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E-Bayesian Inference for Exponential Transformed Inverse Rayleigh Distribution under Progressive Type-II Censoring with Binomial Removals

Satya Prakash Mishra
Department of Statistics
Central University of Haryana,
Mahendergarh, India

Manoj Kumar
Department of Statistics
Institute of Eminence,
University of Delhi, Delhi, India

Sanjay Kumar Singh
Department of Statistics
Banaras Hindu University,
Varanasi, India

Anurag Pathak
Department of Community Medicine
MSD Autonomous State Medical College,
Bahraich, India

Sandeep Kumar
Department of Community Medicine
Autonomous State Medical College,
Kaushambi, India

Abstract

In this paper, we consider the comparative analysis of Bayes and E-Bayes estimates for the newly developed exponential transformed inverse Rayleigh (ETIR) distribution under the symmetric and asymmetric loss function for the progressive Type-II censoring with binomial removals. The comparison between the proposed estimators have been drawn on the basis of the simulated risks. The study also examines the expected experiment time. The suitability of the model and proposed methodology have been through demonstrated using a precipitation data set.

Keywords: ETIR, progressive type-II censoring, E-Bayes estimation, loss function, bootstrap, binomial removals.

1. Introduction

Now a days, the modelling and analysis of lifetime data have become essential in a wide range of scientific fields, including medicine, finance, and engineering. Several lifetime distributions are available in the literature that can be used to analyze the lifetime data. The ETIR distribution has been proposed by Banerjee and Bhunia (2022). The cumulative density function (CDF) and probability density function (PDF) of ETIR distribution is given by equations (1) and (2) respectively,

$$F(x;\beta) = \phi \left\{ e^{e^{-\left(\frac{\beta}{x}\right)^2}} - 1 \right\}; \quad x > 0 \quad \beta > 0, \tag{1}$$

$$f(x;\beta) = 2\beta^2\phi\frac{1}{x^3}e^{e^{-\left(\frac{\beta}{x}\right)^2}}e^{-\left(\frac{\beta}{x}\right)^2}; \quad x > 0 \quad \beta > 0, \quad \text{where} \quad \phi = \frac{1}{e-1}. \tag{2}$$

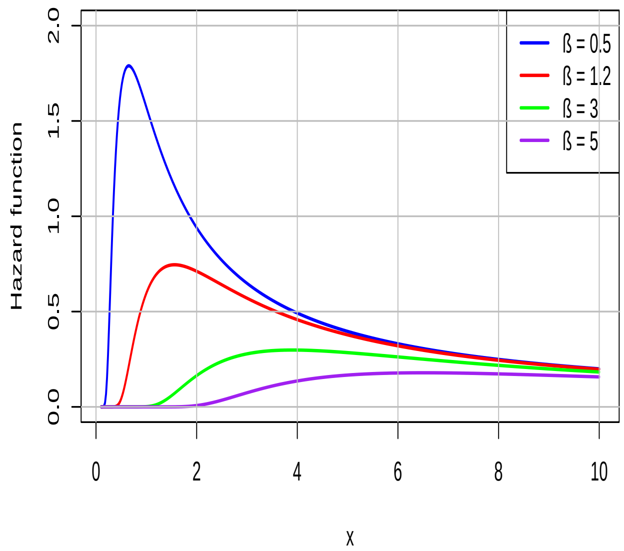

Figure 1: Hazard rate function for various values of scale parameter

Here, β is a scale parameter of the ETIR distribution. The ETIR distribution is an extension of the inverse Rayleigh distribution. This has accommodated a non-monotone hazard rate, see figure 1. Ali, Khalil, Ijaz, and Saeed (2021) has proposed a alpha power exponentiated inverse Rayleigh distribution, and presented its statistical properties. Before this, Aryal and Tsokos (2009) discussed the transmuted extreme value distribution with application to real life problems.

In life testing experiments, incomplete observations are a common issue that many researchers face due to time and cost constraints. Therefore, various types of censoring methods and their applications exist in the literature; see Singh, Gupta, and Upadhyay (2005), Joarder, Krishna, and Kundu (2011). The Type-I and Type-II censoring are applied when the number of failures observed before time T_0 and when the m number of failures observed out of n units, respectively. In some cases when, some experimental units are removed at the intermediate stage. In that case, a progressive censoring scheme is suitable, as it allows the removal of surviving units before the completion of the experiment. When the number of removals at each stage follows a binomial distribution with fixed probability, such censoring is called progressive Type-II censoring with binomial removals (PT-II CBRs). The mathematical formulation of PT-II CBRs, see in detail, Balakrishnan and Aggarwala (2000), and Pathak, Kumar, Singh, Singh, Tiwari, and Kumar (2023).

Bayesian methods in statistical inference rely heavily on the choice of prior distributions. In many cases, the parameters of a prior distribution themselves depend on other unknown quantities, known as hyper parameters. The hierarchical Bayesian method is commonly employed to handle such situations. Lindley and Smith (1972) first introduced the concept of hierarchical prior distributions. This method involves a two-stage process for specifying prior distributions, which adds flexibility and makes the approach more robust than traditional Bayesian methods. In recent years, hierarchical Bayesian methods have been widely applied in various areas of data analysis. For further reading and examples, see the works of Zellner (1986), Han (2007), Yousefzadeh (2017). Despite their advantages, hierarchical Bayesian approaches often involve evaluating complex integrals, which can be computationally challenging. To address this difficulty, numerical methods such as Markov Chain Monte Carlo (MCMC) are frequently used; see Gelman, Carlin, Stern, Dunson, Vehtari, and Rubin (2013).

The Bayes rule estimates an unknown parameter in the Bayesian framework by assigning a prior distribution. However, choosing a specific prior with fixed hyper parameters can be challenging, especially when prior knowledge is vague or uncertain. Expected Bayes (E-Bayes) have been developed to address this problem. This approach considers a class of prior distributions rather than a single fixed prior. The main objective of the E-Bayes method is to estimate the unknown parameter by incorporating uncertainty in the choice of the hyper parameters. Han has extensively used the E-Bayes estimation of parameters for various lifetime distributions (see in a series of papers Han (1997), Han (2007), Han (2009), Han (2011)). In the case of progressive Type-II censoring scheme, various authors have shown the comparison of Bayes and E-Bayes estimation see, Kalantan, Swielum, AL-Sayed, EL-Helbawy, AL-Dayian, and Abd Elaal (2024), Koul (2024), Shojaee, Zarei, and Naruei (2024). Recently, Pathak *et al.* (2023), Pathak *et al.* (2023) obtained the E-Bayes estimators for Poisson inverse exponential and xgamma distribution, respectively, under PT-II CBRs.

We are motivated to do the current work for the following reasons: (1) To the best of our knowledge, no work has been done on the Bayes and E-Bayes estimation of ETIR distribution under censoring mechanism. (2) The superiority of ETIR distribution in fitting real life meteorological data was demonstrated and compared with several other lifetime distributions. Hence, we proposed the E-Bayes estimators under PT-II CBRs for various loss functions like Squared error loss function (SELF), General entropy loss function (GELF), and Linear exponential (LINEX) loss function. The comparison between the proposed estimators was made on the basis of simulated risks by using PT-II CBRs samples.

The remainder of the paper is organized as follows. Section 2 shows the estimation of scale parameter of ETIR using frequentist estimation along with asymptotic and bootstrap confidence intervals. Bayes and E-Bayes estimators under the SELF, GELF, and LINEX loss functions are shown in Sections 3 and 4, respectively. Section 5 shows the expected experiment time. Section 6 shows a comparison between the obtained estimators. An illustrative example has been given in Section 7 to validate the opted methodology. Finally, concluding remarks have been presented in Section 8.

2. ML estimation of β

A lifetime experiment begins with n units. At the first failure time X_1, R_1 units ($0 \leq R_1 \leq n-m$) are removed from the surviving units. At the second failure time X_2, R_2 units ($0 \leq R_2 \leq n-m-R_1$) are removed, and this process continues until the m^{th} failure is observed, i.e., at the m^{th} failure, all the remaining $R_m = n-m-\sum_{i=1}^{m-1} R_i$ units are removed. While m is fixed, the number of removals R_i ($i = 1, 2, 3, \ldots$) is not determined in advance by the experimenter. Instead, each unit in the experiment is subject to a constant but unknown probability of removal. To simplify, it is assumed that every unit remaining in the experiment has an equal likelihood of being removed, represented by p. Consequently, the number of units removed at the i^{th} failure is modeled by a binomial distribution Tse, Yang, and Yuen (2000). Given the observed sample ($X_i, R_i = r_i$), for $i = 1, 2, 3, \ldots, m$, the sample corresponds to a PT-II CBRs sample as described by equation (2), where $X_i < X_{i+1}$, for $i = 1, 2, 3, \ldots, m-1$. It may be noted here that the number of items removed at i^{th} stage, R_i is the random variable following binomial distribution, as discussed above. Hence, the conditional likelihood function can be expressed as follows (see, Cohen (1963)).

$$L(\beta|\mathbf{x}, \mathbf{R}) = k \prod_{i=1}^{m} f(x_i; \beta) \left[1 - F(x_i; \beta)\right]^{r_i}, \tag{3}$$

where

$$n = m + \sum_{i=1}^{m} r_i, \quad n, m \in \mathbb{N}, \quad r_i \in \mathbb{N}_0, \quad 1 \le i \le m,$$

$$r_i \sim \text{Binomial}\left(n - m - \sum_{l=0}^{i-1} r_l, p\right) \quad \text{for} \quad i = 1, 2, 3, \ldots, m-1,$$

with $r_0 = 0$ and $k = \prod_{i=1}^{m} \lambda_i$, where $\lambda_i = \sum_{j=i}^{m}(r_j + 1)$ and $\lambda_1 = n$. Substituting $F(x_i)$ and $f(x_i)$ from equations (1) and (2) into equation (3), it simplifies to:

$$l(\beta \mid x, R) = \left(2\beta^2\phi\right)^m \prod_{i=1}^{m} \left(\frac{1}{x_i^3}\right) e^{\sum_{i=1}^{m} e^{-\left(\frac{\beta}{x_i}\right)^2}} e^{-\beta^2 \sum_{i=1}^{m} \frac{1}{x_i^2}} \left[(1+\phi)^m - (\phi)^m \left\{e^{\sum_{i=1}^{m} e^{-\left(\frac{\beta}{x_i}\right)^2}}\right\}\right]^{r_i}. \tag{4}$$

As mentioned earlier, R_i follows a binomial distribution with parameters $\left(n - m - \sum_{l=1}^{i} r_l, p\right)$. Where, R_i represents the random variable of removals and r_i represents the realised value of R_i. Hence

$$P(R_1 = r_1 \mid p) = \binom{n-m}{r_1} p^{r_1}(1-p)^{n-m-r_1}, \tag{5}$$

and for $i = 2, 3, \ldots, m-1$.

$$P(R_i \mid p) = P(R_i = r_i \mid R_{i-1} = r_{i-1}, \ldots, R_1 = r_1) = \binom{n - m - \sum_{l=0}^{i-1} r_l}{r_i} p^{r_i}(1-p)^{n-m-\sum_{l=1}^{i} r_l}. \tag{6}$$

Here all R_i's are independent of X_i's for all i. Thus, joint likelihood function becomes

$$l(\beta, p \mid x) = l(\beta \mid R) P(R = r \mid p), \tag{7}$$

where,

$$\begin{aligned} P(R = r \mid p) = P(R_1 = r_1) P(R_2 = r_2 \mid R_1 = r_1) P(R_3 = r_3 \mid R_2 = r_2, R_1 = r_1) \cdots \\ P(R_{m-1} = r_{m-1} \mid R_{m-2} = r_{m-2}, \cdots, R_1 = r_1). \end{aligned} \tag{8}$$

Putting the equations (5) and (6) into equation (8), we get

$$P(R = r \mid p) = \frac{(n-m)! p^{\sum_{i=1}^{m-1} r_i} (1-p)^{(m-1)(n-m) - \sum_{i=1}^{m-1}(m-i) r_i}}{\left(n - m - \sum_{i=1}^{m-1} r_i\right)! \prod_{i=1}^{m-1} r_i!}. \tag{9}$$

Now making use of the equations (4), (7) and (9), we can write the full likelihood as,

$$l(\beta, p \mid x) = \zeta l_1(\beta) l_2(p) \tag{10}$$

where, ζ represents the,

$$\zeta = \frac{c(n-m)!}{\left(n - m - \sum_{i=1}^{m-1} r_i\right)! \prod_{i=1}^{m-1} r_i!},$$

$$l_1(\beta) = \left(\frac{2\beta^2}{e-1}\right)^m \prod_{i=1}^{m} \left(\frac{1}{x_i^3}\right) e^{\sum_{i=1}^{m} e^{-\left(\frac{\beta}{x_i}\right)^2}} e^{-\beta^2 \sum_{i=1}^{m} \frac{1}{x_i^2}} \left[(1+\phi)^m - (\phi)^m \left\{e^{\sum_{i=1}^{m} e^{-\left(\frac{\beta}{x_i}\right)^2}}\right\}\right]^{r_i}, \tag{11}$$

and

$$l_2(p) = p^{\sum_{i=1}^{m-1} r_i} (1-p)^{(m-1)(n-m) - \sum_{i=1}^{m-1}(m-i) r_i}. \tag{12}$$

Now, the log-likelihood can be written by making use of equation (11) by which we can find the maximum likelihood estimate (MLE) of β. Thus log-likelihood is written as

$$
\begin{aligned}
logl_1(\beta) = nlog\,(2\phi) + 2mlog\beta + \sum_{i=1}^{m} log\left(\frac{1}{{x_i}^3}\right) + \sum_{i=1}^{m} e^{-}\left(\frac{\beta}{x_i}\right)^2 - \beta^2 \sum_{i=1}^{m} \frac{1}{{x_i}^2} \\
+ \sum_{i=1}^{m} r_i log\left[(1+\phi) - \phi\left\{e^{e^{-\left(\frac{\beta}{x_i}\right)^2}}\right\}\right].
\end{aligned}
\tag{13}
$$

The normal equation can be obtained by differentiating the equation (13) w.r.t. β and equating it to zero for MLE, as given below

$$
\frac{\partial logl_1(\beta)}{\partial\beta} = \frac{2m}{\beta} + \sum_{i=1}^{m} e^{-\left(\frac{\beta}{x_i}\right)^2}\left(\frac{-2\beta}{{x_i}^2}\right) - 2\beta\sum_{i=1}^{m}\frac{1}{x_i^2} + \sum_{i=1}^{m} r_i \frac{\phi e^{e^{\left(-\frac{\beta}{x_i}\right)^2}} e^{-\left(\frac{\beta}{x_i}\right)^2} \frac{-2\beta}{x_i^2}}{1+\phi(1-e^{e^{-\left(\frac{\beta}{x_i}\right)^2}})} = 0. \tag{14}
$$

Equation (14) can not be solved analytically. Therefore, we have used numerical technique to estimate β.

2.1. Asymptotic confidence interval

An asymptotic confidence interval can be obtained by using the distribution of the MLE, which follows aymptotic normal distribution. Hence, the variance of MLE is given by:

$$
\mathrm{Var}(\hat{\beta}_M) = \frac{1}{I(\beta)},
$$

where $I(\beta) = E\left(-\frac{\partial^2 \ln l_1(\beta)}{\partial\beta^2}\right)$ represents the Fisher information. After obtaining the MLE of β as $\hat{\beta}_M$, we can estimate $I(\beta)$ by:

$$
\hat{I}(\beta) = E\left(-\frac{\partial^2 \log l_1(\beta)}{\partial\beta^2}\right)\bigg|_{\beta=\hat{\beta}_M}, \tag{15}
$$

and the estimate of $\mathrm{Var}(\hat{\beta}_M)$ is given by $\hat{\mathrm{Var}}(\hat{\beta}_M) = \frac{1}{\hat{I}(\hat{\beta})}$. Therefore, the $(1-\alpha)100\%$ asymptotic confidence interval for β can be expressed as:

$$
\left(\hat{\beta}_M \pm z_{\alpha/2}\sqrt{\hat{\mathrm{Var}}(\hat{\beta}_M)}\right),
$$

where $z_{\alpha/2}$ denotes the upper $(\alpha/2)^{\mathrm{th}}$ quantile of the standard normal distribution.

2.2. Bootstrap confidence interval

In general we have seen that the asymptotic confidence interval work on the large sample sizes, but in the censoring scenario it is not possible. Therefore one can opt for the alternative method namely bootstrap confidence interval for β see, Tibshirani and Efron (1993). The bootstrap confidence interval algorithm is as follows:

Algorithm for Boot-p

1. Generate PT-II CBRs samples from ETIR, $\left(X_{(1)}, X_{(2)}, X_{(3)}, \ldots, X_{(m)}\right)$, MLE of β is computed by solving equation (14), which provides an estimate of β.

2. Using $\hat{\beta}_M$, generate a sample $\left(X^*_{(1)}, X^*_{(2)}, X^*_{(3)}, \ldots, X^*_{(n)}\right)$ of size n, referred to as the bootstrap sample. Next, extract a PT-II CBRs random sample $\left(X^*_{(1)}, X^*_{(2)}, X^*_{(3)}, \ldots, X^*_{(m)}\right)$ of size m and calculate the bootstrap estimate β^* for β.

3. Repeat the above step, say B (larger number of times).

4. Let $\hat{\beta}_B^{(\alpha)}$ represent the empirical percentile of the ordered B values of β^*. Consequently, the $(1-\alpha)100\%$ Bootstrap-p confidence interval is expressed as

$$\left(\hat{\beta}_B^{(\alpha/2)}, \hat{\beta}_B^{(1-\alpha/2)}\right).$$

Algorithm for Boot-t

Steps (1–2) are used same as to the Boot-p method as taken above.

3. For each $T^* = \frac{\beta^* - \beta}{SE(\beta^*)}$ obtained in Step 2, calculate

$$SE(\beta^*) = \sqrt{\hat{I}(\beta^*)^{-1}},$$

where $SE(\beta^*)$ is the estimated standard error of β^*, which can be obtained using equation (15).

4. Let $T_B^{*(\alpha)}$ be the empirical percentile of the ordered B values of T^*. Then, the $(1-\alpha)100\%$ Boot-t confidence interval is given by:

$$\left(\hat{T}_B^{*(\alpha/2)}, \hat{T}_B^{*(1-\alpha/2)}\right).$$

3. Bayes estimate of β

In this section, we have worked to derive the Bayes estimate of the scale parameter β based on PT-II CBRs. We may assume that the random variable β has the informative prior distribution with the prior PDF as:

$$g(\beta \mid a, b) = \frac{b^a}{\Gamma(a)} e^{-b\beta} \beta^{a-1}; \quad \beta > 0, a > 0, b > 0, \tag{16}$$

Using the Bayes theorem, we can obtain the posterior distribution of β by merging likelihood and prior from the equations (11) and (16) respectively. The posterior distribution can be obtained as:

$$\pi(\beta \mid x) \propto l_1(\beta \mid x, R) g(\beta \mid a, b). \tag{17}$$

The equation for posterior distribution is:

$$\pi(\beta \mid x) \propto \beta^{2m+a-1} \prod_{i=1}^{m} \frac{1}{x_i^3} e^{e^{-\left(\frac{\beta}{x_i}\right)^2}} e^{-\beta^2 \sum_{i=1}^{m} \frac{1}{x_i^2} - b\beta} \prod_{i=1}^{m} \left(1 - \phi\left\{e^{e^{-\left(\frac{\beta}{x_i}\right)^2}} - 1\right\}\right)^{r_i}. \tag{18}$$

The loss function is one of the key elements of Bayesian inference. In this manuscript, we consider both symmetric and asymmetric loss functions. When the problem of over estimation and under estimation are equally important, the most widely used choice is SELF. However, in situations where under-estimation is more critical than over-estimation, or vice versa, an asymmetric loss function is more appropriate. In this work, we employ two asymmetric loss functions the GELF and LINEX loss function. Notably, GELF is a modified version of the LINEX loss function. For more details on GELF, see Calabria and Pulcini (1990). LINEX loss function was introduced in the literature by Varian (1975), has since been widely used by researchers. Similarly, Kim, Jung, and Chung (2011) obtained Bayes estimates based on progressively Type-II censored samples under the same loss function. The LINEX loss function is an asymmetric loss function that penalizes heavily on one side of zero, while increasing linearly on the other side. This property makes it particularly useful in contexts where overestimation and underestimation have different consequences. If $\delta > 0$, the LINEX loss function grows exponentially in the positive direction and linearly in the negative direction, Conversely, when $\delta < 0$, the negative side is penalized more heavily. Furthermore, the intensity of the penalty increases as the absolute value of δ also increases.

Bayes estimate $\hat{\beta_S}$ of β under SELF is given by

$$\hat{\beta_S} = E(\beta \mid x) = \int_0^\infty \beta\pi(\beta \mid x)\, d\beta. \tag{19}$$

Bayes estimate of β under GELF and LINEX is given by the equations (20) and (21) respectively

$$\hat{\beta_G} = (E(\beta^{-\delta} \mid x))^{-1/\delta} = \left(\int_0^\infty \beta^{-\delta}\pi(\beta \mid x)\, d\beta\right)^{-1/\delta} \quad \delta \neq 0, \tag{20}$$

$$\hat{\beta_L} = -\frac{1}{\delta} log E(e^{-\delta\beta} \mid x) = -\frac{1}{\delta} log\left(\int_0^\infty e^{-\beta\delta}\pi(\beta \mid x)\, d\beta\right) \quad \delta \neq 0 \tag{21}$$

where, δ is the shape parameter.

4. E-Bayes estimate of β

In this section, we obtain the E-Bayes estimates of β based on the symmetric loss function (SELF) and the two asymmetric loss functions (GELF and LINEX) under the PT-II CBRs. According to the study by Han and Ding (2004) the hyper parameter should be selected in such a way that the equation (16) becomes the decreasing function of β. The derivative of equation (16) w.r.t. β is given by:

$$\frac{d\left[g_1(\beta \mid a, b)\right]}{d\beta} = \frac{b^a\beta^{a-2}e^{-b\beta}}{\Gamma(a)}[(a-1) - b\beta], \tag{22}$$

From equation (22) it can be observed that when $\beta > 0$, $a > 0$ and $b > 0$, and partiular when $0 < a < 1, b > 0$ the derivative $\frac{dg_1(\beta|a,b)}{d\beta} < 0$. Therefore, $g_1(\beta \mid a, b)$ is a decreasing function of β. Assuming that a and b are independent, then

$$\pi(a, b) = \pi_1(a)\pi_2(b), \tag{23}$$

the E-Bayes estimate is defined as the expectation of the Bayes estimate of β. It is given by:

$$\hat{\beta}_{EB} = E(\beta \mid x) = \int\int_\Theta \hat{\beta}_B(a, b) da db, \tag{24}$$

where, $\hat{\beta}_B$ is the Bayes estimate of β under the SELF, GELF and LINEX loss functions, as defined in equations (19), (20) and (21) respectively. For further details on E-Bayes estimation, see Han (2011),Pathak, Kumar, Singh, and Singh (2022). To study the influence of different prior distribution on the E-Bayes estimate of β, we consider the following three classes of joint prior distribution for a and b.

$$\left.\begin{array}{rcl} \pi_1(a, b) & = & \frac{1}{c(B(u,v))} a^{(u-1)}(1-a)^{(v-1)}; \quad 0 < a < 1; 0 < b < c \\ \pi_2(a, b) & = & \frac{2(c-b)}{c^2(B(u,v))} a^{(u-1)}(1-a)^{(v-1)}; \quad 0 < a < 1; 0 < b < c \\ \pi_3(a, b) & = & \frac{2b}{c^2(B(u,v))} a^{(u-1)}(1-a)^{(v-1)}; \quad 0 < a < 1; 0 < b < c \end{array}\right\}. \tag{25}$$

Using these priors, the E-Bayes estimate of β corresponding to SELF, GELF and LINEX loss functions are given by equations (26), (27) and (28) respectively.

$$\hat{\beta}_{EBSi} = \iint_{\Theta} \hat{\beta}_S \pi_i(a,b) dadb; \quad i = 1,2,3; \Theta \in \{(a,b) : 0 < a < 1, 0 < b < c\}. \tag{26}$$

$$\hat{\beta}_{EBGi} = \iint_{\Theta} \hat{\beta}_G \pi_i(a,b) dadb; \quad i = 1,2,3; \Theta \in \{(a,b) : 0 < a < 1, 0 < b < c\}. \tag{27}$$

$$\hat{\beta}_{EBLi} = \iint_{\Theta} \hat{\beta}_L \pi_i(a,b) dadb; \quad i = 1,2,3; \Theta \in \{(a,b) : 0 < a < 1, 0 < b < c\}. \tag{28}$$

Here $\hat{\beta}_{EBSi}$, $\hat{\beta}_{EBGi}$ and $\hat{\beta}_{EBLi}$ $\quad i = 1,2,3$ represent the E-Bayes estimate of β under the SELF, GELF and LINEX loss functions respectively. The above expressions for Bayes and E-Bayes estimates of β do not have closed-form solutions. To address this, we employ the MCMC method. For details on the MCMC algorithm and its implementation, refer to Gelman *et al.* (2013).

5. Expected experiment time

It is well known that the cost of life testing experiments is directly proportional to the duration of the experiment. Therefore, effective planning of the experiment requires knowledge of the expected duration, which under PT-II CBRs, can be defined as follows

$$\begin{aligned} E\left[X_m\right] &= E_R\left[E\left[X_m \mid R = r\right]\right] \\ &= \sum_{r_1=0}^{t(r_1)} \sum_{r_2=0}^{t(r_2)} \cdots \sum_{r_{m-1}=0}^{t(r_{m-1})} p(R,p) E\left[X_{m:m:n} \mid R = r\right]. \end{aligned}$$

Where $t\left(r_i\right) = n - m - r_1 - \cdots - r_{i-1}$, and $p(R = r; p)$ are given in equation (9). The expected experiment time (EET) conditioned on R can be written as

$$E\left[X_m \mid R\right] = \int_0^{\infty} x f_{X_m}(x) \mathrm{d}x,$$

where, $f_{X_{(m)}} = K_{m-1} f(x) \sum_{l=1}^{m} s_{l,m} (1 - F(x))^{\beta_l}, \quad 1 \leq m \leq n$ and $K_{m-1} = \prod_{i=1}^{m} \beta_i, \quad 1 \leq m \leq n$ and $s_{l,m} = \prod_{i=1}^{m} \frac{1}{\beta_i - \beta_l}; i \neq l, 1 \leq l \leq m \leq n$. Using the above procedure, the EET under the PT-II CBRs is computed for various values of n and m, as shown in Table 1. The value of p is considered as $p = 0.2, 0.4, 0.6, 0.8$ and the model parameter β is arbitrarily taken as 1.2 and 5.0. The simulated results are presented in Table 1. Figure 2 illustrates the EET for the different values of n, p and β. From figure 2 it can be seen that as the sample size increases, the experiment time also increases. However, for moderately large sample sizes (>25), the impact of the probability of removals on experiment time becomes negligible. Figure 3 displays the plot of EET against p for different combinations of n and m, with fixed model parameter as $\beta = 1.2$ and $\beta = 5.0$. From all these plots it is evident that, EET initially increases with p, but eventually stabilizes for all values of m as p increases. For larger values of m, the EET shows minimal fluctuations and remains relatively stable even at lower values of p. Based on these observations, we conclude that a reasonable choice for the probability of removals lies within the range $0.4 < p < 0.6$ for the ETIR distribution.

Table 1: Expected experiment time under PT-II CBRs

		$\beta = 1.2$				$\beta = 5.0$			
n	m	p=0.2	p=0.4	p=0.6	p=0.8	p=0.2	p=0.4	p=0.6	p=0.8
20	3	1.03471	1.16152	1.50601	2.46287	4.32782	4.86278	6.35234	10.50771
	5	1.41956	2.19154	3.66998	4.68264	5.87190	8.97976	15.89252	19.85091
	10	3.90533	7.25334	7.80413	8.27636	16.09941	28.93354	32.98939	33.09820
	15	9.33822	9.99408	10.92547	10.63651	35.92306	40.36930	42.44598	43.02016
	20	11.95227	12.56463	12.65000	12.82211	49.29612	49.92593	49.89615	49.38047
30	10	2.548522	5.89194	7.254399	8.053022	10.45061	25.26909	31.08312	32.90047
	15	6.08596	9.52255	10.04082	10.56614	25.77053	38.9196	41.53427	41.69655
	20	10.31602	11.68463	11.88960	11.86442	43.37574	48.12426	48.89304	49.24499
	25	12.79080	13.45363	12.98110	12.54933	53.74780	54.75631	55.28570	55.91145
	30	13.56775	14.22961	14.39785	14.84849	62.61221	61.24512	61.07817	62.23401
40	20	9.03422	11.35687	11.55664	11.75521	36.68733	46.71055	47.02474	47.77659
	25	11.60922	12.98373	13.07572	13.1992	50.57364	53.17413	53.63851	54.17846
	30	13.82824	14.45575	14.50128	14.5532	57.86165	58.85321	60.52269	62.73048
	35	15.25063	15.7778	16.11343	16.16325	62.58953	62.85825	65.80011	67.5494
	40	16.06167	16.20211	16.86399	16.88511	68.11582	69.75615	70.87254	71.22504

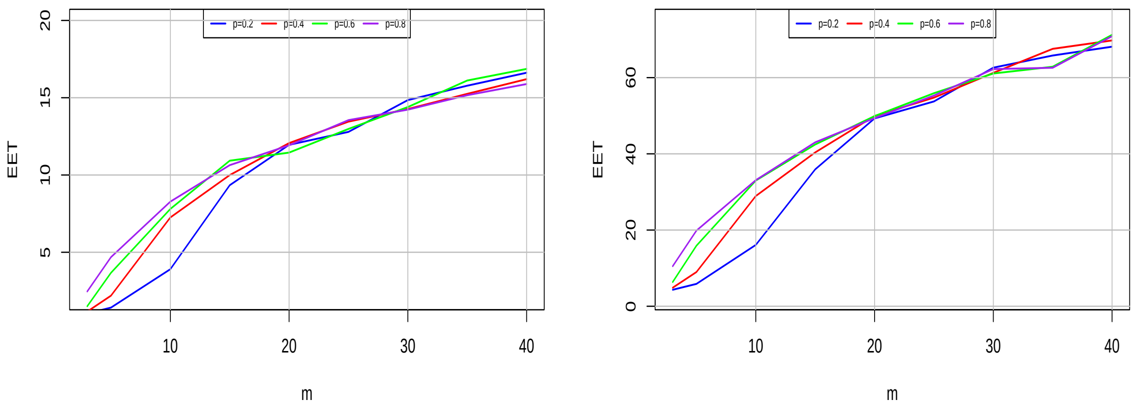

Figure 2: EET plot for fixed $\beta = 1.2$ (left column) and $\beta = 5.0$ (right column) for $\beta = 5.0$ for different values of p

6. Comparison of estimators

This section compares the ML, Bayes, and E-Bayes estimators obtained under PT-II CBRs from previous sections. The $\hat{\beta_M}$ represents the MLE of β and $\hat{\beta_S}$, $\hat{\beta_G}$ and $\hat{\beta_L}$ shows the Bayes estimates of β under the SELF, GELF and LINEX loss functions, respectively. Although EBS_i, EBG_i and $EBL_i \quad (i = 1, 2, 3)$ represent the E-Bayes estimators under PT-II CBRs for SELF, GELF and LINEX loss functions, respectively. We have compared the E-Bayes estimators with the corresponding Bayes estimators under the SELF, GELF, and LINEX loss functions, as well as with the MLEs. These comparisons are made on the basis of the simulated risks for each loss function. It is important to note that exact expressions for the risks cannot be derived, as the estimators do not have closed-form expressions. Consequently, the risks are estimated through a Monte Carlo simulation study using 15000 samples. To

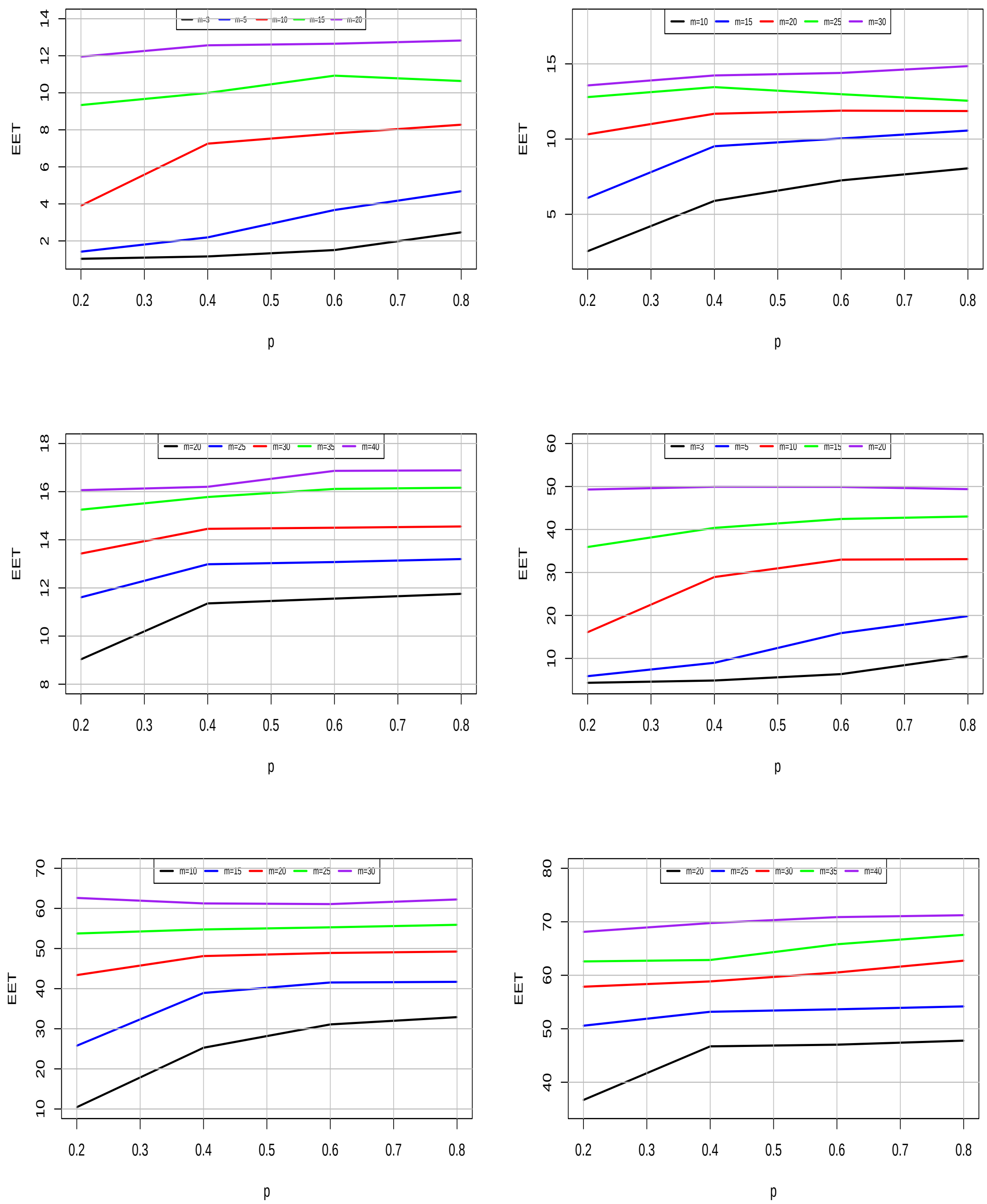

Figure 3: EET plot for fixed $\beta = 1.2$ (left column) and $\beta = 5.0$ (right column) for different values of effective sample size m

obtain the risk of the various estimators, which are the functions of n, m, β, δ and c. To examine the impact of varying the total sample size n in combination with different effective sample sizes m, we have calculated the simulated risks for $\beta = 1.5$ with $n = 20$, $m = 15, 18$; $n = 30$, $m = 23, 27$ and $n = 40$, $m = 30, 36$, under PT-II CBRs. For the consideration of over estimation, the loss parameter δ under GELF and LINEX has been taken as $\delta = 0.5$, and $\delta = -0.5$ for under estimation. After the generation of PT-II CBRs samples from ETIR distribution, PT-II samples were obtained with the help of Balakrishnan and Sandhu (1995) algorithm. An upper limit for the hyperparameter b has been chosen as $c = 3$, where the value of c was arbitrarily chosen. The results obtained are reported in Table 2 and Table 3. Table 2 represents the ML, Bayes, and E-Bayes estimates under the SELF, GELF and LINEX loss functions, while Table 3 shows the risk of the ML, Bayes, and E-Bayes estimates for the loss parameter $\delta = 0.5$ (over estimation is more severe than under estimation) and $\delta = -0.5$ (under estimation is more serious than over estimation)
Furthermore, from Table 3, we can observe that the risk of all estimators decreases as the effective sample size increases for all combinations of n and m. However, we have also seen that the numerical value of the risk under GELF has the least value for all choices of (n, m). From Table 3, we found that the E-Bayes estimator with prior $\pi_1(a, b)$ has the minimum risk in most cases. Based on the simulated risk, we can say that the proposed EBG_1 is the best estimator compared to the other proposed estimators. In Table 2, we have presented the 95% confidence intervals (CI) of the proposed estimators. The length of all intervals decreases as the effective sample size m, increases and the Boot-t interval has the minimum length. The length of the highest posterior density (HPD) intervals is always less than the CI for all combinations of (n, m). In addition, we have also observed from Table 2, that the coverage probabilities (CP) of all the proposed estimators are above 90%.

7. Illustrative example

For Illustrative purpose, we have taken a data set of thirty successive March precipitation observations (in inches) originally given by Hinkley (1977). This data set was also used by Banerjee and Bhunia (2022) in the context of the ETIR distribution. To check the superiority of the ETIR distribution based on the complete precipitation data set, we will compare it with other lifetime distributions in the literature, named:

1. Exponentiated inverse Rayleigh distribution (EIR(α_1, β_1) by Rao and Mbwambo (2019);
2. Inverse Rayleigh distribution (IR(α_2)) by Voda (1972);
3. Rayleigh distribution (R(α_3, β_3)) by Dey, Dey, and Kundu (2014).

To select the best distribution among ETIR and its competitors, we considered the following criterion:

1. Estimated log-likelihood($\mathcal{L}^*$), where $\mathcal{L}^* = \log \mathrm{L}$;
2. Akaike information criterion (AIC), where AIC $= 2(k - \mathcal{L}^*)$;
3. Corrected information criterion (AICc), where AICc $=$ AIC$+2 \times p(p+1)/(n-p-1)$;
4. Consistent information criterion (CAIC), where CAIC $= k(1 + log(n)) - 2\mathcal{L}^*$;
5. Bayesian information crietrion (BIC), where BIC $= k(1 + log(n)) - 2\mathcal{L}^*$;
6. Hannan-Quinn information criterion (HQIC), where HQIC $= 2klog(log(n)) - 2\mathcal{L}^*$;
7. The Kolmogorov-Smirnov (KS) statistic is defined as

$$Z = \sup_x \{F_n(x) - F_X(x)\} = max\{D^+, D^-\}$$

Table 2: ML, Bayes and E-Bayes estimates of β with PT-II CBRs censoring under SELF, GELF and LINEX along with coverage probability (CP), CI, Boot-p, Boot-t and HPD intervals and average length (AL) with fixed value, $\beta = 1.5$, $a = 0.90$ and $b = 0.60$

			$\hat{\beta_M}$	$\hat{\beta_S}$	$\hat{\beta_G}$ (CP)		$\hat{\beta_L}$ (CP)		CI	Boot-p	Boot-t	HPD
n	m		(CP)	(CP)	$\delta = 0.5$	$\delta = -0.5$	$\delta = 0.5$	$\delta = -0.5$	AL	AL	AL	AL
20	15	B	1.599145	1.571303	1.551467	1.564768	1.551467	1.581722	(1.134871, 1.943419)	(1.544118,1.654172)	(1.590343,1.607946)	(1.178208, 1.964133)
			(0.98)	(0.98)	(0.99)	(0.98)	(0.98)	(0.97)	0.808548	0.110054	0.017603	0.785925
		EB_1		1.573765	1.553898	1.56722	1.56335	1.56335				
		EB_2		1.57132	1.551483	1.564785	1.560919	1.560919				
		EB_3		1.568875	1.549069	1.56235	1.558489	1.558489				
	18	B	1.58996	1.548444	1.53042	1.542499	1.53042	1.557729	(1.137897, 1.902024)	(1.577401,1.602519)	(1.586688,1.593232)	(1.76075, 1.921324)
			0.94	0.94	0.95	0.94	0.94	0.94	0.764127	0.025118	0.003456	0.745249
		EB_1		1.548035	1.530015	1.542092	1.538773	1.557317				
		EB_2		1.549773	1.531734	1.543824	1.540502	1.559066				
		EB_3		1.551512	1.533453	1.545556	1.54223	1.560815				
30	23	B	1.57899	1.549776	1.53636	1.54534	1.53636	1.556713	(1.200381, 1.857598)	(1.477157,1.680822)	(1.569090,1.588889)	(1.228149, 1.873204)
			(0.98)	(0.98)	(0.99)	(0.98)	(0.97)	(0.98)	0.657217	0.203665	0.019798	0.645056
		EB_1		1.546795	1.533405	1.542368	1.539877	1.55372				
		EB_2		1.548948	1.535539	1.544515	1.542022	1.555881				
		EB_3		1.551101	1.537673	1.546661	1.544167	1.558042				
	27	B	1.569925	1.548988	1.536637	1.544902	1.536637	1.555377	(1.21488, 1.84497)	(1.533078,1.606771)	(1.565641,1.574207)	(1.24115, 1.860526)
			(0.93)	(0.93)	(0.94)	(0.93)	(0.94)	(0.94)	0.63009	0.073693	0.008566	0.619376
		EB_1		1.549076	1.536725	1.54499	1.542693	1.555465				
		EB_2		1.548366	1.53602	1.544282	1.541986	1.554752				
		EB_3		1.547656	1.535316	1.543574	1.541279	1.554039				
40	30	B	1.560142	1.535646	1.525369	1.532243	1.525369	1.540887	(1.234418, 1.805866)	(1.518309,1.601974)	(1.553831,1.566452)	(1.255894, 1.818028)
			0.92	0.92	0.91	0.94	0.94	0.93	0.571448	0.083665	0.00631	0.562134
		EB_1		1.537346	1.527058	1.533939	1.532102	1.542594				
		EB_2		1.535429	1.525153	1.532025	1.530191	1.540669				
		EB_3		1.533511	1.523248	1.530112	1.52828	1.538745				
	36	B	1.542878	1.526634	1.517369	1.523564	1.517369	1.531334	(1.242096, 1.78366)	(1.486321,1.599433)	(1.532872,1.552882)	(1.260658, 1.794297)
			(0.94)	(0.94)	(0.92)	(0.93)	(0.93)	(0.92)	0.541565	0.129314	0.010188	0.533639
		EB_1		1.52659	1.517325	1.523519	1.521893	1.531289				
		EB_2		1.527741	1.518469	1.524669	1.52304	1.532445				
		EB_3		1.528893	1.519614	1.525818	1.524188	1.5336				

Table 3: Risk of the estimators of β with PT-II CBRs censoring under SELF, GELF and LINEX with fixed value, $\beta = 1.5$, $a = 0.90$ and $b = 0.60$

n	m		SELF					GELF								LINEX						
			$\hat{\beta_M}$	$\hat{\beta_S}$	$\hat{\beta_G}$	$\hat{\beta_L}$		$\hat{\beta_M}$	$\hat{\beta_S}$	$\hat{\beta_G}$	$\hat{\beta_L}$	$\hat{\beta_S}$	$\hat{\beta_G}$	$\hat{\beta_L}$		$\hat{\beta_M}$	$\hat{\beta_S}$	$\hat{\beta_G}$	$\hat{\beta_L}$	$\hat{\beta_S}$	$\hat{\beta_G}$	$\hat{\beta_L}$
									$\delta = 0.5$			$\delta = -0.5$					$\delta = 0.5$			$\delta = -0.5$		
20	15	R_S	0.057477	0.055053	0.05541	0.055504	R_G	0.003852	0.002872	0.002847	0.002867	0.002759	0.002727	0.002851	R_L	0.00955	0.00812	0.00752	0.007686	0.007219	0.007069	0.007568
		R_{EBS1}		0.052863	0.052645	0.052788	R_{EBG1}		0.001888	0.001758	0.001781	0.002722	0.002689	0.002816	R_{EBL1}		0.008058	0.007448	0.007619	0.00719	0.006996	0.007503
		R_{EBS2}		0.053878	0.053749	0.053847	R_{EBG2}		0.002851	0.002868	0.002859	0.002744	0.002701	0.002837	R_{EBL2}		0.008098	0.007487	0.007659	0.007194	0.007041	0.007551
		R_{EBS3}		0.053063	0.053019	0.053073	R_{EBG3}		0.002807	0.002853	0.002804	0.002737	0.00271	0.002833	R_{EBL3}		0.008097	0.007476	0.007653	0.007149	0.007034	0.007549
	18	R_S	0.041263	0.040698	0.040203	0.040795	R_G	0.002058	0.002042	0.001971	0.001977	0.001953	0.001932	0.002015	R_L	0.005828	0.005686	0.005313	0.005409	0.005084	0.004986	0.005317
		R_{EBS1}		0.038453	0.038977	0.038563	R_{EBG1}		0.00203	0.00195	0.001966	0.001933	0.001914	0.001992	R_{EBL1}		0.005652	0.005281	0.005376	0.005019	0.004926	0.005244
		R_{EBS2}		0.039659	0.039449	0.039734	R_{EBG2}		0.002037	0.001963	0.001971	0.001939	0.001919	0.001999	R_{EBL2}		0.005684	0.005353	0.005456	0.00504	0.004953	0.005269
		R_{EBS3}		0.039057	0.039119	0.039096	R_{EBG3}		0.002031	0.001956	0.001969	0.001945	0.001926	0.002004	R_{EBL3}		0.005642	0.005304	0.005403	0.005047	0.004945	0.005273
30	23	R_S	0.050006	0.04158	0.039753	0.040253	R_G	0.003141	0.002159	0.002106	0.002112	0.002118	0.002105	0.00216	R_L	0.005558	0.005275	0.005091	0.00517	0.005061	0.004992	0.005219
		R_{EBS1}		0.040124	0.038212	0.03876	R_{EBG1}		0.002071	0.002012	0.002021	0.002027	0.002011	0.002072	R_{EBL1}		0.005109	0.004901	0.004986	0.004876	0.004804	0.005039
		R_{EBS2}		0.04091	0.039019	0.039553	R_{EBG2}		0.00211	0.002054	0.002062	0.002068	0.002054	0.00212	R_{EBL2}		0.005257	0.005001	0.005068	0.004959	0.004888	0.005119
		R_{EBS3}		0.040772	0.038903	0.039427	R_{EBG3}		0.002118	0.002061	0.002069	0.002076	0.002061	0.002112	R_{EBL3}		0.005177	0.004986	0.005084	0.004976	0.004904	0.005138
	27	R_S	0.041991	0.041148	0.039568	0.04004	R_G	0.002094	0.002089	0.002037	0.002044	0.002031	0.002017	0.00207	R_L	0.005188	0.005076	0.005074	0.005097	0.004981	0.004916	0.005129
		R_{EBS1}		0.040298	0.039142	0.039599	R_{EBG1}		0.002074	0.002023	0.00203	0.002018	0.002005	0.002058	R_{EBL1}		0.005004	0.00501	0.005028	0.004935	0.004872	0.005081
		R_{EBS2}		0.040918	0.039419	0.039891	R_{EBG2}		0.002077	0.002026	0.002033	0.002019	0.002005	0.002058	R_{EBL2}		0.005063	0.005057	0.005046	0.004938	0.004874	0.005106
		R_{EBS3}		0.040834	0.039209	0.039672	R_{EBG3}		0.002078	0.002028	0.002035	0.002026	0.002013	0.002064	R_{EBL3}		0.005024	0.005023	0.005084	0.004958	0.004894	0.005085
40	30	R_S	0.022985	0.022837	0.022417	0.022722	R_G	0.001234	0.001216	0.001201	0.001208	0.001215	0.001206	0.00124	R_L	0.002988	0.002934	0.002851	0.002896	0.002868	0.00283	0.00295
		R_{EBS1}		0.022173	0.021893	0.022158	R_{EBG1}		0.001205	0.001178	0.001184	0.001198	0.001184	0.001214	R_{EBL1}		0.002808	0.00278	0.00282	0.002797	0.002762	0.002874
		R_{EBS2}		0.023391	0.022191	0.022339	R_{EBG2}		0.001208	0.001185	0.001185	0.001192	0.001184	0.001216	R_{EBL2}		0.002914	0.002802	0.002845	0.002803	0.002767	0.002883
		R_{EBS3}		0.022782	0.021905	0.022054	R_{EBG3}		0.00121	0.001179	0.001191	0.001193	0.001189	0.001222	R_{EBL3}		0.002906	0.002782	0.002825	0.00282	0.002783	0.002899
	36	R_S	0.020932	0.020247	0.020214	0.020422	R_G	0.001081	0.001045	0.001058	0.001062	0.001054	0.001048	0.001072	R_L	0.002691	0.002603	0.002588	0.00262	0.00255	0.002522	0.002612
		R_{EBS1}		0.019628	0.019053	0.019236	R_{EBG1}		0.001017	0.000997	0.001	0.000992	0.000986	0.001009	R_{EBL1}		0.002506	0.00244	0.002468	0.002401	0.002375	0.00246
		R_{EBS2}		0.019977	0.019559	0.019754	R_{EBG2}		0.001037	0.001024	0.001027	0.001009	0.001003	0.001026	R_{EBL2}		0.002567	0.002504	0.0025	0.002435	0.002409	0.002494
		R_{EBS3}		0.019713	0.019313	0.019496	R_{EBG3}		0.001033	0.001013	0.001016	0.001019	0.001013	0.001037	R_{EBL3}		0.002535	0.002471	0.002534	0.002466	0.002439	0.002527

Table 4: Model comparison metrics for March precipitation data set

Distribution	MLE	logL	AIC	AICc	CAIC	BIC	HQIC	K-S	p-value
ETIR	$\hat{\beta} = 0.8544$	-51.292	104.585	104.728	106.987	105.987	105.0342	0.1366	0.8039
EIR	$\hat{\alpha_1} = 0.1818$, $\hat{\beta_1} = 0.0897$	-106.926	155.795	156.239	160.597	158.597	156.691	0.3873	0.1522
IR	$\hat{\alpha_2} = 0.2084$	-99.974	269.331	269.473	271.732	270.732	269.779	0.7626	6.53×10^{-9}
R	$\hat{\alpha_3} = 0.0157$, $\hat{\beta_3} = 0.2189$	-245.79	304.906	305.049	307.307	306.307	305.354	0.8221	9.23×10^{-11}

where $D^+ = \max_{i=1,\ldots,n} \left\{ \left(\frac{i}{n}\right) - F_X(x_i) \right\}$ and $D^- = \max_{i=1,\ldots,n} \left\{ F_X(x_i) - \left(\frac{i-1}{n}\right) \right\}$, such that the P-value is given by

$$P\text{-value} = \Pr(Z \leq x) = 1 - 2\sum_{i=1}^{\infty} (-1)^{i-1} \exp(-2(ix)^2),$$

where n is the total sample size and p is the number of parameters in the distribution.

The result of the model comparison metric is presented in Table 4. Apart from the highest p-value, the most suitable distribution is the one that yields the lowest values across all other goodness-of-fit statistics. Table 4 shows that the ETIR distribution has the lowest value of all model comparison criterion and has the highest p-value.

We have shown samples that have been generated under the various censoring schemes in the Table 5 under the PT-II CBRs for the real data set. Schemes has been developed at the 100%, 90% and 80% of the total sample size, with $m = 30, 27, 24$.

In the context of an idea linked to failure rate, we explored a graphical technique using Total Time on Test (TTT) plots as a basic indicator see, Aarset (1987). The empirical TTT is given by

$$T(r/n) = \frac{\sum_{i=1}^{r} x_{(i)} + (n-r)x_{(r)}}{\sum_{i=1}^{n} x_{(i)}},$$

where $r = 1, 2, \cdots, n$ and $x_{(r)}$ is the order statistics of the sample. Figure 4 shows the K-S plots and TTT plots for different schemes of the data set. It exhibit that for all these schemes data is also suitable for the considered distribution. From K-S test, it has been verified the suitability of different schemes. Further, we notice that from figure 4, the TTT plots for different schemes of the data set have increasing failure rate. Hence, this data set are used for the further analysis purpose.

Using the data set, we obtained the ML, Bayes, and E-Bayes estimates for different censoring schemes under symmetric and asymmetric loss function. We have also obtained the CI, Boot-p, Boot-t and HPD interval. Moreover, we have obtained the length of intervals. The Newton-Raphson method has been used to compute the MLE of β. The Newton-Raphson method requires initial guess value for the solution of the normal equation as given in equation (10). The guess values for calculating the MLE has been taken with the help of contour plot, see figure 6. Hence, we have obtained the ML estimate of the parameter, which can be seen in Table 6.

In the Bayesian paradigm, when we have no information about the parameters, we use a non-informative prior for the parameters. Therefore the value of hyper parameter for β is taken as $(a = 0.0001, b = 0.0001)$ for the computation of Bayesian estimates. Now we have implemented the MCMC algorithm, the initial guess was often set to the MLE, and various diagnostic plots such as trace and kernel density plots see figure 8 are generated and scrutinized to confirm the convergence of the Markov chain. Table 6 shows, all the estimates

Table 5: The precipitation data set under PT-II CBRs for different censoring schemes with $n = 30$ and $p = 0.5$

(n,m)																
(30,30)	i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
	r	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	x	0.32	0.47	0.52	0.59	0.77	0.81	0.81	0.9	0.96	1.18	1.2	1.2	1.31	1.35	1.43
	i	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
	r	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	x	1.51	1.62	1.74	1.87	1.89	1.95	2.05	2.1	2.2	2.48	2.81	3	3.09	3.37	4.75
(30,27)	i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
	r	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0
	x	0.32	0.52	0.81	0.81	0.9	0.96	1.18	1.2	1.2	1.31	1.35	1.43	1.51	1.62	1.74
	i	16	17	18	19	20	21	22	23	24	25	26	27			
	r	0	0	0	0	0	0	0	0	0	0	0	0			
	x	1.87	1.89	1.95	2.05	2.1	2.2	2.48	2.81	3	3.09	3.37	4.75			
(30,24)	i	1	2	3	4	5	6	7	8	9	10	11	12			
	r	3	2	1	0	0	0	0	0	0	0	0	0			
	x	0.32	0.77	0.9	1.18	1.2	1.2	1.31	1.35	1.43	1.51	1.62	1.74			
	i	13	14	15	16	17	18	19	20	21	22	23	24			
	r	0	0	0	0	0	0	0	0	0	0	0	0			
	x	1.87	1.89	1.95	2.05	2.1	2.2	2.48	2.81	3	3.09	3.37	4.75			

and intervals with their average length that have been obtained for the data set.
Also, from Table 6 we observed, that the E-Bayes estimates for all the schemes and under all the priors under consideration have almost the same estimate for both the symmetric and asymmetric loss functions. The length of all the intervals continues to increase as the effective sample size i.e. m decreases. We can also observe from Table 6 that the Boot-t interval has the least length in most cases. Also, we noticed that for the least mentioned value of m, HPD interval has the least length compared to the other intervals.

Table 6: ML, Bayes and E-Bayes estimates of β with PT-II CBRs censoring under SELF, GELF and LINEX along with, CI, Boot-p, Boot-t and HPD intervals and AL with fixed value $a = 0.90$ and $b = 0.60$ for data set

				$\hat{\beta}_G$		$\hat{\beta}_L$		CI	Boot-p	Boot-t	HPD
(n,m)		$\hat{\beta}_M$	$\hat{\beta}_S$	$\delta = 0.5$	$\delta = -0.5$	$\delta = 0.5$	$\delta = -0.5$	length	length	length	length
(30,30)		0.829313						(0.668792, 0.989833)	(1.12952, 0.959049)	(0.630848, 0.840212)	(0.738049, 1.060323)
	B		0.870806	0.864975	0.86892	0.864975	0.872376	0.321040	2.088566	0.209364	0.322273
	EB_1		1.020184	1.013354	1.017975	1.018318	1.022024				
	EB_2		1.025923	1.019054	1.023701	1.024046	1.027773				
	EB_3		1.031662	1.024754	1.029427	1.029774	1.033522				
(30,27)		0.913318									
	B		0.94621	0.940284	0.94424	0.940284	0.948074	(0.730907, 1.095728)	(1.33246, 1.031708)	(0.626358, 0.924588)	(0.784617, 1.119041)
	EB_1		1.107878	1.10094	1.105572	1.105703	1.110061	0.364821	2.364164	0.298230	0.334423
	EB_2		1.106835	1.099904	1.104531	1.104663	1.109016				
	EB_3		1.105793	1.098868	1.103491	1.103622	1.107971				
(30,24)		0.999414									
	B		1.026424	1.018996	1.023957	1.018996	1.028951	(0.794841, 1.203986)	(1.50956, 1.244843)	(0.626958, 1.025928)	(0.836236, 1.228296)
	EB_1		1.191798	1.183174	1.188934	1.188873	1.194733	0.409144	2.754403	0.398970	0.392060
	EB_2		1.20508	1.19636	1.202184	1.202122	1.208048				
	EB_3		1.218362	1.209546	1.215434	1.215371	1.221362				

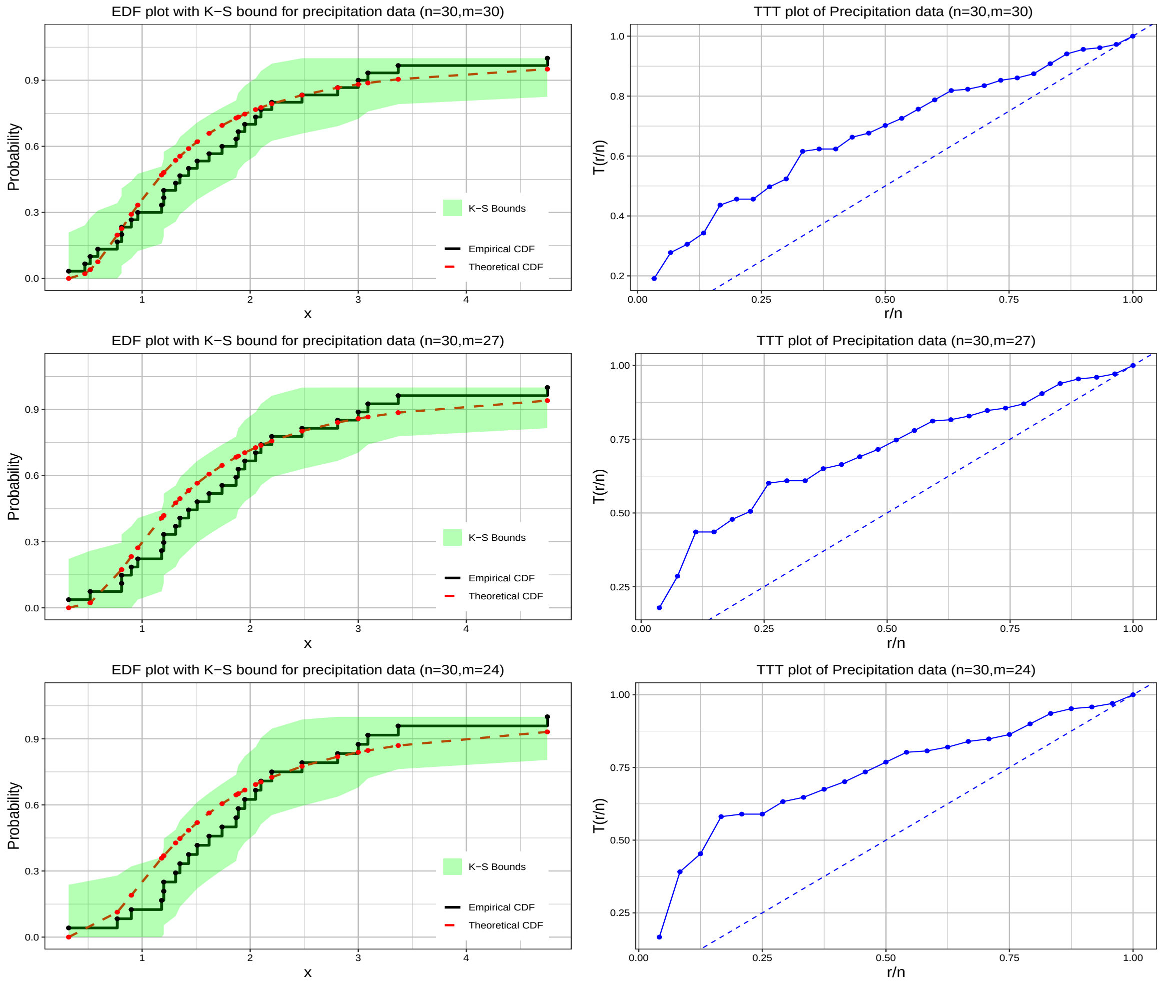

Figure 4: K-S Plots (left column) and TTT Plots (right column) for different schemes of the data set

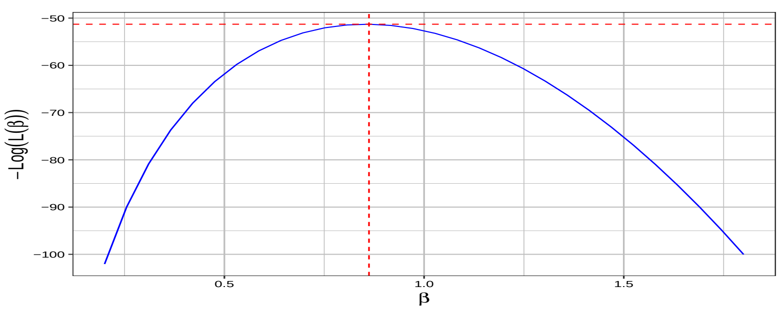

Figure 5: Log-Likelihood plot of β based on data set

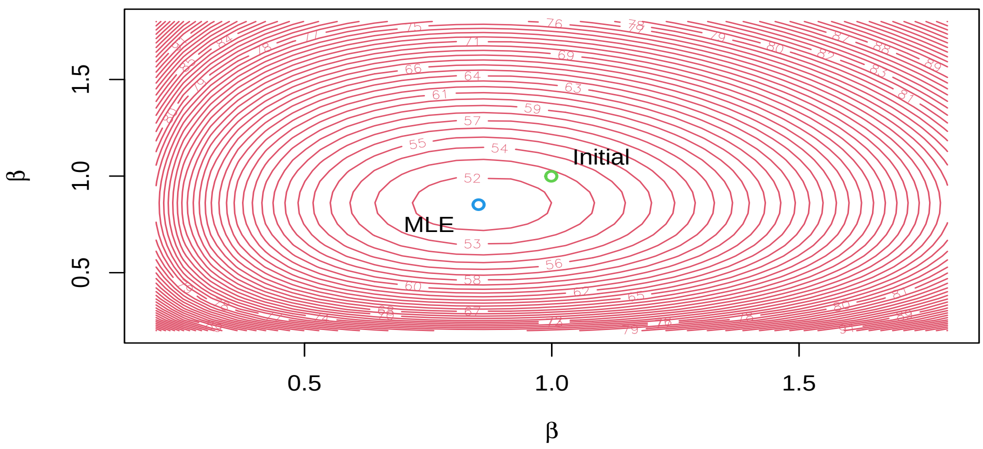

Figure 6: Contour plot of β based on data set

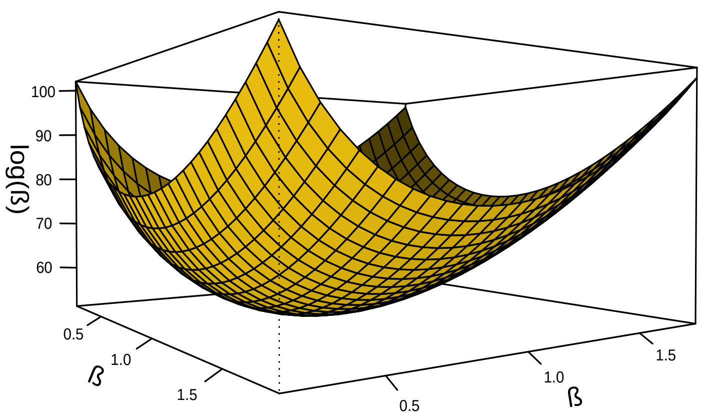

Figure 7: The 3D plot of β based on data set

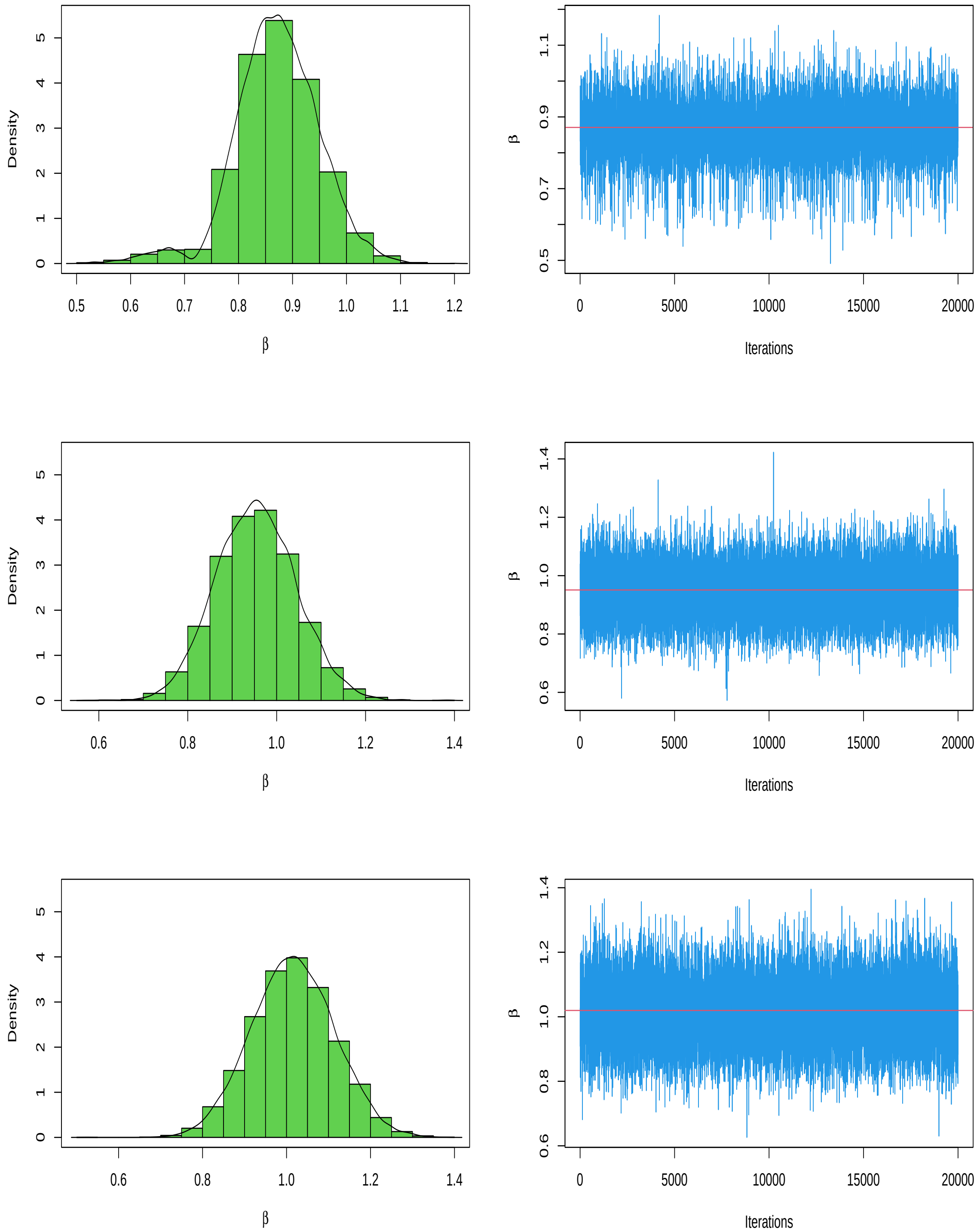

Figure 8: Density and trace plots for different schemes (first row:(n=30,m=30), second row (n=30,m=27) and third row (n=30,m=24)) for data set

8. Concluding remarks

Based on the discussion of the results presented in the previous sections, we may conclude that out of all proposed estimators, the $\hat{\beta}_{EBGi} \quad i = 1, 2, 3$ comes out be the best for $\delta > 0$ and $\delta < 0$. In addition, in most cases the numerical value of the simulated risk of $\hat{\beta}_{EBG1}$ is minimum compared to the other estimators. Thus, the proposed $\hat{\beta}_{EBG1}$ is recommended under the symmetric and asymmetric loss functions. Moreover, real data set has been used to validate the proposed methodology. As the distribution is one parameter and flexible in nature, so it can be used for further research in reliability, survival analysis, engineering, medical etc.

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Affiliation:

Manoj Kumar
Department of Statistics
University of Delhi
Delhi-110007, India
E-mail: manustats@gmail.com

Satya Prakash Mishra
Department of Statistics
Central University of Haryana
Mahendergarh-123031, India
E-mail: satymishra77@gmail.com

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