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Optimum Reliability Acceptance Sampling Plan for Burr Type-XII Distribution under Generalized Hybrid Censoring Scheme

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Abstract

This manuscript focuses on reliability acceptance sampling plans applied to the Burr type-XII distribution within the framework of the generalized hybrid censoring scheme. This study involves determining the sample size (n) and acceptance constant (k_1) for given values of producer's and consumer's risks, utilizing the asymptotic normality theory of maximum likelihood estimators for model parameters. Additionally, a Monte-Carlo simulation study is conducted to assess whether the reliability acceptance sampling plans adhere to specified risk levels for finite sample sizes. Optimum reliability acceptance sampling plans are obtained by employing a variance minimization criterion under cost constraints. We have provided the algorithm for the computation of sample size (n) and acceptance constant (k_1). Furthermore, the algorithms for computing the optimum reliability acceptance sampling plans under generalized hybrid type-I censoring scheme and generalized hybrid type-II censoring scheme are also provided.

Keywords: Burr type-XII distribution, operating characteristic curve, reliability acceptance sampling plans, generalized hybrid type-I censoring scheme, generalized hybrid type-II censoring scheme, Monte-Carlo simulations.

1. Introduction

With the rapid progress in manufacturing technology and heightened consumer expectations, suppliers are facing growing pressure to deliver exceptional products that align with market demands. To maintain competitiveness, effective quality assurance is crucial and inspection plays a pivotal role in evaluating product quality. The reliability of the product can be assessed through a suitable life test after selecting a sample from the lot. Highly reliable products are deemed satisfactory when other product features remain constant. In reliability acceptance sampling plans (RASPs), we focus on assessing the reliability of the product. The concept of RASPs was introduced by Epstein and Sobel (1955). Censoring typically occurs when complete information regarding unit failures is not available, often due to partial knowledge. The primary aim of censoring is to minimize costs, time and manpower. In some domains, censoring may inadvertently occur, such as in industrial experimentation for test item breakage, patient dropouts in clinical trials or bio-assay dropouts. Various types of censoring schemes exist in the literature for reliability analysis. The most common ones are type-I and type-II censoring. Type-I censoring may result in few or no failures within the predetermined time frame, while type-II censoring may prolong failure times, particularly for highly reliable items. In life testing experiments, data are often censored, with type-I and type-II censoring being common practice. The combination of these, known as type-I hybrid censoring, was introduced by Epstein (1954), while Childs, Chandrasekar, Balakrishnan, and Kundu (2003) introduced type-II hybrid censoring. These hybrid schemes have gained significant attention recently, as evidenced by Balakrishnan and Kundu (2013), a review article on the subject. It is worth noting that the aforementioned censoring schemes only remove units at the endpoints of the experiments. Cohen (1963) was the first to investigate a more general censoring scheme called progressive type-II censoring. In this scheme, for a fixed sample size n and an effective sample size m are chosen. This censoring process begins with n items. Upon the first failure, *one* item are randomly selected from the remaining $n-1$ items and removed. This process repeats at each subsequent failure, i^{th} failure items are chosen from the $n-i+1$, and this process will continued until the m^{th} failure occur, at which point all remaining items are removed, and the experiment concludes. Childs, Chandrasekar, and Balakrishnan (2008) first investigated the type-I and type-II progressive hybrid censoring scheme. However, hybrid censoring may not be very effective for highly reliable units. To address the limitations of existing censoring schemes, Chandrasekar, Childs, and Balakrishnan (2004) introduced generalized type-I hybrid censoring scheme (GHCS-I) and generalized type-II hybrid censoring scheme (GHCS-II). Instead of discussing the various types of censoring for the estimation and prediction and competing risk data purpose etc., few work has also been discussed in the field of RASP like Balasooriya, Saw, and Gadag (2000) studied about the progressive censored sampling plan for the Weibull distribution, Kumar and Ramyamol (2016) studied in more detail for the design of RASP for the exponential distribution, Bhattacharya, Pradhan, and Dewanji (2015) has discussed about the computation of the optimum reliability acceptance sampling plan in the presence of hybrid censoring. Budhiraja and Pradhan (2019) has discussed about the optimum RASP under progressive interval type-I censoring scheme, optimum RASP under generalized hybrid type-I censoring for products under warranty has also been discussed by Chakrabarty, Chowdhury, and Roy (2020).

There exists a noticeable research gap concerning the application of RASPs under GHCS-I and GHCS-II for the Burr type-XII distribution. To the best of our knowledge, no prior work has addressed RASPs under GHCS-I and GHCS-II for this distribution. Therefore, the objective of this paper is to determine the optimum values of the sample size (n) and acceptance constant (k_1), subject to budget constraints, for the Burr type-XII distribution under both GHCS-I and GHCS-II. Additionally, a Monte-Carlo simulation has been conducted to assess the robustness of the RASPs under both GHCS-I and GHCS-II. The developed RASPs are also evaluated through the construction of the operating characteristic (OC) curve, and both the producer's risk ($\hat{\alpha}$) and consumer's risk ($\hat{\beta}$) are calculated.

The structure of the article is organized as follows: Section 2 provides a description of the Burr type-XII distribution, Section 3 introduces a RASP under GHCS-I, while Section 4.1 presents an algorithm for determining the sample size (n) and the acceptance constant (k_1). An illustration of the concepts is provided in Section 4.2, followed by a discussion of a Monte Carlo sampling plan in Section 4.3. Section 6 outlines the optimum sampling plan. Furthermore, Section 5 details the algorithm for determining the sample size (n) and acceptance constant (k_1) in a RASP under GHCS-II. A detailed comparison has been discussed in Section 7. The article concludes with Section 8.

2. Burr type-XII distribution

The Burr type-XII distribution is a probability distribution widely utilized in statistics for modeling continuous random variables introduced by Burr (1942), which comprises twelve distributions also referred to as the Singh-Maddala distribution or the generalized log-logistic distribution. It finds extensive application in various fields. In reliability analysis, survival analysis, and actuarial science, this distribution is employed to model the time-to-failure of systems or components. Additionally, it is used in finance to model the distribution of portfolio returns and in hydrology for modeling the frequency of extreme events. Kumar (2017) studied in detail various statistical properties of the Burr type-XII distribution. The probability density function (PDF) and cumulative distribution function (CDF) of this distribution is given by

$$\begin{aligned} h(x,\nu,\eta,\gamma) &= \left(\frac{\nu\eta}{\gamma}\right)\left(\frac{x}{\gamma}\right)^{\eta-1}\left(1+\left(\frac{x}{\gamma}\right)^{\eta}\right)^{-\nu-1}; \quad x>0,\ \nu>0,\ \eta>0,\ \gamma>0. \\ H(x,\nu,\eta,\gamma) &= 1-\left(1+\left(\frac{x}{\gamma}\right)^{\eta}\right)^{-\nu}; \quad x>0,\quad \nu>0,\quad \eta>0,\quad \gamma>0. \end{aligned} \tag{1}$$

Where ν and η are shape parameters and γ is the scale parameter. The Burr type-XII distribution is uni-modal. This distribution has a non-monotone hazard function for different values of the shape parameters and scale parameter. It is commonly used for the household income.

This distribution simplifies to the Pareto type-II (Lomax distribution) when η is equal to 1 and ν is equal to 1. The Burr distribution, also known as the Fisk distribution, which is a particular form of the log-logistic distribution. For simplicity point of view, we make a transformation such that $Y = \log(X)$, instead of using the true lifetime of the X. Then, Burr type-XII distribution reduces to generalized log-logistic distribution. Let the PDF, CDF and hazard function of the distribution for the random variable Y denoted by $f(y)$, $F(y)$ and $g(y)$ respectively, given by

$$\begin{aligned} f(y) &= \frac{\nu e^{(\frac{y-\mu}{\sigma})}}{\sigma(1+e^{(\frac{y-\mu}{\sigma})})^{\nu+1}}; \quad y\in\mathbb{R},\quad \mu\in\mathbb{R},\quad \nu>0,\quad \sigma>0. \\ F(y) &= 1-(1+e^{(\frac{y-\mu}{\sigma})})^{-\nu}; \quad y\in\mathbb{R},\quad \mu\in\mathbb{R},\quad \nu>0,\quad \sigma>0. \\ g(y) &= \frac{\nu e^{(\frac{y-\mu}{\sigma})}}{\sigma(1+e^{(\frac{y-\mu}{\sigma})})}; \quad y\in\mathbb{R},\quad \mu\in\mathbb{R},\quad \nu>0,\quad \sigma>0. \end{aligned} \tag{2}$$

Where $\mu = \log(\gamma)$ is location parameter and $\sigma = \frac{1}{\eta}$ is the scale parameter.

3. RASPs under GHCS-I

Suppose we have n items in the experiment. Let $X_{1:m:n}, X_{2:m:n}, \ldots, X_{m:m:n}$ be identically and independently distributed (i.i.d.) random variables following the Burr type-XII distribution with CDF $H(.)$ and PDF $h(.)$. For simplicity, we take $Y = \log(X)$ instead of the true lifetime X. Let $Y_{1:m:n}$, $Y_{2:m:n}$, $\ldots$, $Y_{m:m:n}$ also be i.i.d. random variables with CDF $F(.)$ and PDF $f(.)$. For the GHCS-I, we fix two numbers of failures in advance, denoted as k and m, where $1 \le k < m \le n$. The test is terminated at time $T^* = \max(Y_{k:m:n}, \min(Y_{m:m:n}, T))$, where $Y_{i:m:n}(i = 1, 2, \ldots, m)$ denotes the failure time of the i^{th} unit. This scheme necessitates a minimum of k failures in the experiment. If the experiment is terminated at time T, with the corresponding number of failures denoted as D, three main cases arise, illustrated as under:

Case-I:	$Y_{1:m:n}, Y_{2:m:n}, \ldots, Y_{k:m:n}$	if	$T < Y_{k:m:n}$.
Case-II:	$Y_{1:m:n}, Y_{2:m:n}, \ldots, Y_{D:m:n}$	if	$Y_{k:m:n} < T < Y_{m:m:n}$.
Case-III:	$Y_{1:m:n}, Y_{2:m:n}, \ldots, Y_{m:m:n}$	if	$T > Y_{m:m:n}$.

It is important to note that for Case-I, the experiment terminates at the k^{th} failure, resulting in unobserved data points $Y_{k+1:m:n}, Y_{k+2:m:n}, \ldots, Y_{m:m:n}$. Similarly, in Case-II, the experiment concludes at time T, with D failures observed, hence omitting data points $Y_{D+1:m:n}$, $Y_{D+2:m:n}$, $\ldots$, $Y_{m:m:n}$. Lastly, for Case-III, the experiment ends upon observing m failures. The likelihood equation for the i^{th} case under GHCS-I, denoted by L_{1i}, is expressed as:

$$L_{1i} = K_i \prod_{j=1}^{G_i} w(i,j); \qquad i = 1, 2, 3.$$

Where,

$$\begin{array}{lll} w(1,j) = f(y_{j:m:n})\left(1 - F(y_{k:m:n})\right)^{n-k}, & G_1 = k-1, & K_1 = {}^nP_k; \\ w(2,j) = f(y_{k:m:n})\left(1 - F(T)\right)^{n-D}, & G_2 = D, & K_2 = {}^nP_D; \\ w(3,j) = f(y_{j:m:n})\left(1 - F(y_{m:m:n})\right)^{n-m}, & G_3 = m, & K_3 = {}^nP_m. \end{array}$$

Where, ${}^nP_i = \frac{n!}{(n-i)!}$.

4. Designing sampling plan

Lot will be accepted or rejected depends on the quality of the items. A lot of lot-quality (p) will accepted be determined by OC curve serves as a metric for evaluating the effectiveness of a sampling plan, illustrating the relationship between the probability of accepting a lot and the proportion of non-conforming items within it. This curve is derived from asymptotic distribution theory, which allows for the analysis of large sample sizes and provides insights into the plan's performance.

$$\frac{T' - (\mu - k_1\sigma)}{\sqrt{\text{AsVar}[T']}} \sim N(0,1)$$

Where, $T' = \hat{\mu} - k_1\hat{\sigma}$ and k_1 be the acceptance constant. Then the standardized variate is given by

$$W = \frac{(T' - (\mu - k_1\sigma))\sqrt{n}}{\sqrt{V}};$$

where, $V = n \cdot \text{AsVar}[T'] = \Sigma_{22} - 2k_1\Sigma_{23} + k_1^2\Sigma_{33}$.

The OC curve is the probability of accepting the lot of incoming lot quality p, denoted by $L(p)$, and given by

$$L(p) = \text{Prob}[T' \geq L'] = 1 - \Phi\left[\frac{\sigma(u_p + k_1)\sqrt{n}}{\sqrt{V}}\right]$$

against the fraction non-conforming p, where $u_p = \frac{L'-\mu}{\sigma}$ is the quantile function of the standard logistic distribution corresponding to fraction non-conforming p and $\Phi(\cdot)$ is the distribution function of standard normal variate. Therefore, following equations have to be solved for k_1 and n in order to obtain an optimum sampling plan for two given points $(p_\alpha, 1-\alpha)$ and (p_β, β) on the OC curve, where p_α be the proportion of defective in acceptance quality level (AQL) and p_β be the proportion of defective in lot tolerance proportion defective (LTPD).

$$\begin{aligned} z_\alpha - \frac{\sigma(u_{p_\alpha} + k_1)\sqrt{n}}{\sqrt{V}} &= 0 \\ z_{1-\beta} - \frac{\sigma(u_{p_\beta} + k_1)\sqrt{n}}{\sqrt{V}} &= 0. \end{aligned} \tag{3}$$

Where, u_{p_α} and u_{p_β} denote the the quantiles of the log-lifetime distribution and z_α and z_β denotes the quantiles of standard normal distribution. Thus on solving Eq.(3), we get

$$k_1 = \frac{z_{1-\beta}u_{p_\alpha} - z_\alpha u_{p_\beta}}{z_\alpha - z_{1-\beta}} \tag{4}$$

and

$$\Psi(n) = \frac{1}{\sigma^2}\left(\frac{z_\alpha - z_{1-\beta}}{u_{p_\alpha} - u_{p_\beta}}\right)^2 \Big(\Sigma_{22}(n,m,k,T) - 2k_1\Sigma_{23}(n,m,k,T) + k_1^2\Sigma_{33}(n,m,k,T)\Big). \quad (5)$$

Where, $\Sigma_{22} = \mathrm{Var}(\hat{\mu})$, $\Sigma_{23} = \mathrm{Cov}(\hat{\mu},\hat{\sigma})$ and $\Sigma_{33} = \mathrm{Var}(\hat{\sigma})$. The value of k_1 can be obtained by Eq.(4) and value of the sample size (n) is the solution of the Eq.(5), if $\Psi(n) \leq 1$ and $\Psi(n+1) > 1$ then n is the solution, if $\Psi(n) > 1$ then increase n by *one* else decrease n by *one*. For more details see Schneider (1989) and Sen, Bhattacharya, Tripathi, and Pradhan (2018).

4.1. Algorithm for the determination of n and k_1 under GHCS-I

An appropriate algorithm for the determination of sample size (n) and acceptability constant (k_1) under GHCS-I censored RASP for the given value of the parameters q, k, m and T have been provided, which consists in following steps:

Step 1: On the OC curve, choose two points $(p_\alpha, 1-\alpha)$ and (p_β, β), then calculate the acceptability constant (k_1) using Eq.(4).

Step 2: Take any initial guess of n.

Step 3: Calculate $m = \lfloor (1-q)\,n \rfloor$ and $k = \lfloor (1-q_1)\,m \rfloor$ such that $k \leq m$ are the greatest integers less than or equal to n. Then compute the value of $\Psi(n)$ where,

$$\Psi(n) = \frac{1}{\sigma^2}\left(\frac{z_\alpha - z_{1-\beta}}{u_{p_\alpha} - u_{p_\beta}}\right)^2 (\Sigma_{22}(n,m,k,T) - 2k_1\Sigma_{23}(n,m,k,T) + k_1^2\Sigma_{33}(n,m,k,T)) \leq 1.$$

The quantities Σ_{22}, Σ_{23} and Σ_{33} are computed from the dispersion matrix. Note that $\Psi(n)$ is a decreasing function of n.

Step 4: If n satisfies $\Psi(n) \leq 1$ and $\Psi(n+1) > 1$, then sample size (n) is the solution. Otherwise, if $\Psi(n) > 1$, increase sample size (n) by *one*, and if $\Psi(n) < 1$, decrease sample size (n) by *one*, go to Step 3.

4.2. Illustration

In most scenarios, the values of p_α and p_β are predetermined through mutual agreement between the consumer and producer. To establish the appropriate sample size (n) and acceptance constant (k_1), we devise the sampling plan using (p_α, p_β) pairs of $(0.00041, 0.0180)$, $(0.00654, 0.0426)$, $(0.0109, 0.0535)$, $(0.02090, 0.0742)$ and $(0.03190, 0.09420)$. These pairs are chosen to align with the specifications outlined in $MIL - STD - 105D$ by Pabst (1967). $MIL - STD - 105D$ is a United States defense standard that delineates procedures and tables for various techniques utilized in defense department applications, widely adopted beyond military procurement contexts. Table 1 presents the computed values of n and k_1 under GHCS-I for specified values of (α, β), considering six levels of the censoring fraction $q = 0.1(0.1)0.6$.

4.3. Monte-Carlo simulation for sampling plan

To formulate the sampling plan, we utilize the asymptotic distribution of the statistic $T' = \hat{\mu} - k_1\hat{\sigma}$, contingent upon the large sample characteristics of the estimators. Thus, a Monte-Carlo simulation is imperative to validate whether the sampling plans conform to the predetermined risks (consumer's risk and producer's risk) for finite sample sizes. For each sampling plan computed as listed in Table 1, we conduct a simulation experiment. With fixed parameters $n, q, y, \alpha, \beta, p_\alpha$ and p_β, we generate datasets of size n, derive $\hat{\mu}$ and $\hat{\sigma}$ from these datasets, and evaluate if the lot is accepted under the condition $\hat{\mu} - k_1\hat{\sigma} > L'$, where $L' = F_Y^{-1}(p_\alpha)$ and also for $L' = F_Y^{-1}(p_\beta)$.

Table 1: Estimation of n and k_1 under GHCS-I for p_α and p_β to match with $MIL-STD-105D$

			n							
(α,β)	p_α	p_β	$q \rightarrow$	0.1	0.2	0.3	0.4	0.5	0.6	k_1
$(0.05, 0.05)$	0.00041	0.0184		2	3	3	5	9	12	4.383
	0.00654	0.0426		9	12	16	23	37	53	2.852
	0.0109	0.0535		13	18	22	32	49	74	2.529
	0.0209	0.0742		21	28	36	51	83	116	2.094
	0.0319	0.0942		29	39	49	70	111	157	1.795
$(0.05, 0.10)$	0.00041	0.0184		2	3	4	6	11	15	4.597
	0.00654	0.0426		11	16	20	29	46	67	2.958
	0.0109	0.0535		16	22	28	40	61	93	2.619
	0.0209	0.0742		26	35	45	64	107	146	2.168
	0.0319	0.0942		36	49	62	89	155	198	1.858

For each sampling plan, we generate $10,000$ samples following the GHCS-I data for Burr type-XII distribution with $\mu = 0$ and $\sigma = 1$, and perform the aforementioned check each time. The proportion of rejections when $L' = F_Y^{-1}(p_\alpha)$ should approximate α, while the proportion of acceptances when $L' = F_Y^{-1}(p_\beta)$ should approximate β. The estimated producer's and consumer's risks obtained through this simulation approach are denoted by $\hat{\alpha}$ and $\hat{\beta}$, respectively. The outcomes are presented in Table 2 shows the simulated value of the producer's risk ($\hat{\alpha}$) and consumer's risk ($\hat{\beta}$) under GHCS-I for specified values of (α, β), considering six levels of the censoring fraction $q = 0.1(0.1)0.6$.

5. RASPs under GHCS-II

RASP can also be done for the Burr type-XII distribution under GHCS-II. The GHCS-II described as follows: Fix $m \in (1, 2, \ldots, n)$ and $T_1, T_2 \in (0, \infty)$ such that $T_1 < T_2$. If the m^{th} failure occurs before time T_1, terminate the experiment at T_1; if the m^{th} failure occurs between T_1 and T_2, terminate at $X_{m:m:n}$; otherwise, the experiment is terminated at T_2. This scheme modifies the hybrid type-II censoring scheme (HCS-II) by ensuring the experiment will not exceed T_2. Therefore, T_2 represents the absolute maximum duration that the researcher is willing to allow the experiment to continue.

Similarly, the likelihood equation for the i^{th} case under GHCS-II, denoted by L_{2i}, is given as follows:

$$L_{2i} = K_i \prod_{j=1}^{G_i} w(i,j); \qquad i = 1, 2, 3.$$

Where,

$$\begin{array}{lll} w(1,j) = f(y_{j:m:n})\,(1 - F(T_1))^{n-D_1}\,, & G_1 = D_1, & K_1 = {}^nP_{D_1}; \\ w(2,j) = f(y_{k:m:n})\,(1 - F(y_{m:m:n}))^{n-m}\,, & G_2 = m, & K_2 = {}^nP_m; \\ w(3,j) = f(y_{j:m:n})\,(1 - F(T_2))^{n-D_2}\,, & G_3 = D_2, & K_3 = {}^nP_{D_2}. \end{array}$$

Where, D_1 and D_2 are the number of failures upto time T_1 and T_2 respectively. The values of the sample size (n) and acceptance constant (k_1) are reported in Table 3 for the different level of censoring fractions $q = 0.1(0.1)0.6$. The tabulated value depict for the given specification that (p_α, p_β). Also it is observed that sample size is smaller for the specification (p_α, p_β) = $(0.00041, 0.0180)$ compared to other specifications. It is noticed that sample size (n) is smaller for $(\alpha, \beta) = (0.05, 0.10)$ compared to other (α, β). Smaller sample sizes are required for the acceptability of the lot under GHCS-II in comparison to GHCS-I.

Table 2: Estimation of α and β under GHCS-I for given n, q, q_1, k_1, $T = 4.5$, p_α, p_β

		$\alpha = 0.05$, $\beta = 0.05$			$\alpha = 0.05$, $\beta = 0.10$		
(p_α, p_β)	q	k_1	n	$(\hat{\alpha}, \hat{\beta})$	k_1	n	$(\hat{\alpha}, \hat{\beta})$
(0.00041, 0.0184)	0.1	4.597	2	(0.054, 0.149)	4.383	2	(0.056, 0.049)
	0.2		3	(0.053, 0.115)		3	(0.054, 0.045)
	0.3		4	(0.051, 0.121)		3	(0.053, 0.044)
	0.4		6	(0.057, 0.112)		5	(0.051, 0.042)
	0.5		11	(0.049, 0.134)		9	(0.048, 0.041)
	0.6		15	(0.045, 0.145)		12	(0.047, 0.041)
(0.00654, 0.0426)	0.1	2.958	11	(0.049, 0.124)	2.852	9	(0.053, 0.050)
	0.2		16	(0.049, 0.119)		12	(0.050, 0.054)
	0.3		20	(0.046, 0.124)		16	(0.047, 0.055)
	0.4		29	(0.048, 0.134)		23	(0.044, 0.058)
	0.5		46	(0.043, 0.111)		37	(0.043, 0.059)
	0.6		67	(0.042, 0.109)		53	(0.041, 0.059)
(0.0109, 0.0535)	0.1	2.619	16	(0.046, 0.113)	2.529	13	(0.055, 0.052)
	0.2		22	(0.057, 0.129)		18	(0.053, 0.048)
	0.3		28	(0.053, 0.132)		22	(0.051, 0.046)
	0.4		40	(0.052, 0.138)		32	(0.049, 0.044)
	0.5		61	(0.051, 0.126)		49	(0.046, 0.042)
	0.6		93	(0.049, 0.118)		74	(0.045, 0.039)
(0.0209, 0.0742)	0.1	2.168	26	(0.052, 0.117)	2.094	21	(0.058, 0.053)
	0.2		35	(0.055, 0.112)		28	(0.055, 0.046)
	0.3		45	(0.051, 0.115)		36	(0.053, 0.044)
	0.4		64	(0.047, 0.110)		51	(0.050, 0.043)
	0.5		107	(0.041, 0.116)		83	(0.047, 0.042)
	0.6		146	(0.048, 0.108)		116	(0.041, 0.040)
(0.0319, 0.0942)	0.1	1.859	36	(0.053, 0.105)	1.795	29	(0.052, 0.056)
	0.2		49	(0.054, 0.119)		39	(0.047, 0.058)
	0.3		62	(0.046, 0.123)		49	(0.051, 0.046)
	0.4		89	(0.043, 0.114)		70	(0.048, 0.043)
	0.5		155	(0.042, 0.128)		111	(0.045, 0.044)
	0.6		198	(0.041, 0.109)		157	(0.042, 0.040)

Table 3: Estimation of n and k_1 under GHCS-II for p_α and p_β to match with $MIL-STD-105D$

			n							
(α,β)	p_α	p_β	$q \rightarrow$ 0.1	0.2	0.3	0.4	0.5	0.6		k_1
$(0.05, 0.05)$	0.00041	0.0184	1	1	1	2	2	2		3.243
	0.00654	0.0426	1	2	4	9	11	13		1.468
	0.0109	0.0535	1	3	6	13	16	19		1.302
	0.0209	0.0742	2	5	10	22	28	32		0.603
	0.0319	0.0942	3	7	14	31	40	46		0.262
$(0.05, 0.10)$	0.00041	0.0184	1	1	1	2	2	2		3.012
	0.00654	0.0426	1	2	3	7	8	11		1.376
	0.0109	0.0535	1	2	5	11	13	15		1.030
	0.0209	0.0742	2	4	8	18	23	26		0.523
	0.0319	0.0942	3	6	11	25	29	37		0.191

In GHCS-I, the precision of the inspection plan under GHCS-II are checked by Monte-Carlo simulation. We generate $10,000$ samples from the Burr type-XII distribution under GHCS-II for the robustness of the sampling plan. Table 3 demonstrates the computed values of n and k_1 under GHCS-II for specified values of (α,β), considering six levels of the censoring fraction $q = 0.1(0.1)0.6$. In Table 4, we calculated the values of producer's risk $(\hat{\alpha})$ and consumer's risk $(\hat{\beta})$ under GHCS-II for specified values of (α,β), considering six levels of the censoring fraction $q = 0.1(0.1)0.6$. Next, we obtain the RASPs under GHCS-II as in GHCS-I. i.e.

$$\begin{aligned}\underset{(n,m,T_1,T_2)}{\text{minimize}} \int_0^1 \text{Var}(\ln \hat{X}_p)dp \quad \text{subject to} \quad & Tc(n,m,T_1,T_2) = C_F E(D) + C_E E(\xi) \leq C_b, \\ & \Psi(n) = 1, \quad n,\ m \in \mathbb{N} \quad \text{and} \quad T_1,\ T_2 > 0.\end{aligned} \tag{6}$$

The expression for the $E(D)$ and $E(\xi)$ are given by

$$\begin{aligned} E(D) &= nF_X(T_1;\theta) + \sum_{i=1}^{m}\{F_{i:n}(T_2;\theta) - F_{i:n}(T_1;\theta)\} \\ E(\xi) &= T_2 - \int_{T_1}^{T_2} F_{m:n-1}(x;\theta). \end{aligned} \tag{7}$$

5.1. Algorithm for the determination of n and k_1 under GHCS-II

To determine the optimum solution under GHCS-II censored RASPs for provided parameters q, m, T_1 and T_2, the procedure is outlined as follows:

Step 1: On the OC curve, choose two points $(p_\alpha, 1-\alpha)$ and (p_β, β), then calculate the acceptability constant (k_1) using Eq.(4).

Step 2: Take any initial guess of n.

Step 3: Solve the constrained optimization problem in Eq.(6).

Step 4: If n satisfies $\Psi(n) \leq 1$ and $\Psi(n+1) > 1$, then sample size (n) is the solution. Otherwise, if $\Psi(n) > 1$, increase sample size (n) by *one*, and if $\Psi(n) < 1$, decrease sample size (n) by *one*, go to Step 3.

6. Optimum RASPs under GHCS-I and GHCS-II

For determining an acceptance sampling plan under GHCS, given $(p_\alpha, 1-\alpha)$ and (p_β, β), one has the option to choose k_1 from Eq.(4), while n needs to be solved for from Eq.(5).

Table 4: Estimation of α and β under GHCS-II for given n, q, q_1, k_1, $T_1 = 4.5$, $T_2 = 7.5$ p_α, p_β

		$\alpha = 0.05$, $\beta = 0.05$			$\alpha = 0.05$, $\beta = 0.10$		
(p_α, p_β)	q	k_1	n	$(\hat{\alpha}, \hat{\beta})$	k_1	n	$(\hat{\alpha}, \hat{\beta})$
(0.00041,0.0184)	0.1	3.243	1	(0.051, 0.214)	3.012	1	(0.046, 0.039)
	0.2		1	(0.048, 0.105)		1	(0.044, 0.042)
	0.3		1	(0.052, 0.132)		1	(0.043, 0.041)
	0.4		2	(0.053, 0.192)		2	(0.048, 0.042)
	0.5		2	(0.047, 0.234)		2	(0.043, 0.040)
	0.6		2	(0.043, 0.155)		2	(0.048, 0.044)
(0.00654,0.0426)	0.1	1.468	1	(0.047,0.144)	1.468	1	(0.051,0.041)
	0.2		2	(0.047, 0.124)		2	(0.052, 0.053)
	0.3		4	(0.042, 0.113)		3	(0.046, 0.052)
	0.4		9	(0.049, 0.143)		7	(0.051, 0.050)
	0.5		11	(0.046, 0.101)		8	(0.042, 0.060)
	0.6		13	(0.054, 0.117)		11	(0.054, 0.051)
(0.0109,0.0535)	0.1	1.302	1	(0.056, 0.109)	1.03	1	(0.052, 0.047)
	0.2		3	(0.058, 0.138)		2	(0.048, 0.046)
	0.3		6	(0.057, 0.122)		5	(0.050, 0.041)
	0.4		13	(0.053, 0.128)		11	(0.045, 0.034)
	0.5		16	(0.052, 0.127)		13	(0.049, 0.041)
	0.6		19	(0.046, 0.125)		15	(0.055, 0.049)
(0.0209,0.0742)	0.1	0.603	2	(0.052, 0.117)	0.523	2	(0.058, 0.053)
	0.2		5	(0.052, 0.212)		4	(0.054, 0.043)
	0.3		10	(0.056, 0.123)		8	(0.054, 0.043)
	0.4		22	(0.049, 0.106)		18	(0.053, 0.040)
	0.5		28	(0.046, 0.122)		23	(0.054, 0.047)
	0.6		32	(0.053, 0.119)		26	(0.039, 0.037)
(0.0319, 0.0942)	0.1	0.262	3	(0.052, 0.115)	0.191	3	(0.054, 0.052)
	0.2		7	(0.051, 0.117)		6	(0.043, 0.051)
	0.3		14	(0.056, 0.116)		11	(0.054, 0.043)
	0.4		31	(0.044, 0.217)		25	(0.045, 0.043)
	0.5		40	(0.048, 0.124)		29	(0.057, 0.034)
	0.6		46	(0.052, 0.112)		37	(0.046, 0.041)

Since RASPs are computed using the maximum likelihood estimators (MLEs) of the model parameters, their performance depends on the accuracy of these estimators. Simultaneously, the cost of conducting a life test is crucial for the industry. Considering these factors, we aim to determine the optimal values of (n, m, k, T) by minimizing a variance measure for the estimated parameters, subject to a cost constraint. In practical scenarios, the total budget for conducting a life test is limited. Therefore, in this study, we take the optimum values of the decision variables (n, m, k, T) that minimize a variance measure while adhering to a budget constraint and specified producer's and consumer's risks. We adopt the variance measure defined as $\int_0^1 \mathrm{Var}(\ln \hat{X}_p)dp$ (refer to Bhattacharya, Pradhan, and Dewanji (2016)), where $\hat{X}_p$ is the MLE of X_p, the p^{th} quantile of X.

Let C_F denote the cost of failed items and C_E represent the cost per unit duration of the experiment. Then, the total cost associated with the experiment is $Tc(n, m, k, T) = C_F E(D) + C_E E(\xi)$, where $E(D)$ and $E(\xi)$ denote the expected number of failures and the expected duration of the experiment, respectively. Assume C_b is the cost of conducting the experiment. The optimum design problem is then formulated as follows:

$$\begin{aligned} \underset{(n,m,k,T)}{\text{minimize}} \int_0^1 \mathrm{Var}(\ln \hat{X}_p)dp \quad \text{subject to} \quad & Tc(n, m, k, T) = C_F E(D) + C_E E(\xi) \leq C_b, \\ & \Psi(n) = 1, \quad n, m, k \in \mathbb{N} \quad \text{and} \quad T > 0. \end{aligned} \tag{8}$$

It is worth noting that $\ln(\hat{X}_p) = \hat{\mu} + \hat{\sigma} h(p)$, where $h(p) = \ln((1-p)^{(-1/\alpha)} - 1)$. Obtaining the exact expression for $\int_0^1 \mathrm{Var}(\ln(\hat{X}_p))dp$ is challenging, but an asymptotic expression can be derived using the delta method. This yields:

$$\int_0^1 \mathrm{Var}(\ln \hat{X}_p)dp = \Sigma_{22} + \Sigma_{33} \int_0^1 (h(p))^2 dp + 2\Sigma_{23} \int_0^1 h(p)dp.$$

In the above expression, the decision variables n, m and k are positive integers, while T is a positive real number. The objective function and constraints involve non-linear functions of these decision variables. Thus, the optimization problem described above is categorized as a mixed integer non-linear programming problem. To tackle this, we employ a straightforward technique outlined in Taha (2013) to derive optimal solutions. To solve the optimization problem given in Eq.(8), the following results are essential.

Result 1. For a fixed n, m and k, the expected number of failures $E(D)$ increases with T, where

$$E(D) = k + \sum_{i=k+1}^{m} F_{i:n}(T; \theta). \tag{9}$$

Arguing the same argument for the calculation of E(ξ), which is given by

$$E(\xi) = \int_0^\infty x f_{k:n-1}(x; \theta)dx + \int_0^T (F_{k:n-1}(x; \theta) - F_{m:n-1}(x; \theta))dx. \tag{10}$$

The CDF of $X_{m:n-1}$ is given by

$$F_{m:n-1}(x; \theta) = \sum_{i=m}^{n-1} \binom{n-1}{i} F_X(x; \theta)^i (1 - F_X(x; \theta))^{n-i-1}. \tag{11}$$

For more details see Sen *et al.* (2018).

6.1. Algorithm for the optimum solutions

Here, we present a summary of the results obtained for the computation of the sampling plan as follows:

Table 5: Optimum RASP under GHCS-I with $C_f = 10$ and $C_t = 15$

(α, β)	(p_α, p_β)	C_b	(n, m, k, T)	$\int_0^1 \text{Var}(\ln(\hat{X}_p))$	k_1
(0.05, 0.05)	(0.00041, 0.0184)	250	(80, 34, 24, 1.326)	0.166	4.597
		270	(66, 29, 21, 1.533)	0.219	4.597
		290	(53, 33, 18, 1.683)	0.266	4.597
	(0.00654, 0.0426)	310	(92, 43, 28, 1.431)	0.197	2.958
		330	(87, 45, 23, 1.450)	0.236	2.958
		350	(73, 35, 26, 1.652)	0.326	2.958
	(0.0109, 0.0535)	370	(103, 55, 36, 1.445)	0.227	2.619
		390	(89, 44, 31, 1.606)	0.315	2.619
		410	(68, 41, 27, 1.800)	0.388	2.619
	(0.0209, 0.0742)	430	(119, 44, 33, 1.462)	0.246	2.168
		450	(103, 54, 39, 1.630)	0.299	2.168
		470	(87, 47, 38, 1.784)	0.487	2.168
(0.05, 0.10)	(0.00041, 0.0184)	250	(96, 34, 22, 1.242)	0.097	4.383
		270	(88, 36, 24, 1.357)	0.140	4.383
		290	(72, 30, 21, 1.503)	0.276	4.383
	(0.00654, 0.0426)	310	(108, 36, 27, 1.323)	0.387	2.852
		330	(97, 42, 29, 1.440)	0.437	2.852
		350	(83, 37, 32, 1.601)	0.450	2.852
	(0.0109, 0.0535)	370	(115, 44, 34, 1.413)	0.340	2.529
		390	(101, 47, 36, 1.538)	0.458	2.529
		410	(84, 44, 28, 1.637)	0.568	2.529
	(0.0209, 0.0742)	430	(135, 54, 33, 1.375)	0.209	2.094
		450	(123, 48, 34, 1.466)	0.379	2.094
		470	(98, 52, 37, 1.676)	0.467	2.094

Step 1: Select two points $(p_\alpha, 1-\alpha)$ and (p_β, β) on the OC curve, then calculate the acceptability constant k_1.
Step 2: Start with an initial value for n.
Step 3: Minimize the expression $\int_0^1 \text{Var}(\ln(\hat{X}p))dp$ with respect to T, while ensuring that n and $m = 1(1)n$ satisfy the cost constraint given in Eq.(8). For a fixed n and m, determine T as the solution of the equation $C_F E(D) + C_E E(\xi) = C_b$, taking it as the upper bound. Select the pair (m, T) that minimizes $\int_0^1 \text{Var}(\ln(\hat{X}_p))dp$ over $m = 1(1)n$.
Step 4: Using the obtained values of n and the solution (m, T) from Step 3, calculate the value of $\Psi(n)$, where

$$\Psi(n) = \frac{1}{\sigma^2}\left[\frac{z_\alpha - z_{1-\beta}}{u_{p_\alpha} - u_{p_\beta}}\right]^2 \left(\Sigma_{22}(n,m,k,T) - 2k_1\Sigma_{23}(n,m,k,T) + k_1^2\Sigma_{33}(n,m,k,T)\right).$$

The values of Σ_{22}, Σ_{33} and Σ_{23} are calculated based on the variance-covariance matrix. Additionally, it is noted that $\Psi(n)$ decreases as n increases.
Step 5: If the condition $\Psi(n) \leq 1$ and $\Psi(n+1) > 1$ is fulfilled, then n represents the solution. Otherwise, if $\Psi(n) > 1$, increment n by 1; if $\Psi(n) < 1$, decrement n by 1, and proceed to Step 3.

Table 5 presents the optimum value of (n, m, k, T) at which cost is minimized under GHCS-I for specified values of (α, β), considering six levels of the censoring fraction $q = 0.1(0.1)0.6$. Similar exercise can be performed for the optimum value of (n, m, T_1, T_2) at which cost is minimized under GHCS-II for the specified values of the $\alpha, \beta, p_\alpha, p_\beta, q, q_1$ presented in Table 6. The R code used in this article is available on GitHub for reference and reproducibility at `https://github.com/Shubham1039883/R-code-for-article.`

Table 6: Optimum RASP under GHCS-II with $C_f = 10$ and $C_t = 15$

$(\alpha,\ \beta)$	$(p_\alpha,\ p_\beta)$	C_b	(n, m, T_1, T_2)	$\int_0^1 \text{Var}(\ln(\hat{X}_p))$	k_1
(0.05, 0.05)	(0.00041, 0.0184)	250	(80, 34, 1.326, 1.727)	0.166	3.243
		270	(84, 36, 1.361, 1.861)	0.128	3.243
		290	(67, 42, 1.518, 2.018)	0.108	3.243
	(0.00654, 0.0426)	310	(111, 62, 1.297, 1.797)	0.098	1.468
		330	(101, 54, 1.378, 1.878)	0.091	1.468
		350	(89, 43, 1.487, 1.987)	0.084	1.468
	(0.0109, 0.0535)	370	(131, 64, 1.322, 1.822)	0.082	1.302
		390	(113, 54, 1.462, 1.962)	0.080	1.302
		410	(96, 43, 1.522, 2.022)	0.076	1.302
	(0.0209, 0.0742)	430	(145, 74, 1.370, 1.870)	0.074	0.603
		450	(124, 52, 1.449, 1.949)	0.069	0.603
		470	(99, 56, 1.551, 2.051)	0.054	0.603
(0.05, 0.10)	(0.00041, 0.0184)	250	(109, 42, 1.161, 1.661)	0.216	3.012
		270	(89, 36 ,1.322, 1.822)	0.161	3.012
		290	(67, 42, 1.592, 2.092)	0.125	3.012
	(0.00654, 0.0426)	310	(129, 48, 1.208, 1.708)	0.083	1.376
		330	(114, 47, 1.330, 1.830)	0.065	1.376
		350	(94, 52, 1.500, 2.000)	0.052	1.376
	(0.0109, 0.0535)	370	(142, 42, 1.264, 1.764)	0.037	1.030
		390	(126, 54, 1.363, 1.863)	0.031	1.030
		410	(102, 62, 1.569, 2.069)	0.022	1.030
	(0.0209, 0.0742)	430	(165, 72, 1.258, 1.758)	0.014	0.523
		450	(134, 84, 1.445, 1.945)	0.009	0.523
		470	(97, 84, 1.710, 2.210)	0.005	0.523

7. A comparative study

We have undertaken a detailed comparative study of the sampling plans derived from our proposed methodology with those obtained using the approach developed by Schneider (1989) under the framework of type-II censoring. Specifically, our study focuses on the computation of sampling plans under GHCS-I and GHCS-II. A key distinction between the two methods lies in the treatment of the Fisher information matrix while Schneider's methodology relies on an approximate version of the Fisher information matrix, our approach is based on the exact Fisher information matrix, which enhances the precision of the inferential procedures.

Although we do not present the detailed numerical results in this paper, our findings indicate that the sampling plans obtained through both methods exhibit marginal equivalence in their outcomes. It is worth noting, however, that the work of Sen *et al.* (2018) also employs the exact Fisher information matrix for constructing the sampling plan. Nevertheless, the sample sizes and acceptance constants determined through our model are comparatively smaller than those reported in the studies by Sen *et al.* (2018), Bhattacharya *et al.* (2016) and Bhattacharya *et al.* (2015) thereby suggesting potential efficiency gains in the implementation of our model. Furthermore, the values of the producer's risk($\hat{\alpha}$) and consumer's risk($\hat{\beta}$) obtained through our model are comparatively smaller than those reported in Sen *et al.* (2018). In addition, the optimal values of the design parameters—namely, (n, m, k, T) in case of GHCS-I and (n, m, T_1, T_2) in case of GHCS-II are also observed to be lower in our model, thereby indicating improved efficiency in terms of both cost and testing time.

8. Conclusion

In the present manuscript, a RASPs has been formulated specifically for the Burr type-XII distribution within the framework of GHCS-I and GHCS-II. The determination of the sample size (n) and acceptance threshold (k_1) for this sampling scheme has been accomplished through a proper algorithm. Furthermore, the article presents the devised RASPs under GHCS-I and GHCS-II. Moreover, optimal RASPs have been computed via Monte-Carlo simulation, employing a suitable algorithm for precise calculations. From the Table 1 and Table 3, it is observed that sample sizes (n) and acceptance constants (k_1) under GHCS-II are less than under GHCS-I. This work can be extended for various lifetime models under different censoring schemes for classical approach as well as Bayesian paradigm.

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