The model with the least values of AIC, $CAIC$, BIC, KS, A^* and W^*, and greatest value for KS p-value, fit the data better than the others. Correspondingly, a brief description of the applications of each distributions is presented. Section 4 presents a comparative study on some of the selected models from section 3 using a real dataset. The review ends with the conclusion in Section 4.

2. Trigonometric distributions in former times

- von Mises (1918) introduced the von Mises distribution ($M(\mu,\kappa)$) to analyse circular and directional data. It is symmetric and unimodal, considered to serve as the circular analogue of the Normal distribution. It has the following probability density function (pdf):

  $$f(\theta) = \frac{1}{2\pi I_0(\kappa)} \exp\left(\kappa \cos(\theta-\mu)\right); \quad 0 < \theta \leq 2\pi, 0 \leq \mu < 2\pi.$$

  where $I_0(\kappa) = \sum_{r=0}^{\infty} \frac{1}{(r!)^2}\left(\frac{\kappa}{2}\right)^{2r}$ is known as the modified Bessel function of the first kind and order zero. $\kappa \geq 0$ is the concentration parameter and μ is called the mean direction.

  The von Mises distribution can be approximated by cardioid distribution, wrapped normal distribution and wrapped Cauchy distribution.
  This distribution was originally derived to investigate the deviations of measured atomic weights of elements from integral values (the fractional parts of the atomic weights were converted to angles). However from the perspective of statistical inference, it is seen that the von Mises distribution is the most useful distribution on the circle (see Mardia, Jupp, and Mardia (2000)).

- If the Normal distribution is used for the observations on the line, von Mises distribution is to model directions on the plane (circle). Similarly Fisher (1953) derived the Fisher Distribution to model directions on the space (sphere). It is a generalisation of the von Mises distribution where the directions are distributed unimodally and with rotational symmetry. The *pdf* of Fisher distribution in terms of spherical polar coordinates is given by,

  $$f(\theta,\phi)=\frac{\kappa}{4\pi\sinh\kappa}\exp\{\kappa\left[\cos\theta\cos\alpha+\sin\theta\sin\alpha\cos(\phi-\beta)\right]\}\sin\theta$$

  for $0\leq\theta\leq\pi; 0\leq\phi\leq 2\pi$.
  Here θ is the angular displacement from the true position and κ is the measure of precision.

  The author developed the theory based on the measurements of direction of remanent magnetisation in the directly and inversely magnetised lava flows in Iceland, which was recorded by a geophysicist, J. Hospers.
  The theory seems to be relevant to measurements of position on a sphere.

- Raab and Green (1961) introduced a trigonometric approximation to the Normal distribution with cosine function.

  The *pdf* is given by,

  $$f(x)=\frac{1}{2\pi}(1+\cos x);\quad -\pi<x<\pi$$

  This distribution is symmetric and bell shaped with unit area. The cosine approximation has the benefit of simplifying the evaluation of the expressions of normal distribution algebraically.
  The authors developed this distribution as a useful tool for psychological research.

- Nadarajah and Kotz (2006) introduced six new beta trigonometric (BT) distributions - three of which involve the cosine function and other three are complementary distributions involving sine function – as a generalization of beta distribution. The *pdfs* of BT distributions are as follows :

  1. BT Cos I distribution

     $$f(x)=C_1\ x^{\nu-1}(1-x)^{\mu-1}\cos(ax);\ 0<x<1,\ \nu>0,\ \mu>0,\ 0\leq a<\frac{\pi}{2}$$

  2. BT Sin I distribution

     $$f(x)=C_2\ x^{\nu-1}(1-x)^{\mu-1}\sin(ax);\ 0<x<1,\ \nu>-1(\nu\neq 0),\ \mu>0,\ 0<a<\pi$$

  3. BT Cos II distribution

     $$f(x)=C_3\ x^{2\nu-1}(1-x^2)^{\mu-1}\cos(ax);\ 0<x<1,\ \nu>0,\ \mu>0, 0\leq a<\frac{\pi}{2}$$

  4. BT Sin II distribution

     $$f(x)=C_4\ x^{2\nu-1}(1-x^2)^{\mu-1}\sin(ax);\ 0<x<1,\ \nu>\frac{-1}{2},\ \mu>0,\ 0<a<\pi$$

  5. BT Cos III distribution

     $$f(x)=C_5\ x^{\nu-1}\exp(-\beta x)\cos(ax);\ 0<x<1,\ \nu>0,\ \ -\infty<\beta<\infty,\ 0\leq a<\frac{\pi}{2}$$

  6. BT Sin III distribution

     $$f(x)=C_6\ x^{\nu-1}\exp(-\beta x)\cos(ax);\ 0<x<1,\ \nu>-1,\ \ -\infty<\beta<\infty,\ 0<a<\pi$$

Here $C_1,\ C_2,\ C_3,\ C_4,\ C_5$ and C_6 are constants in terms of the respective parameters. The authors derived various properties of BT distributions including their moments, method of moment estimators, maximum likelihood estimators and the Fisher information matrices. The authors also discussed the possible extensions of the six BT distributions over any finite interval instead of just considering the unit interval.
With a data on consumer expenditure, the authors showed that these distributions better fit for economic data than standard beta distribution.

3. Recent advancements in trigonometrically transformed distributions

3.1. Distributions based on sine function

- Almetwally and Meraou (2022) used a combination of sin-G family and NH distribution to produce the sin-NH distribution which have the following *cdf* and *pdf* :

$$F(x) = \sin\left[\frac{\pi}{2}\left(1-\exp\left(1-(1+\lambda x)^{\alpha}\right)\right)\right];\quad x>0,\alpha,\lambda>0 \quad \text{and}$$

$$f(x) = \frac{\pi}{2}\alpha\lambda(1+\lambda x)^{\alpha-1}\exp\left(1-(1+\lambda x)^{\alpha}\right)\cos\left[\frac{\pi}{2}\left(1-\exp\left(1-(1+\lambda x)^{\alpha}\right)\right)\right]$$
$$;\quad x>0,\alpha,\lambda>0$$

The shapes of *pdf*s and hrfs are plotted of different values of the parameters. The parameter estimates are obtained using maximum likelihood and maximum product spacing methods which, the latter produced better results. A simulation study is performed to compare the values of the parameter estimates. The authors applied this model to an environmental data.

Application

The authors found this distribution best suited for decreasing *pdf* types. Thus a data set is taken having 68 values measured using line-transect distance sampling method for estimating population densities of wild animals on a 1km distance which is 20m wide. Standard errors, AIC, BIC, A^*, W^* and KS values of this distribution are compared with many alternatives and found that sin-NH distribution well fits the data.

- Tomy, G., and Chesneau (2021) created a one parameter survival distribution called the sine modified Lindley distribution (S-ML) which has

the *cdf* and *pdf* respectively given by,

$$F_{S-ML}(x;\theta) = \cos\left[\frac{\pi}{2}\left(1+\exp\left(-\theta x\right)\frac{x\theta}{1+\theta}\right)\exp\left(\theta x\right)\right];x>0 \quad \text{and}$$

$$f_{S-ML}(x;\theta) = \frac{\pi}{2}\frac{x\theta}{1+\theta}\exp\left(-2\theta x\right)\left[(1+\theta)\exp\left(\theta x\right)+2x\theta-1\right]$$
$$\sin\left[\frac{\pi}{2}\left(1+\exp\left(-\theta x\right)\frac{x\theta}{1+\theta}\right)\exp\left(-\theta x\right)\right];x>0,\theta>0$$

The authors studied the functionality, reliability and moments of the distribution. Maximum likelihood estimation method is used to estimate the parameter θ.

Application

This distribution can be used to model lifetime data which has right-skewed and leptokurtic nature and 'increasing-reverse bathtub-constant' hrf. Simulation studies and applications demonstrate the utility of the model under consideration. The authors analysed a right-skewed and leptokurtic data of precipitation (in inches) during one month to illustrate the functionality of the model. Another dataset containing the time between failures for repairable items was also considered. With the datasets, the S-ML distribution is compared with some of the well known alternatives. The goodness of fit statistics like AIC, BIC, A^*, W^* and KS values and corresponding p-values were computed and the authors concluded that the new distribution outperforms the alternatives and provides best fit for the data under consideration. The authors extended the studies on its applications in Tomy, G., and Chesneau (2022) using real-world data sets representing the life periods of guinea pigs given different doses of tubercle bacilli, estimated time from growth hormone treatment for children until a target age, and the size of tumors in cancer patients of different stages. For these datasets also the S-ML distribution provides best statistical results compared to the well known alternatives. Thus the model is assumed to be effective in modeling the survival times of diseases related to cancer.

- Muhammad, Alshanbari, Alanzi, Liu, Sami, Chesneau, and Jamal (2021a) proposed a new family of distributions called the exponentiated sine-G family and the study emphasised on a particular member of the family called exponentiated sine Weibull (ESW) distribution.

The *cdf* and *pdf* of exponentiated sine-G family (ESG) are given by,

$$
\begin{aligned}
F(x) &= \left[\sin\left(\frac{\pi}{2}G(x)\right)\right]^{\alpha}; \quad \alpha > 0 \\
f(x) &= \frac{\alpha\pi}{2} g(x) \cos\left(\frac{\pi}{2}G(x)\right)\left[\sin\left(\frac{\pi}{2}G(x)\right)\right]^{\alpha-1}
\end{aligned}
$$

The authors derived all the important mathematical and statistical properties of the distribution specially stress-strength reliability and Rényi entropy. Stress-strength reliability parameters have application in engineering, biology, and finance. Model parameters are estimated using both maximum likelihood and Bayes estimation methods.

Application

The authors found that if the failure rate has monotonically increasing or decreasing or bathtub or upside-down bathtub shapes, the ESW model is useful. Three data sets are used to study the applicability of the distribution. The first data set consists of the total milk output from the first birth of 107 SINDI race cows which showed an increasing hrf. The second is the famous bladder cancer patients dataset which showed an upside-down bathtub hrf. The third and fourth datasets respectively represent strength data for single fibres tested under tension at guage length of 10mm and impregnated tows of 1000 fibres tested at guage length of 20mm. By the values of $\hat{l}$, AIC, $CAIC$, BIC, A^*, W^* and KS statistics, the authors proved that the ESW distribution provides good fit to the datasets. Thus all these datasets are found to be better modelled with new ESW distribution than some of the well known alternative models.

- Jamal, Chesneau, and Aidi (2021a) introduced the sine extended odd Fréchet-G (SEOF-G) family of distributions, as a combination of the sine-G and the extended odd Fréchet-G families. The *cdf* and *pdf* of SEOF-G family are respectively,

$$F(x) = \sin\left[\frac{\pi}{2}\exp\left(-\left(\frac{1-G(x)^{\alpha}}{G(x)^{\alpha}}\right)^{\theta}\right)\right]; x \in \mathbb{R} \quad \text{and}$$

$$f(x) = \frac{\pi}{2}\alpha\theta g(x)\frac{[1-G(x)^{\alpha}]^{\theta-1}}{G(x)^{\alpha\theta+1}}\exp\left(-\left(\frac{1-G(x)^{\alpha}}{G(x)^{\alpha}}\right)^{\theta}\right)$$
$$\cos\left[\frac{\pi}{2}\exp\left(-\left(\frac{1-G(x)^{\alpha}}{G(x)^{\alpha}}\right)^{\theta}\right)\right]; x \in \mathbb{R}$$

As a particular case, a novel member of this family is derived with the Nadarajah-Haghighi (NH) distribution as a baseline, called the sine extended odd Fréchet Nadarajah-Haghighi (SEOFNH) distribution. The authors studied the asymptotic properties, quantile functions, moments and linear representations of the family. Maximum likelihood estimation of parameters and simulation studies are carried out for complete as well as right censored data.

Application

This distribution can be applied to highly right skewed and leptokurtic data whose hazard rate function (hrf) has bathtub, upside down bathtub, decreasing and increasing shapes. To assess the functionality of this model, two complete datasets and one right censored data set are used. 100 values of annual maximum temperature (inches) for one rain gauge in Colorado is taken as the first data set and 59 observations on monthly actual taxes revenue (in 1000 million Egyptian pounds) in Egypt is taken as the second data set. The right censored data contained the survival times of 26 psychiatric inpatients admitted to the hospitals of Iowa City. With the help of goodness of fit statistics like AIC, BIC, A^*, W^* and KS and p-values computed from these datasets, the authors were able to prove that the new model provides a satisfactory alternative to the extended odd Fréchet Nadarajah-Haghighi model and some other alternatives.

- Mahmood, Chesneau, and Tahir (2019) introduced a new sine generated family of distributions (N-sine-G) with the same number of parameters as the Sin-G family.

The *cdf* and corresponding *pdf* of the N-sine-G family is given as

$$F(x) = \sin\left(\frac{\pi}{4}G(x)\left(G(x)+1\right)\right); \quad x \in \mathbb{R}$$
$$f(x) = \frac{\pi}{4}g(x)[2G(x)+1]\cos\left(\frac{\pi}{4}G(x)(G(x)+1)\right); \quad x \in \mathbb{R}$$

where G(x) is the *cdf* of the baseline distribution.
The N-sine-G family connects the sine-G family with the transmuted G family by Shaw and Buckley (2009). The authors presented a special member of the family named N-sine-IW distribution using the *cdf* of inverse Weibull distribution as the baseline G(x).

Application

The authors found this distribution as a useful model for real life data with heavy right tail. It is also found that the new distribution has decreasing and upside-down bathtub hazard rates. Various structural properties like asymptotes and shapes, quantile functions, moments and moment generating functions are investigated. Maximum likelihood estimates of the parameters were obtained which are found to be asymptotically unbiased and normal. Two practical datasets are used to analyse the capability of the proposed model. 72 observations on survival times in days of guinea pigs that willingly received various tubercle bacilli dosages are represented in the first data set whereas 23 values of million revolutions before a ball bearing fails are included in the second data set. The N-sine-IW model is compared against some strong competing models including the sin-IW model. Several goodness-of-fit statistics such as $\hat{l}$, AIC, A^*, W^* and KS values showed that the suggested model produces superior fits than the alternatives. The newly derived family is believed to be applicable in areas such as reliability, engineering, hydrology, economics and survival analysis.

3.2. Distributions based on cosine function

- Mahmood, M Jawa, Sayed-Ahmed, Khalil, Muse, Tolba *et al.* (2022) introduced the Extended Cosine Generalized Family of Distributions as an extension to Cos-G class of distributions by utilizing parameter induction method.

  The *cdf* and *pdf* are, respectively,

$$F(x) = \left[1 - \cos\left(\frac{\pi}{2} G(x)^{\alpha}\right)\right]^{\beta} ; x \in \mathbb{R}$$

$$f(x) = \frac{\pi}{2}\alpha\beta g(x) G(x)^{\alpha-1} \sin\left(\frac{\pi}{2} G(x)^{\alpha}\right) \left[1 - \cos\left(\frac{\pi}{2} G(x)^{\alpha}\right)\right]^{\beta-1}$$

  Considering Weibull distribution as G(x) the authors developed the extended Cosine Weibull (ECW) distribution which is a four parameter model. This new model is derived with symmetrical, right-skewed, left-skewed, reversed-J, and bimodal reversed-J density shapes and increasing, decreasing, bathtub, and upsidedown bathtub hazard rate shapes. Several mathematical, numerical, and structural properties such as linear representation for cdf and pdf, moments and weighted moments, quantile function, order statistics, entropies, and stochastic ordering are also derived. The parametric estimation is performed using the method of maximum likelihood.

  Application

  To demonstrate the applicability of the model three datasets were taken. The first dataset comprised of the survival times of 121 breast cancer patients during a specific period, the second dataset consists of 60 observations of number of 1000s of cycles to failure for electrical appliances in a life test and third dataset represents the failure time (in weeks) of 50 industrial components that are used concurrently. The proposes model outperforms the cosine Weibull model, Weibull model and several other competing models according to the values of $\hat{l}$, A^*, W^* and KS statistics. Hence the authors found that this distribution is best suited to survival and failure datasets.

- Chesneau, Tomy, and Gillariose (2021) introduced a new distribution based on inverse cosine and power function. The *cdf* and *pdf* are:

$$F(x) = x^{\alpha} \arccos(x^{\alpha}) + 1 - \sqrt{1 - x^{2\alpha}} \ ; \quad x \in (0,1) \quad \textit{and}$$

$$f(x) = \alpha x^{\alpha-1} \arccos(x^{\alpha}) \ ; \quad x \in (0,1)$$

  The authors studied the mathematical and statistical properties of the distribution, the asymptotes and shape properties of the *pdf* and hrf and therefore noticed that this distribution is best suited for decreasing and left skewed data types and with increasing and bathtub shaped hrf.

  Application

  The authors applied this model to two real datasets where the first comprised 28 observations on times to infection of kidney dialysis patients in months and the second comprised of 30 observations on failure times of the air conditioning system of an airplane in hours. With the values of the estimates, $\hat{l}, AIC$, BIC, A^*, W^* and KS statistics, the model is proved to be better than power model for analysing decreasing and left skewed data sets.

- Muhammad, Bantan, Liu, Chesneau, Tahir, Jamal, and Elgarhy (2021b) introduced New Extended Cosine-G (ECSG) family of distributions as an extension to Sin-G family by an addition of one parameter to the distribution. Let $s(x)$ be the survival function (sf) related to a baseline *cdf* of a continuous distribution, denoted by $G(x)$. That is, $s(x) = 1 - G(x)$. Let $g(x)$ be the *pdf* corresponding to $G(x)$. Then, the *cdf* of the ECSG family with parameter $\alpha > 0$ and the corresponding *pdf* are given as:

$$F(x) = 1 - \left[1 - \cos\left(\frac{\pi}{2}s(x)\right)\right]^{\alpha}$$

$$f(x) = \alpha\frac{\pi}{2}g(x)\sin\left(\frac{\pi}{2}s(x)\right)\left[1 - \cos\left(\frac{\pi}{2}s(x)\right)\right]^{\alpha-1}$$

Both functions characterise the entire ECSG family in the probabilistic sense. As

$$\cos\left(\frac{\pi}{2}(1-x)\right) = \sin\left(\frac{\pi}{2}x\right),$$

we have,

$$\cos\left(\frac{\pi}{2}s(x)\right) = \sin\left(\frac{\pi}{2}G(x)\right)$$

which corresponds to the *cdf* of the Sin-G family. Some important mathematical and statistical properties were derived, such as asymptotic results, quantile function, the series representation of the cumulative distribution and probability density functions, moments, moments of residual life, reliability parameter, and order statistics and some special members of the ECSG family namely, the extended cosine Weibull (ECSW), extended cosine power (ECSP), and extended cosine generalized half-logistic (ECSGHL) distributions were derived and discussed. The parameters are estimated using the maximum likelihood estimation, least-square estimation, percentile estimation, and Bayes estimation under the square error loss function. A simulation was considered to study the Bayes estimates of the stress–strength parameter.

Application

The authors observed that the ECSW model can be used for the analysis of lifetime data with decreasing or right-skewed unimodal structures, the ECSGHL distribution can be used to model lifetime data with decreasing and right-skewed unimodal structures, with various symmetric and platykurtic features whereas the ECSP model is highly flexible to model increasing, decreasing, unimodal left-skewed, unimodal right-skewed, and bathtub shaped data. Two real datasets consisted of strength data for single carbon fibres tested under tension at 20mm and 10 mm guage lengths expressed in GPA were taken to model the ECSW distribution. Hundred observations on the breaking stress of carbon fibers (in Gba) are taken to illustrate the applicability of the ECSGHL model. And a real dataset consisting of the lifetimes of fifty devices were taken to demonstrate the ECSP model. These models are compared with some existing well known models and the values of $\hat{l}$, AIC, $CAIC$, BIC, A^*, W^* and KS statistics showed that new models outperformed the alternatives. It is also observed that trigonometric regression models, skewed trigonometric distributions and copulas can be constructed using the ECSG family.

3.3. Distributions based on tan function

- Kumar, Chaurasia, Kumar, and Chaurasia (2022) introduced a new transformation using inverse trigonometric function given by

  $$F(x) = K\tan^{-1} G(x); \quad \text{where} \quad K = \frac{1}{\tan^{-1} 1}$$

  The function G(x) possesses the properties of a *cdf* and the author applied this transformation to uniform, logistic, Cauchy, normal, exponential and Lindley distributions. The authors found wide applications to this transformed Lindley distribution. Hence the *cdf* and *pdf* of inverse trigonometric Lindley distribution $IT_{lin}(\theta)$ are

  $$F(x) = K\tan^{-1}\left[1 - \left(\left(1 + \frac{\theta x}{\theta+1}\right)\exp(-\theta x)\right)\right]$$
  $$f(x) = K\frac{\theta^2}{\theta+1}\frac{(1+x)\exp(-\theta x)}{1 + \left[1 - \left(\left(1 + \frac{\theta x}{\theta+1}\right)\exp(-\theta x)\right)\right]^2}$$

  The applications of inverse trigonometric Lindley distribution $IT_{lin}(\theta)$ are studied. Some of its important various statistical mathematical properties including shape, survival function, hazard rate, moments and associated measures, order statistics are discussed and Rényi entropy of the proposed distribution have been derived. The parameter θ is estimated using maximum likelihood estimator method.

  Application

  The model is suited for increasing, bathtub and non-monotone hazard rate shapes. Two real data sets of survival of head and neck cancer patients were considered to illustrate the applicability of the discussed model. The first dataset contains 51 patients treated with radiotherapy alone and second dataset consisted of 45 patients treated with combined radiotherapy and chemotherapy. The authors computed the $\hat{l}$, AIC, BIC and KS statistics to show that the new model provides better fit than the inverse Rayleigh distribution and other alternative models.

- Karakaya, Korkmaz, Chesneau, and Hamedani (2022) introduced a new unit-Lindley distribution (NwUL) by combining the joint functionalities of the hyperbolic tan transformation and Lindley distribution on the unit interval, with the Weibull distribution as the baseline distribution. The hyperbolic tangent function transformation of the Lindley distribution is used to create a unit distribution. The *cdf* and *pdf* are respectively,

  $$F(x) = 1 - \left(1 + \frac{\theta\,\text{arctanh}(x)}{1+\theta}\right)\exp(-\theta\,\text{arctanh}(x))$$
  $$f(x) = \frac{\theta^2}{(\theta+1)(1-x^2)}(1 + \text{arctanh}(x))\ \exp(-\theta\,\text{arctanh}(x))\,; x \in (0,1)$$

  The authors described the characterizations based on two truncated moments and hazard rate function. The model parameters are estimated using six different point estimation methods such as maximum likelihood estimation, ,maximum product spacing method, least square method, weighted least square estimation, Anderson-Darling estimation, the Cramér-von Mises estimation.

  Application

  The model is found useful for increasing, decreasing, U-shaped and inverse N-shaped *pdf* forms. A real data which describes 38 values of the number of people who have been out of work for a year or more as a percentage of the labor force was taken. With the dataset, the authors computed the parameter estimates, $\hat{l}$, AIC, BIC, A^*, W^* and KS statistics, and proved that the NwUL distribution is superior to two existing unit-Lindley distributions and other alternative models.

- Alkhairy, Nagy, Muse, and Hussam (2021) derived the arctan-X family of distributions which is obtained using an arctangent function and have the following *cdf* and *pdf*:

  $$F(x) = \frac{4}{\pi}\arctan(G(x)); \quad x \in \mathbb{R}$$
  $$f(x) = \frac{4}{\pi}\frac{g(x)}{1+G(x)^2}; \quad x \in \mathbb{R}$$

  where $G(x)$ is the *cdf* of some baseline distribution.The general properties are discussed. The authors introduced 15 special members of the arctan-X family with Weibull, Gompertz, log-logistic, Lomax, Kumaraswamy, Pareto, normal, Dagum, Burr-XII, Rayleigh, gamma, Lindley, exponential, Gumbel, and uniform distributions as baseline functions. The arctan -Weibull (AT-W) distribution was discussed in detail and its mathematical and statistical properties are derived. A simulation study is conducted to validate the parameter estimates.

  Application

  These distributions were found to be useful for modelling economic as well as acturial data sets for these data are mostly heavy tailed in nature. For the purpose of illustration the authors took an insurance dataset containing 58 observations on monthly unemployment insurance metric. The AT-W distribution was found to be outperforming than the considered alternative distributions.

3.4. Distributions based on sec function

- Souza, de Oliveira, de Brito, Chesneau, Fernandes, and Ferreira (2022) introduced the secant generated (Sec-G) family of distributions described by the following *cdf* and *pdf* respectively given by the functions,

  $$F(x) = \sec\left(\frac{\pi}{3}G(x)\right) - 1; x \in \mathbb{R} \quad \text{and}$$
  $$f(x) = \frac{\pi}{3}g(x)\sec\left(\frac{\pi}{3}G(x)\right)\tan\left(\frac{\pi}{3}G(x)\right); x \in \mathbb{R}$$

  By defining Kumaraswamy Weibull (Kum - W) distribution as the baseline, a new member of the family is derived namely the Sec-Kum-W distribution. The authors investigated mathematical and statistical properties of the new distribution including shape, moments and associated measures, order statistics, reliability measure and Rényi entropy. Parameter estimation is performed using maximum likelihood method and simulation studies are conducted to validate the estimates.

  Application

  The distribution is found to be useful in analysing data with highly right-skewed and near symmetric *pdf* forms and constant, increasing, decreasing and bathtub hrf forms. Applications are illustrated using three real data sets: lifetime of 50 devices, 101 observations of fatigue life of 6061-T6 aluminum, sums of skin folds for 100 Australian female athletes. Parameter estimates and statistical measures like AIC, BIC, $CAIC$, $HQIC$ and p-values are computed. The authors proved that the proposed model outperforms some of the strong competing existing models.

3.5. Fresh inclusions

In addition, a large number of newly created distributions in 2023 making use of trigonometric extensions indicates that this area of distribution theory is starting to set trends in the discipline. A selection of them are listed below.

- Sapkota, Kumar, and Kumar (2023) investigated the mathematical and statistical features of the new class of sine-generated family (NCS-G) distributions and characterised them. A new class sine inverse Weibull (NCS-IW) distribution,a peculiar member of the family, is presented and demonstrated with the help of a simulation study and two real datasets. The first dataset contained 44 observations on patients with neck and head cancer who received radiotherapy, and the second dataset contained the relief times of 20 patients who took an analgesic. The hrf can have an increasing, inverted bathtub, or reverse-J form. The parameters are estimated using maximum likelihood estimation method.
- Benchiha, Sapkota, Al Mutairi, Kumar, Khashab, Gemeay, Elgarhy, and Nassr (2023) developed the alpha sine-G family of distributions and studied its statistical properties. Using Weibull distribution as baseline, the authors developed alpha-sine Weibull distribution, a special member of the family which is suited for fitting highly skewed heterogeneous datasets. The parameter estimates are obtained using maximum likelihood estimation, maximum product spacing, least square estimation and minimum distance methods. Simulation studies are conducted for comparing the parameter estimates and the model is applied to two real datasets : first is annual rainfall data with 37 values and second is failure time data with 24 observations.
- Elgarhy, Alsadat, Hassan, and Chesneau (2023) proposed a one parameter sine truncated Lomax (STLo) distribution which is ideal for right skewed and reverse-J shaped data. The parameters were estimated using the maximum likelihood and Bayesian methods, and the results were compared. A real dataset comprising of inflation rates of 45 nations in Asia was used for demonstration.
- Mustapha, Isa, Sule, and Itopa (2023) investigated the statistical properties of a two parameter sine Lomax distribution which has a right skewed pdf shape. The maximum likelihood estimation approach is used to estimate the parameters. The model is applied to bladder cancer patients dataset and guinea pigs' survival times dataset. A comparative analysis is performed with some of the existing models.
- Nasiru, Abubakari, and Chesneau (2023) studied the characteristics of a two parameter arctan power distribution in the unit interval to model left skewed, right skewed, symmetric, J and reverse-J shaped datasets. A bivariate extension of this distribution was also created by the authors. A medical dataset that shows the recovery rates of 239 individuals who consented to an autologous peripheral blood stem cell transplant following myeloablative chemotherapy dosages is used to illustrate the concept. Maximum likelihood, least square and weighted least square approaches, Cramér-Von Mises estimation, Anderson-Darling estimation, percentile estimation, maximum product spacing, and Bayes estimation techniques are used in the parameter estimation process.

4. Comparisons

In this section, a real dataset is taken to illustrate some of the above discussed distribution models. The dataset is taken from Lee and Wang (2003) that comprises of remission times (in months) of 128 bladder cancer patients and the values are given as:
0.08, 0.20, 0.40, 0.50, 0.51, 0.81, 0.90, 1.05, 1.19, 1.26, 1.35, 1.40, 1.46, 1.76, 2.02, 2.02, 2.07, 2.09, 2.23, 2.26, 2.46, 2.54, 2.62, 2.64, 2.69, 2.69, 2.75, 2.83, 2.87, 3.02, 3.25, 3.31, 3.36, 3.36,

3.48, 3.52, 3.57, 3.64, 3.70, 3.82, 3.88, 4.18, 4.23, 4.26, 4.33, 4.34, 4.40, 4.50, 4.51, 4.87, 4.98, 5.06, 5.09, 5.17, 5.32, 5.32, 5.34, 5.41, 5.41, 5.49, 5.62, 5.71, 5.85, 6.25, 6.54, 6.76, 6.93, 6.94, 6.97, 7.09, 7.26, 7.28, 7.32, 7.39, 7.59, 7.62, 7.63, 7.66, 7.87, 7.93, 8.26, 8.37, 8.53, 8.65, 8.66, 9.02, 9.22, 9.47, 9.74, 10.06, 10.34, 10.66, 10.75, 11.25, 11.64, 11.79, 11.98, 12.02, 12.03, 12.07, 12.63, 13.11, 13.29, 13.80, 4.24, 14.76, 14.77, 14.83, 15.96, 16.62, 17.12, 17.14, 17.36, 18.10, 19.13, 20.28, 21.73, 22.69, 23.63, 5.74, 25.82, 26.31, 32.15, 34.26, 36.66, 43.01, 46.12, 79.05.

Table 1 provides the descriptive measures of the dataset.

Table 1: Descriptive measures

Mean	**Median**	**Mode**	**Variance**	**Skewness**	**Kurtosis**	**Minimum**	**Maximum**
9.3656	6.395	5	110.425	3.2866	15.483	0.08	79.05

It can be seen from Table 1 that the data is right skewed, leptokurtic and has a large variance. The TTT plot displayed in Figure 1 is first concave and eventually becomes convex, demonstrates that the dataset exhibits an inverse bathtub hazard rate function.

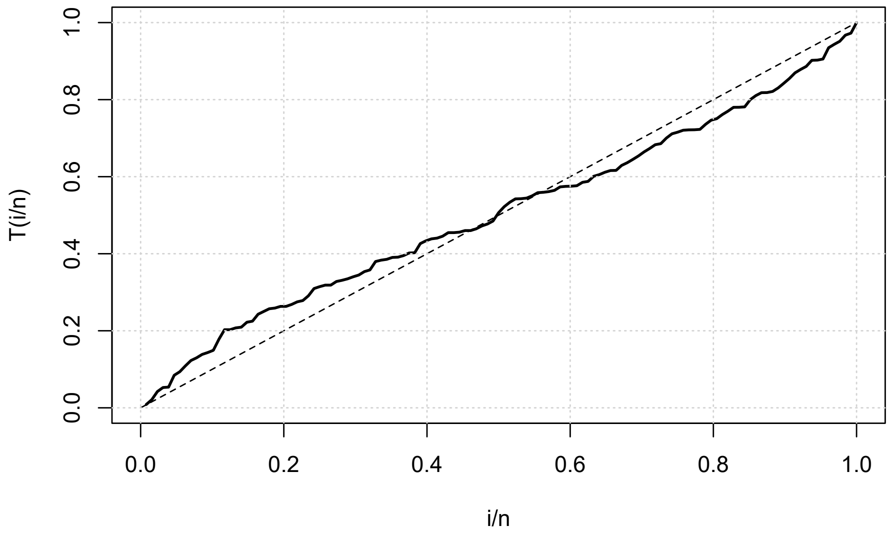

Figure 1: TTT plot of the Dataset

Based on these characteristics, we have selected seven models for illustration from section 3. The following are the models with the parameters (in parenthesis) :
ESW(p, q, r), ECW(p, q, r, s), AT-W(p, q), ECSW(p, q, r), SEOFNH(p, q, r, s), Sin-NH (p, q) and S-ML(p).

We have employed the statistical programme R (see Team *et al.* (2018)) to analyse data both numerically and graphically. The chosen models are compared using the $goodness.fit$ function from the AdequacyModel package(see Marinho, Silva, Bourguignon, Cordeiro, and Nadarajah (2019)). The function returns values of the information criteria metrics including AIC, BIC, $CAIC$ and $HQIC$ as well as goodness of fit statistics A^*, W^* and $K-S$ with a p-value and the -log-likelihood statistic -$\hat{l}$. The maximum likelihood estimates of the model parameters are also provided, which may be further used to construct the estimated *pdf*s and *cdf*s.

Table 2 displays the information criteria measurements, log-likelihood value, and goodness of fit statistics with a p-value for the study at hand. Table 3 displays the model parameters' maximum likelihood estimates together with their standard errors (in parenthesis).

Table 2: AIC, $CAIC$, BIC, $HQIC$, $\hat{l}$ and goodness-of-fit statistics with p-value

Model	AIC	$CAIC$	BIC	$HQIC$	$\hat{l}$	A^*	W^*	KS	p-**value**
ESW	826.98	827.18	835.54	830.46	-410.49	0.2599	0.0391	0.0442	0.9637
ECW	828.80	829.15	840.23	833.46	-410.41	0.2475	0.0370	0.0455	0.9533
AT-W	840.41	840.50	846.11	842.72	-418.20	0.5575	0.0916	0.0663	0.6256
ECSW	832.76	832.95	841.31	836.23	-413.38	0.6817	0.1133	0.0667	0.6190
SEOFNH	836.97	837.29	848.38	841.61	-414.48	0.7897	0.1184	0.0738	0.4895
Sin-NH	830.75	830.88	836.48	833.10	-413.39	0.5363	0.0893	0.0867	0.2906
S-ML	854.26	854.29	857.12	855.43	-426.13	0.9056	0.1534	0.1466	0.0081

Table 3: Parameter estimates with standard errors (in parenthesis)

Model	p	q	r	s
ESW	2.8360 (1.0317)	0.6081 (0.1011)	0.2765 (0.1086)	...
ECW	3.2633 (2.5119)	0.4019 (0.2887)	0.4049 (0.1936)	0.6638 (0.1219)
AT-W	0.0847 (0.0072)	1.1169 (0.0702)	...	...
ECSW	0.2739 (0.0280)	0.2039 (0.0282)	1.0036 (0.0194)	...
SEOFNH	0.4255 (0.2516)	0.9437 (0.4815)	1.1363 (0.4153)	0.0272 (0.0204)
Sin-NH	0.7523 (0.1445)	0.0902 (0.0269)	...	...
S-ML	0.0864 (0.0053)	...	...	...

From Table 2 and Table 3, it is observed that the ESW and ECW models have lower values for AIC, BIC, $CAIC$, $HQIC$, A^*, W^*, KS statistic and $\hat{l}$ than AT-W, ECSW, SEOFNH, Sin-NH and S-ML models. Additionally, both models' p-values for the KS statistic are high. The ESW model has been shown to be the best performing model for bladder cancer patient data in Muhammad *et al.* (2021a). With the help of our findings, it can be observed that the ECW model likewise offers an excellent fit to the same data.

The plots of estimated *pdf* and *cdf* over the histogram and empirical *cdf*, respectively, are generated using the parameter estimates derived in Table 3. Figure 2 displays the data's histogram and predicted pdf plots. Figure 3 shows the estimated *cdf* curves for various models under investigation along with the empirical *cdf* derived from the data.

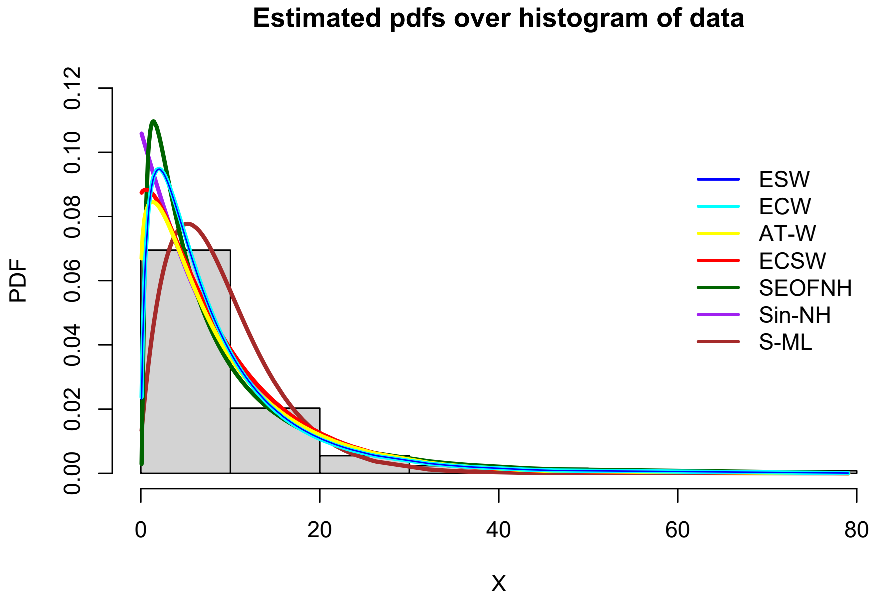

Figure 2: Estimated *pdf* over the histogram of data

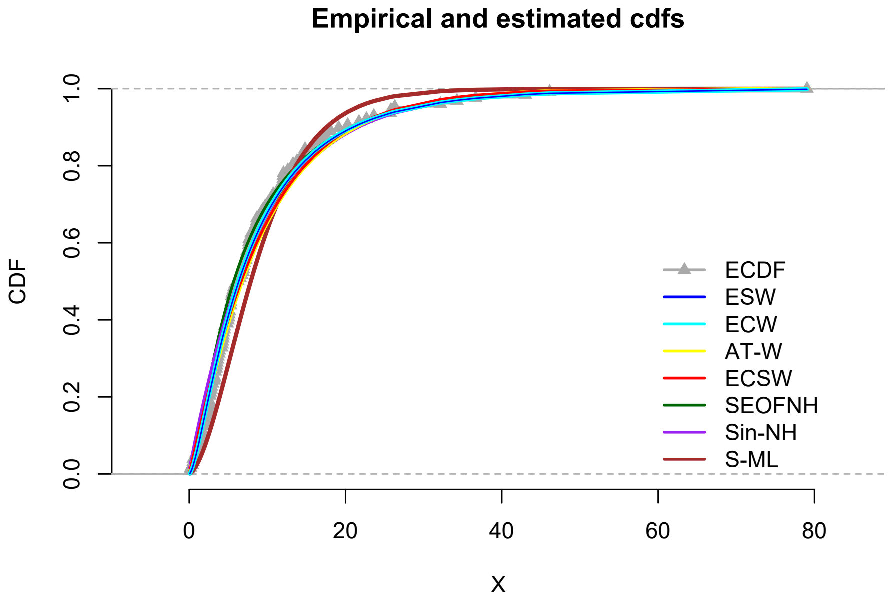

Figure 3: Estimated *cdf* over empirical *cdf* of data

5. Conclusion

The focus of the current study was mostly on various distributions produced recently utilising trigonometric modifications. Initially trigonometric distributions were mostly defined for the purpose of analysing circular and directional data. The recent trends reveal that the new distributions developed using trigonometric transformations can be applied to every fields of study especially to model life time distributions. It is observed that each of the distributions were developed for modelling data having specific pdf and hrf forms. The pdf shapes can be right or left skewed, almost symmetric, increasing, or decreasing, whereas the hrf can take increasing, decreasing, constant, bathtub, or reverse bathtub shapes. Using graphical tools like histograms and TTT plots, these shapes can be identified. The authors of each of the distributions discussed in this article demonstrated that these models are more flexible and offer a better fit to the data in the relevant study when compared to many of the existing standard distributions. Statistical tools like $AIC, BIC, CAIC, HQIC, A^*, W^*, K-S$ and $\hat{l}$ values were computed for such comparisons.

With the aid of the bladder cancer patients dataset, we investigated the main features of seven recent distributions extracted from section 3. To assess the selected models, goodness of fit tests are carried out and maximum likelihood estimates of the model parameters are generated. The review clearly conveys that the new distributions and extensions of distributions based on trigonometric functions find high use in modelling data arising in our daily lives.

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