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Asymptotic Normality of Parameter Estimators for Mixed Fractional Brownian Motion with Trend

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Abstract

We investigate the mixed fractional Brownian motion of the form $X_t = \theta t + \sigma W_t + \kappa B_t^H$, driven by a standard Brownian motion W and a fractional Brownian motion B^H with Hurst parameter H. We consider strongly consistent estimators of unknown model parameters $(\theta, H, \sigma, \kappa)$ based on the equidistant observations of a trajectory. Joint asymptotic normality of these estimators is proved for $H \in (0, \frac{1}{2})$.

Keywords: Fractional Brownian motion, Wiener process, mixed model, asymptotic distribution.

1. Introduction

The standard Brownian motion is traditionally applied to model many real-world processes that evolve over time. At the same time, it cannot be used for accurate modeling of some time series with properties of long-range dependence, self-similarity and complex correlation structures (Dai and Singleton 2000; Ding, Granger, and Engle 1993; Henry 2002; Paxson and Floyd 1995; Rostek and Schöbel 2013; Sadique and Silvapulle 2001; Zhang, Xiao, and He 2009). Alternatively, a fractional Brownian motion can be used which has correlated increments that imply short-range ($H < 1/2$) or long-range ($H > 1/2$) dependence. Recently, the so-called mixed fractional Brownian motion, i.e. a combination of the Wiener process and a fractional Brownian motion, has attracted significant interest. This process was introduced by Cheridito (2001) with the aim to consider models of financial markets that are simultaneously arbitrage-free and have a memory. Mixed models often have better modeling properties than "pure" models (driven by standard or fractional Brownian motion only) because they can take into account both the independence of process increments over short time intervals and the availability of memory at longer intervals. Further details on the properties of the mixed fractional Brownian motion can be found in Zili (2006); some examples of practical applications are provided in Filatova (2008); Sun (2013); Xiao, Zhang, Zhang, and Zhang (2012); Zhang *et al.* (2009).

In the present paper we consider the following mixed fractional Brownian motion with a linear trend:

$$X_t = \theta t + \sigma W_t + \kappa B_t^H, \quad t \geq 0, \tag{1}$$

where W is a Wiener process, B^H is a fractional Brownian motion with Hurst index H, B^H is independent of W. Parameter estimation problem for the model (1) was studied in Cai, Chigansky, and Kleptsyna (2016); Mishura, Ralchenko, and Shklyar (2017); Dufitinema, Pynnönen, and Sottinen (2022); Kukush, Lohvinenko, Mishura, and Ralchenko (2022). Cai *et al.* (2016) considered the estimation of the drift parameter θ assuming that H and σ are known and $\kappa = 1$. The drift parameter estimation for a more general model driven by a Gaussian process was discussed in Mishura *et al.* (2017), a similar problem for the model with two fractional Brownian motions was studied in Mishura and Voronov (2015) and Mishura, Ralchenko, and Zhelezniak (2022), the case of multifractional noise was considered in Dozzi, Kozachenko, Mishura, and Ralchenko (2018).

The simultaneous estimation of all four parameters the model (1) was studied in Dufitinema *et al.* (2022) by the maximum likelihood method, and in Kukush *et al.* (2022), where the approach of Dozzi, Mishura, and Shevchenko (2015) (based of power variations) was generalized and, moreover, ergodic-type estimators were proposed. Xiao, Zhang, and Zhang (2011); Zhang, Sun, and Xiao (2014); Dozzi *et al.* (2015); Filatova (2008) studied the parameter estimation for the mixed fractional Brownian motion without trend, i.e., for the process $X_t = \sigma W_t + \kappa B_t^H$, $t \geq 0$. In Xiao *et al.* (2011) the maximum likelihood approach was applied for the estimation of parameters σ and κ and providing asymptotic properties of these estimators. In Zhang *et al.* (2014) the authors also applied the maximum likelihood approach but it was combined with Powell's optimization method to efficiently compute estimators of σ and κ and those results were obtained for the case $H > 1/2$. The estimators constructed do not have explicit forms and thus their theoretical properties were not proved. In addition to that the Hurst parameter H was assumed to be known in both Xiao *et al.* (2011); Zhang *et al.* (2014). In Filatova (2008) authors developed the parameter estimation method based on the least-squares method and constructed estimators do not have explicit forms, thus the paper does not provide any theoretical properties. Finally, in Dozzi *et al.* (2015) the asymptotic properties of mixed power variations were studied and used to construct consistent estimators of the parameters H, σ and κ.

It is well known that the properties of a fractional Brownian motion substantially depend on the value of H, namely, one should distinguish between the cases of short ($H < 1/2$) and long ($H > 1/2$) memory. Unlike the recent paper Dufitinema *et al.* (2022) (where the case $H > 1/2$ was considered), in the present paper we focus on the case $H < 1/2$, which is much less studied in the literature. Note that the model is non-identifiable in the case $H = \frac{1}{2}$, so we should exclude this value.

We aim to estimate unknown parameters $(\theta, H, \sigma, \kappa)$ based on observed $\{X_{kh}, k = 0, 1, 2, \dots\}$, $h > 0$. Following Kukush *et al.* (2022), we introduce the next four statistics

$$
\begin{aligned}
\phi_N &:= \frac{X_{Nh}}{N} = \frac{1}{N} \sum_{k=0}^{N-1} \left(X_{(k+1)h} - X_{kh}\right), \\
\xi_N &:= \frac{1}{N} \sum_{k=0}^{N-1} \left(X_{(k+1)h} - X_{kh}\right)^2, \\
\eta_N &:= \frac{1}{N} \sum_{k=0}^{N-1} \left(X_{(k+1)h} - X_{kh}\right) \left(X_{(k+2)h} - X_{(k+1)h}\right), \\
\zeta_N &:= \frac{1}{N} \sum_{k=0}^{N-1} \left(X_{(k+2)h} - X_{kh}\right) \left(X_{(k+4)h} - X_{(k+2)h}\right),
\end{aligned}
\tag{2}
$$

and consider the following estimators for the parameters $(\theta, H, \sigma^2, \kappa^2)$:

$$\begin{aligned} \hat{\theta}_N &= \frac{\phi_N}{h}, & \widehat{H}_N &= \frac{1}{2}\log_{2+}\frac{\zeta_N - 4\phi_N^2}{\eta_N - \phi_N^2}, \\ \hat{\kappa}_N^2 &= \frac{\eta_N - \phi_N^2}{h^{2\widehat{H}_N}(2^{2\widehat{H}_N-1}-1)}, & \hat{\sigma}_N^2 &= \frac{1}{h}\left(\xi_N - \phi_N^2 - \hat{\kappa}_N^2 h^{2\widehat{H}_N}\right), \end{aligned} \tag{3}$$

with

$$\log_{2+} x = \begin{cases} \log_2 x, & \text{if } x > 0, \\ 0, & \text{if } x \le 0. \end{cases}$$

The strong consistency of these estimators was proved in Kukush *et al.* (2022). The goal of the present paper is to establish their joint asymptotic normality.

The paper is organized as follows. In Section 2 we prove the asymptotic normality of statistics (2) and evaluate their asymptotic covariance matrix. Thereafter, in Section 3 we obtain the main result on asymptotic normality of (3) by applying the delta-method. In Section 4 we investigate the behavior of the estimators numerically, using Monte Carlo simulations.

We use the following notation. The symbol **cov** stands for the covariance of two random variables and for the covariance matrix of a random vector. In the paper, all the vectors are column ones. The superscript $\top$ denotes transposition. Weak convergence in distribution is denoted by $\xrightarrow{\mathrm{d}}$.

2. Asymptotic normality of vector $(\phi_N, \xi_N, \eta_N, \zeta_N)$

We will start with a study of the asymptotic properties of the vector

$$\tau_N = (\phi_N, \xi_N, \eta_N, \zeta_N). \tag{4}$$

According to (Kukush *et al.* 2022, Lemma 3.2), for any $H \in (0,1)$,

$$\tau_N \to (\mathsf{E}\,\phi_N, \mathsf{E}\,\xi_N, \mathsf{E}\,\eta_N, \mathsf{E}\,\zeta_N) =: \tau_0$$

a. s., as $N \to \infty$, where

$$\begin{aligned} \mathsf{E}\,\phi_N &= \theta h, & \mathsf{E}\,\xi_N &= \theta^2 h^2 + \sigma^2 h + \kappa^2 h^{2H}, \\ \mathsf{E}\,\eta_N &= \theta^2 h^2 + \kappa^2 h^{2H}\left(2^{2H-1}-1\right), & \mathsf{E}\,\zeta_N &= 4\theta^2 h^2 + \kappa^2 h^{2H} 2^{2H}\left(2^{2H-1}-1\right). \end{aligned}$$

Further, let us denote the increment

$$\Delta X_k \coloneqq X_{(k+1)h} - X_{kh} = \theta h + \sigma \Delta W_k + \kappa \Delta B_k^H = \theta h + \Delta Z_k,$$

where $\Delta Z_k \coloneqq \sigma\Delta W_k + \kappa \Delta B_k^H$. It is well known that $\{\Delta X_k\}$ and $\{\Delta Z_k\}$ are stationary Gaussian sequences with the following autocovariance function

$$\begin{aligned} \tilde{\rho}(i) &= \mathbf{cov}(\Delta X_0, \Delta X_i) = \mathbf{cov}(\Delta Z_0, \Delta Z_i) \\ &= \sigma^2\,\mathbf{cov}\left(\Delta W_0, \Delta W_i\right) + \kappa^2\,\mathbf{cov}\left(\Delta B_0^H, \Delta B_i^H\right) \\ &= \sigma^2 h \mathbb{1}_{\{i=0\}} + \kappa^2 h^{2H}\rho(i), \quad i \in \mathbb{Z}, \end{aligned} \tag{5}$$

where

$$\rho(i) \coloneqq \frac{1}{2}\left(|i+1|^{2H} - 2|i|^{2H} + |i-1|^{2H}\right) \tag{6}$$

denotes the autocovariance function of the stationary sequence $\{B_{k+1}^H - B_k^H, k \ge 0\}$, which is known as a fractional Gaussian noise. The following statement provides some important properties of the sequences $\tilde{\rho}(i)$ and $\rho(i)$, $i \in \mathbb{Z}$, defined by (5) and (6) respectively.

Lemma 2.1. *Let $H \in (0, \frac{1}{2})$. Then the following statements hold.*

1. $\rho(i) \sim H(2H-1)|i|^{2H-2}$ *as* $|i| \to \infty$.
2. $\sum_{i=-\infty}^{\infty} \rho^2(i) < \infty$.
3. $\sum_{i=-\infty}^{\infty} \rho(i) < \infty$ *and*

$$\sum_{i=-\infty}^{+\infty} \tilde{\rho}(i) < \infty. \tag{7}$$

4. *For all* $\alpha, \beta \in \mathbb{Z}$,

$$\sum_{i=-\infty}^{+\infty} \tilde{\rho}(i+\alpha)\tilde{\rho}(i+\beta) < \infty. \tag{8}$$

Proof. First two statements are well known, see e. g., Nourdin (2012). In order to prove the third one we use the first property $\rho(i) \sim H(2H-1)|i|^{2H-2}$ as $|i| \to \infty$. Using the usual criterion for convergence of the Riemann sums, we deduce that $\sum_{i=-\infty}^{\infty} \rho(i)$ is finite if and only if $2H-2 < 1$, i.e., $H < \frac{1}{2}$. Therefore

$$\sum_{i=-\infty}^{\infty} \tilde{\rho}(i) = \sigma^2 h + \kappa^2 h^{2H} \sum_{i=-\infty}^{\infty} \rho(i) < \infty,$$

In order to prove the fourth statement, we observe that

$$\sum_{i=-\infty}^{\infty} \tilde{\rho}^2(i) = \sigma^4 h^2 + 2\sigma^2\kappa^2 h^{2H+1} + \kappa^4 h^{4H} \sum_{i=-\infty}^{\infty} \rho^2(i) < \infty,$$

by the second statement. Then, using the Cauchy–Schwarz inequality, we get for all $\alpha, \beta \in \mathbb{Z}$,

$$\sum_{i=-\infty}^{+\infty} \tilde{\rho}(i+\alpha)\tilde{\rho}(i+\beta) \le \sqrt{\sum_{i=-\infty}^{+\infty} \tilde{\rho}^2(i+\alpha) \cdot \sum_{i=-\infty}^{+\infty} \tilde{\rho}^2(i+\beta)} = \sum_{i=-\infty}^{+\infty} \tilde{\rho}^2(i) < \infty,$$

where the equality follows by substitutions $i' = i + \alpha$ and $i' = i + \beta$. □

Now we are ready to prove the main result of this section.

Theorem 2.2. *Let $H \in (0, \frac{1}{2})$. The vector τ_N defined by (4) is asymptotically normal, namely*

$$\sqrt{N}(\tau_N - \tau_0) = \sqrt{N} \begin{pmatrix} \phi_N - \mathsf{E}\,\phi_N \\ \xi_N - \mathsf{E}\,\xi_N \\ \eta_N - \mathsf{E}\,\eta_N \\ \zeta_N - \mathsf{E}\,\zeta_N \end{pmatrix} \xrightarrow{\mathrm{d}} \mathcal{N}(\vec{0}, \widetilde{\Sigma})$$

with the asymptotic covariance matrix $\widetilde{\Sigma}$, which can be presented explicitly as

$$\widetilde{\Sigma} = \begin{pmatrix} \widetilde{\Sigma}_{11} & \widetilde{\Sigma}_{12} & \widetilde{\Sigma}_{13} & \widetilde{\Sigma}_{14} \\ \widetilde{\Sigma}_{12} & \widetilde{\Sigma}_{22} & \widetilde{\Sigma}_{23} & \widetilde{\Sigma}_{24} \\ \widetilde{\Sigma}_{13} & \widetilde{\Sigma}_{23} & \widetilde{\Sigma}_{33} & \widetilde{\Sigma}_{34} \\ \widetilde{\Sigma}_{14} & \widetilde{\Sigma}_{24} & \widetilde{\Sigma}_{34} & \widetilde{\Sigma}_{44} \end{pmatrix}$$

where

$$\widetilde{\Sigma}_{11} = \sum_{i=-\infty}^{+\infty} \tilde{\rho}(i), \quad \widetilde{\Sigma}_{22} = 2\sum_{i=-\infty}^{+\infty} \tilde{\rho}(i)^2 + 4\theta^2 h^2 \sum_{i=-\infty}^{+\infty} \tilde{\rho}(i),$$

$$\widetilde{\Sigma}_{33} = \sum_{i=-\infty}^{\infty} \tilde{\rho}(i)\Big(\tilde{\rho}(i) + \tilde{\rho}(i+2)\Big) + 4\theta^2 h^2 \sum_{i=-\infty}^{+\infty} \tilde{\rho}(i),$$

$$\widetilde{\Sigma}_{44} = \sum_{i=-\infty}^{+\infty} \tilde{\rho}(i)\Big(6\tilde{\rho}(i) + 8\tilde{\rho}(i+1) + 3\tilde{\rho}(i+2)$$
$$+ 4\tilde{\rho}(i+3) + 6\tilde{\rho}(i+4) + 4\tilde{\rho}(i+5) + \tilde{\rho}(i+6)\Big) + 64\theta^2 h^2 \sum_{i=-\infty}^{+\infty} \tilde{\rho}(i),$$

$$\widetilde{\Sigma}_{23} = 2 \sum_{i=-\infty}^{\infty} \tilde{\rho}(i)\tilde{\rho}(i+1) + 4\theta^2 h^2 \sum_{i=-\infty}^{+\infty} \tilde{\rho}(i),$$

$$\widetilde{\Sigma}_{24} = 2 \sum_{i=-\infty}^{+\infty} \tilde{\rho}(i)\Big(\tilde{\rho}(i+1) + 2\tilde{\rho}(i+2) + \tilde{\rho}(i+3)\Big) + 16\theta^2 h^2 \sum_{i=-\infty}^{+\infty} \tilde{\rho}(i),$$

$$\widetilde{\Sigma}_{34} = \sum_{i=-\infty}^{+\infty} \tilde{\rho}(i)\Big(\tilde{\rho}(i) + 2\tilde{\rho}(i+1) + 2\tilde{\rho}(i+2) + 2\tilde{\rho}(i+3) + \tilde{\rho}(i+4)\Big) + 4\theta^2 h^2 \sum_{i=-\infty}^{+\infty} \tilde{\rho}(i),$$

$$\widetilde{\Sigma}_{12} = \widetilde{\Sigma}_{13} = 2\theta h \sum_{i=-\infty}^{+\infty} \tilde{\rho}(i), \quad \widetilde{\Sigma}_{14} = 8\theta h \sum_{i=-\infty}^{+\infty} \tilde{\rho}(i).$$

Proof. The proof consists of two parts: in the first part we compute the asymptotic covariance matrix $\widetilde{\Sigma}$, while the second part contains the proof of asymptotic normality.

Part 1: Identification of the asymptotic covariance matrix. Firstly, we will find the explicit form of covariance matrix $\widetilde{\Sigma}$ by evaluating convergence limits as $N \to \infty$ for the following variances and covariances: $N\,\mathbf{Var}(\xi_N)$, $N\,\mathbf{Var}(\eta_N)$, $N\,\mathbf{Var}(\zeta_N)$, $N\,\mathbf{Var}(\phi_N)$, $N\,\mathbf{cov}(\xi_N,\eta_N)$, $N\,\mathbf{cov}(\xi_N,\zeta_N)$, $N\,\mathbf{cov}(\eta_N,\zeta_N)$, $N\,\mathbf{cov}(\xi_N,\phi_N)$, $N\,\mathbf{cov}(\eta_N,\phi_N)$, $N\,\mathbf{cov}(\zeta_N,\phi_N)$.

1.1 Evaluation of convergence limit for $N\,\mathbf{Var}(\xi_N)$. Using Isserlis' theorem[1], and stationarity of the centered sequence $\{\Delta Z_k\}$, we can write

$$\mathbf{cov}\left((\Delta Z_k)^2, (\Delta Z_j)^2\right) = 2\,\mathbf{cov}\left((\Delta Z_k), (\Delta Z_j)\right)^2 = 2\tilde{\rho}(k-j)^2,$$
$$\mathbf{cov}\left(\Delta Z_k, (\Delta Z_j)^2\right) = 0.$$

Then

$$\begin{aligned}\mathbf{cov}\left((\Delta X_k)^2, (\Delta X_j)^2\right) &= \mathbf{cov}\left((\theta h + \Delta Z_k)^2, (\theta h + \Delta Z_j)^2\right)\\ &= \mathbf{cov}\left(2\theta h \Delta Z_k + \Delta Z_k^2, 2\theta h \Delta Z_j + \Delta Z_j^2\right)\\ &= 4\theta^2 h^2\,\mathbf{cov}\left(\Delta Z_k, \Delta Z_j\right) + \mathbf{cov}\left(\Delta Z_k^2, \Delta Z_j^2\right)\\ &= 4\theta^2 h^2 \tilde{\rho}(k-j) + 2\tilde{\rho}(k-j)^2.\end{aligned}$$

Therefore

$$\begin{aligned}N\,\mathbf{Var}(\xi_N) &= \frac{1}{N}\mathbf{Var}\left(\sum_{k=0}^{N-1} (\Delta X_k)^2\right) = \frac{1}{N}\sum_{k=0}^{N-1}\sum_{j=0}^{N-1} \mathbf{cov}\left((\Delta X_k)^2, (\Delta X_j)^2\right)\\ &= \frac{1}{N}\sum_{k=0}^{N-1}\sum_{j=0}^{N-1}\left(4\theta^2 h^2 \tilde{\rho}(k-j) + 2\tilde{\rho}(k-j)^2\right)\\ &= \frac{4\theta^2 h^2}{N}\sum_{k=0}^{N-1}\sum_{j=0}^{N-1}\tilde{\rho}(k-j) + \frac{2}{N}\sum_{k=0}^{N-1}\sum_{j=0}^{N-1}\tilde{\rho}(k-j)^2. \qquad (9)\end{aligned}$$

[1]Isserlis' theorem Isserlis (1918): if (X_1, X_2, X_3, X_4) is a zero-mean multivariate normal random vector, then $\mathsf{E}(X_1X_2X_3X_4) = \mathsf{E}\,X_1X_2\,\mathsf{E}\,X_3X_4 + \mathsf{E}\,X_1X_3\,\mathsf{E}\,X_1X_4 + \mathsf{E}\,X_1X_4\,\mathsf{E}\,X_2X_4$ and $\mathsf{E}(X_1X_2X_3) = 0$. In particular, $\mathbf{cov}(X_1^2, X_2^2) = 2\,\mathbf{cov}(X_1, X_2)^2$ and $\mathbf{cov}(X_1, X_2^2) = 0$.

Further, by rearranging sums for the second term in the right hand side of (9)

$$\begin{aligned}\frac{2}{N}\sum_{k=0}^{N-1}\sum_{j=0}^{N-1}\tilde{\rho}(k-j)^2 &= \frac{2}{N}\sum_{k=0}^{N-1}\sum_{i=k-N+1}^{k}\tilde{\rho}(i)^2\\ &= \frac{2}{N}\sum_{i=-N+1}^{0}\sum_{k=0}^{N-1+i}\tilde{\rho}(i)^2 + \frac{2}{N}\sum_{i=1}^{N-1}\sum_{k=i}^{N-1}\tilde{\rho}(i)^2\\ &= 2\sum_{i=-N+1}^{N-1}\left(1-\frac{|i|}{N}\right)\tilde{\rho}(i)^2 \to 2\sum_{i=-\infty}^{\infty}\tilde{\rho}(i)^2, \quad \text{as } N\to\infty. \end{aligned} \tag{10}$$

where the passage to the limit can be justified by the dominated convergence theorem due to Lemma 2.1.

Similarly, for the first term in the right hand side of (9)

$$\frac{4\theta^2h^2}{N}\sum_{k=0}^{N-1}\sum_{j=0}^{N-1}\tilde{\rho}(k-j) = 4\theta^2h^2\sum_{i=-N+1}^{N-1}\left(1-\frac{|i|}{N}\right)\tilde{\rho}(i) \to 4\theta^2h^2\sum_{i=-\infty}^{\infty}\tilde{\rho}(i), \text{ as } N\to\infty. \tag{11}$$

Therefore

$$N\,\mathbf{Var}(\xi_N) \to 4\theta^2h^2\sum_{i=-\infty}^{\infty}\tilde{\rho}(i) + 2\sum_{i=-\infty}^{\infty}\tilde{\rho}(i)^2 \quad \text{as } N\to\infty.$$

1.2. Evaluation of convergence limit for $N\,\mathbf{cov}(\xi_N,\eta_N)$. Write

$$\begin{aligned}N\,\mathbf{cov}(\xi_N,\eta_N) &= \frac{1}{N}\,\mathbf{cov}\left(\sum_{k=0}^{N-1}(\Delta X_k)^2, \sum_{j=0}^{N-1}\Delta X_j\Delta X_{j+1}\right)\\ &= \frac{1}{N}\sum_{k=0}^{N-1}\sum_{j=0}^{N-1}\mathbf{cov}\left((\Delta X_k)^2, \Delta X_j\Delta X_{j+1}\right),\end{aligned}$$

Then

$$\begin{aligned}&\mathbf{cov}\left((\Delta X_k)^2, \Delta X_j\Delta X_{j+1}\right)\\ &\quad= \mathbf{cov}\left(2\theta h\Delta Z_k + (\Delta Z_k)^2, \sum_{r=0}^{1}\theta h\Delta Z_{j+r} + \Delta Z_j\Delta Z_{j+1}\right)\\ &\quad= 2\theta^2h^2\sum_{r=0}^{1}\mathbf{cov}\left(\Delta Z_k, \Delta Z_{j+r}\right) + \mathbf{cov}\left((\Delta Z_k)^2, \Delta Z_j\Delta Z_{j+1}\right)\\ &\quad= 2\theta^2h^2\sum_{r=0}^{1}\tilde{\rho}(k-j-r) + \mathbf{cov}\left((\Delta Z_k)^2, \Delta Z_j\Delta Z_{j+1}\right).\end{aligned}$$

By Isserlis' theorem,

$$\begin{aligned}\mathbf{cov}\left((\Delta Z_k)^2, \Delta Z_j\Delta Z_{j+1}\right) &= \mathsf{E}(\Delta Z_k)^2\Delta Z_j\Delta Z_{j+1} - \mathsf{E}(\Delta Z_k)^2\,\mathsf{E}\,\Delta Z_j\Delta Z_{j+1}\\ &= \mathsf{E}\,\Delta Z_k\Delta Z_j\,\mathsf{E}\,\Delta Z_k\Delta Z_{j+1} + \mathsf{E}\,\Delta Z_k\Delta Z_{j+1}\,\mathsf{E}\,\Delta Z_k\Delta Z_j\\ &= 2\,\mathbf{cov}(\Delta Z_k, \Delta X_j)\,\mathbf{cov}(\Delta Z_k, \Delta Z_{j+1}) = 2\tilde{\rho}(j-k)\tilde{\rho}(j-k+1).\end{aligned}$$

Therefore arguing as in (10), we obtain

$$\frac{2}{N}\sum_{j=0}^{N-1}\sum_{k=0}^{N-1}\tilde{\rho}(j-k)\tilde{\rho}(j-k+1) = \frac{2}{N}\sum_{j=0}^{N-1}\sum_{i=j-N+1}^{j}\tilde{\rho}(i)\tilde{\rho}(i+1)$$

$$= 2 \sum_{i=-N+1}^{N-1} \left(1 - \frac{|i|}{N}\right) \tilde{\rho}(i)\tilde{\rho}(i+1) \to 2 \sum_{i=-\infty}^{\infty} \tilde{\rho}(i)\tilde{\rho}(i+1),$$

as $N \to \infty$, where the last series converges according to the Lemma 2.1.
Similarly to (11)

$$\begin{aligned}
\sum_{r=0}^{1} \frac{2\theta^2 h^2}{N} \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} \tilde{\rho}(k-j-r) &= \sum_{r=0}^{1} \frac{2\theta^2 h^2}{N} \sum_{j=0}^{N-1} \sum_{i=j-N+1}^{j} \tilde{\rho}(i-r) \\
&= \sum_{r=0}^{1} 2\theta^2 h^2 \sum_{i=-N+1}^{N-1} \left(1 - \frac{|i|}{N}\right) \tilde{\rho}(i-r) \to \sum_{r=0}^{1} 2\theta^2 h^2 \sum_{i=-\infty}^{\infty} \tilde{\rho}(i-r) \\
&= \sum_{r=0}^{1} 2\theta^2 h^2 \sum_{i=-\infty}^{\infty} \tilde{\rho}(i) = 4\theta^2 h^2 \sum_{i=-\infty}^{\infty} \tilde{\rho}(i), \quad \text{as } N \to \infty.
\end{aligned}$$

Thus

$$N\,\mathbf{cov}(\xi_N, \eta_N) \to 4\theta^2 h^2 \sum_{i=-\infty}^{\infty} \tilde{\rho}(i) + 2 \sum_{i=-\infty}^{\infty} \tilde{\rho}(i)\tilde{\rho}(i+1), \quad \text{as } N \to \infty.$$

1.3. Evaluation of convergence limit for $N\,\mathbf{Var}(\eta_N)$. We have

$$N\,\mathbf{Var}(\eta_N) = \frac{1}{N} \sum_{k=0}^{N-1} \sum_{j=0}^{N-1} \mathbf{cov}(\Delta X_k \Delta X_{k+1}, \Delta X_j \Delta X_{j+1}).$$

By applying transformation $\Delta X_k = \theta h + \Delta Z_k$ we obtain

$$\begin{aligned}
&\mathbf{cov}(\Delta X_k \Delta X_{k+1}, \Delta X_j \Delta X_{j+1}) \\
&\quad = \mathbf{cov}(\theta h \sum_{\ell=0}^{1} \Delta Z_{k+\ell} + \Delta Z_k \Delta Z_{k+1}, \theta h \sum_{r=0}^{1} \Delta Z_{j+r} + \Delta Z_j \Delta Z_{j+1}) \\
&\quad = \sum_{\ell,r=0}^{1} \theta^2 h^2\, \mathbf{cov}(\Delta Z_{k+\ell}, \Delta Z_{j+r}) + \mathbf{cov}(\Delta Z_k \Delta Z_{k+1}, \Delta Z_j \Delta Z_{j+1}) \\
&\quad = \sum_{\ell,r=0}^{1} \theta^2 h^2 \tilde{\rho}(k+\ell-j-r) + \mathbf{cov}(\Delta Z_k \Delta Z_{k+1}, \Delta Z_j \Delta Z_{j+1}).
\end{aligned}$$

By Isserlis' Theorem,

$$\begin{aligned}
&\mathbf{cov}(\Delta Z_k \Delta Z_{k+1}, \Delta Z_j \Delta Z_{j+1}) \\
&\quad = \mathsf{E}\,\Delta Z_k \Delta Z_{k+1} \Delta Z_j \Delta Z_{j+1} - \mathsf{E}\,\Delta Z_k \Delta Z_{k+1}\, \mathsf{E}\,\Delta Z_j \Delta Z_{j+1} \\
&\quad = \mathsf{E}\,\Delta Z_k \Delta Z_{k+1}\, \mathsf{E}\,\Delta Z_j \Delta Z_{j+1} + \mathsf{E}\,\Delta Z_k \Delta Z_j\, \mathsf{E}\,\Delta Z_{k+1} \Delta Z_{j+1} \\
&\quad\quad + \mathsf{E}\,\Delta Z_k \Delta Z_{j+1}\, \mathsf{E}\,\Delta Z_{k+1} \Delta Z_j - \mathsf{E}\,\Delta Z_k \Delta Z_{k+1}\, \mathsf{E}\,\Delta Z_j \Delta Z_{j+1} \\
&\quad = (\mathbf{cov}(\Delta Z_k, \Delta Z_j))^2 + \mathbf{cov}(\Delta Z_k, \Delta Z_{j+1})\, \mathbf{cov}(\Delta Z_{k+1}, \Delta Z_j) \\
&\quad = \tilde{\rho}(k-j)^2 + \tilde{\rho}(k-j+1)\tilde{\rho}(k-j-1).
\end{aligned}$$

Therefore

$$\begin{aligned}
N\,\mathbf{Var}(\eta_N) = \sum_{\ell,r=0}^{1} \frac{\theta^2 h^2}{N} \sum_{k=0}^{N-1} \sum_{j=0}^{N-1} \tilde{\rho}(k+\ell-j-r) + \frac{1}{N} \sum_{k=0}^{N-1} \sum_{j=0}^{N-1} \tilde{\rho}(k-j)^2 \\
+ \frac{1}{N} \sum_{k=0}^{N-1} \sum_{j=0}^{N-1} \tilde{\rho}(k-j+1)\tilde{\rho}(k-j-1). \qquad (12)
\end{aligned}$$

Similarly to (11), the first term in the right hand side of (12)

$$\sum_{\ell,r=0}^{1}\frac{\theta^2h^2}{N}\sum_{k=0}^{N-1}\sum_{j=0}^{N-1}\tilde\rho(k+\ell-j-r)=\sum_{\ell,r=0}^{1}\frac{\theta^2h^2}{N}\sum_{j=0}^{N-1}\sum_{i=j-N+1}^{j}\tilde\rho(i+\ell-r)$$
$$=\sum_{\ell,r=0}^{1}\theta^2h^2\sum_{i=-N+1}^{N-1}\left(1-\frac{|i|}{N}\right)\tilde\rho(i+\ell-r)\to\sum_{\ell,r=0}^{1}\theta^2h^2\sum_{i=-\infty}^{\infty}\tilde\rho(i+\ell-r)$$
$$=\sum_{\ell,r=0}^{1}\theta^2h^2\sum_{i=-\infty}^{\infty}\tilde\rho(i)=4\theta^2h^2\sum_{i=-\infty}^{\infty}\tilde\rho(i),\quad \text{as } N\to\infty. \tag{13}$$

According to (10), the second term in the right hand side of (12)

$$\frac{1}{N}\sum_{k=0}^{N-1}\sum_{j=0}^{N-1}\tilde\rho(k-j)^2=\frac{N}{2}\,\mathbf{Var}(\xi_N)\to\sum_{i=-\infty}^{\infty}\tilde\rho(i)^2,\quad N\to\infty. \tag{14}$$

The third term can be treated similarly to (10) as follows:

$$\frac{1}{N}\sum_{j=0}^{N-1}\sum_{k=0}^{N-1}\tilde\rho(k-j+1)\tilde\rho(k-j-1)=\frac{1}{N}\sum_{j=0}^{N-1}\sum_{i=j-(N-1)}^{j}\tilde\rho(i-1)\tilde\rho(i+1)$$
$$=\sum_{i=-(N-1)}^{N-1}\left(1-\frac{|i|}{N}\right)\tilde\rho(i-1)\tilde\rho(i+1)\to\sum_{i=-\infty}^{\infty}\tilde\rho(i-1)\tilde\rho(i+1)$$
$$=\sum_{i=-\infty}^{\infty}\tilde\rho(i)\tilde\rho(i+2),\quad \text{as } N\to\infty. \tag{15}$$

Combining (12)–(15), we arrive at

$$\lim_{N\to\infty}N\,\mathbf{Var}(\eta_N)=\sum_{i=-\infty}^{\infty}\tilde\rho(i)\Big(\tilde\rho(i)+\tilde\rho(i+2)\Big)+4\theta^2h^2\sum_{i=-\infty}^{\infty}\tilde\rho(i).$$

1.4. Evaluation of convergence limit for $N\,\mathbf{cov}(\xi_N,\zeta_N)$. We rewrite $N\,\mathbf{cov}(\xi_N,\zeta_N)$ in the following form:

$$N\,\mathbf{cov}(\xi_N,\zeta_N)=\frac{1}{N}\,\mathbf{cov}\left(\sum_{k=0}^{N-1}(\Delta X_k)^2,\sum_{j=0}^{N-1}(\Delta X_j+\Delta X_{j+1})(\Delta X_{j+2}+\Delta X_{j+3})\right)$$
$$=\frac{1}{N}\sum_{\alpha,\beta=0}^{1}\sum_{k=0}^{N-1}\sum_{j=0}^{N-1}\mathbf{cov}\left((\Delta X_k)^2,\Delta X_{j+\alpha}\Delta X_{j+\beta+2}\right).$$

We apply transformation $\Delta X_k=\theta h+\Delta Z_k$ with Isserlis' theorem and get

$$\mathbf{cov}\left((\Delta X_k)^2,\Delta X_{j+\alpha}\Delta X_{j+\beta+2}\right)$$
$$=\mathbf{cov}\left((\Delta Z_k)^2,\Delta Z_{j+\alpha}\Delta Z_{j+\beta+2}\right)+\theta^2h^2\sum_{\gamma=0}^{1}\mathbf{cov}(\Delta Z_k,\Delta Z_{j+\alpha+\gamma(\beta+2-\alpha)})$$
$$=2\tilde\rho(j-k+\alpha)\tilde\rho(j-k+\beta+2)+\theta^2h^2\sum_{\gamma=0}^{1}\tilde\rho\left(j-k+\alpha+\gamma(\beta+2-\alpha)\right).$$

According to (15)

$$\frac{2}{N}\sum_{\alpha,\beta=0}^{1}\sum_{k=0}^{N-1}\sum_{j=0}^{N-1}\tilde\rho(j-k+\alpha)\tilde\rho(j-k+\beta+2)$$

$$
\begin{aligned}
&= 2 \sum_{\alpha,\beta=0}^{1} \sum_{i=-(N-1)}^{N-1} \left(1 - \frac{|i|}{N}\right) \tilde{\rho}(i+\alpha)\tilde{\rho}(i+\beta+2) \\
&\to 2 \sum_{\alpha,\beta=0}^{1} \sum_{i=-\infty}^{+\infty} \tilde{\rho}(i+\alpha)\tilde{\rho}(i+\beta+2) = 2 \sum_{\alpha,\beta=0}^{1} \sum_{i=-\infty}^{+\infty} \tilde{\rho}(i)\tilde{\rho}(i+\beta+2-\alpha) \\
&= 2 \sum_{i=-\infty}^{+\infty} \tilde{\rho}(i)\Big(\tilde{\rho}(i+1) + 2\tilde{\rho}(i+2) + \tilde{\rho}(i+3)\Big).
\end{aligned}
$$

Similarly to (11)

$$
\begin{aligned}
&\sum_{\alpha,\beta,\gamma=0}^{1} \frac{2\theta^2 h^2}{N} \sum_{j=0}^{N-1} \sum_{k=0}^{N-1} \tilde{\rho}\left(j-k+\alpha+\gamma(\beta+2-\alpha)\right) \\
&\quad = \sum_{\alpha,\beta,\gamma=0}^{1} \frac{2\theta^2 h^2}{N} \sum_{j=0}^{N-1} \sum_{i=j-N+1}^{j} \tilde{\rho}\left(i+\alpha+\gamma(\beta+2-\alpha)\right) \\
&\quad = \sum_{\alpha,\beta,\gamma=0}^{1} 2\theta^2 h^2 \sum_{i=-N+1}^{N-1} \left(1 - \frac{|i|}{N}\right) \tilde{\rho}\left(i+\alpha+\gamma(\beta+2-\alpha)\right) \\
&\quad \to \sum_{\alpha,\beta,\gamma=0}^{1} 2\theta^2 h^2 \sum_{i=-\infty}^{\infty} \tilde{\rho}\left(i+\alpha+\gamma(\beta+2-\alpha)\right) \\
&\quad = \sum_{\alpha,\beta,\gamma=0}^{1} 2\theta^2 h^2 \sum_{i=-\infty}^{\infty} \tilde{\rho}(i) = 16\theta^2 h^2 \sum_{i=-\infty}^{\infty} \tilde{\rho}(i), \quad \text{as } N \to \infty. \qquad (16)
\end{aligned}
$$

Therefore

$$
\lim_{N\to\infty} N\,\mathbf{cov}(\xi_N, \zeta_N) = 2 \sum_{i=\infty}^{+\infty} \tilde{\rho}(i)\Big(\tilde{\rho}(i+1) + 2\tilde{\rho}(i+2) + \tilde{\rho}(i+3)\Big) + 16\theta^2 h^2 \sum_{i=-\infty}^{\infty} \tilde{\rho}(i).
$$

1.5 Evaluation of convergence limit for $N\,\mathbf{cov}(\eta_N, \zeta_N)$. Similarly to previous considerations,

$$
\begin{aligned}
N\,\mathbf{cov}(\eta_N, \zeta_N) &= \frac{1}{N} \mathbf{cov}\left(\sum_{k=0}^{N-1} \Delta X_k \Delta X_{k+1}, \sum_{j=0}^{N-1} (\Delta X_j + \Delta X_{j+1})(\Delta X_{j+2} + \Delta X_{j+3})\right) \\
&= \frac{1}{N} \sum_{\alpha,\beta=0}^{1} \sum_{k=0}^{N-1} \sum_{j=0}^{N-1} \mathbf{cov}(\Delta X_k \Delta X_{k+1}, \Delta X_{j+\alpha} \Delta X_{j+\beta+2}).
\end{aligned}
$$

Again, by Isserlis' theorem and transformation $\Delta X_k = \theta h + \Delta Z_k$,

$$
\begin{aligned}
&\mathbf{cov}(\Delta X_k \Delta X_{k+1}, \Delta X_{j+\alpha} \Delta X_{j+\beta+2}) \\
&\quad = \mathbf{cov}(\Delta Z_k \Delta Z_{k+1}, \Delta Z_{j+\alpha} \Delta Z_{j+\beta+2}) + \theta^2 h^2 \sum_{\ell,r=0}^{1} \mathbf{cov}(\Delta Z_{k+\ell}, \Delta Z_{j+\alpha+r(\beta+2-\alpha)} \\
&\quad = \sum_{\gamma=0}^{1} \tilde{\rho}(j-k+\alpha-\gamma)\tilde{\rho}(j-k+1+\beta+\gamma) + \theta^2 h^2 \sum_{\ell,r=0}^{1} \tilde{\rho}\left(j-k-\ell+\alpha+r(\beta+2-\alpha)\right).
\end{aligned}
$$

According to (15)

$$
\frac{1}{N} \sum_{\alpha,\beta,\gamma=0}^{1} \sum_{k=0}^{N-1} \sum_{j=0}^{N-1} \tilde{\rho}(j-k+\alpha-\gamma)\tilde{\rho}(j-k+1+\beta+\gamma)
$$

$$= \sum_{\alpha,\beta,\gamma=0}^{1} \sum_{i=-(N-1)}^{N-1} \left(1 - \frac{|i|}{N}\right) \tilde{\rho}(i+\alpha-\gamma)\tilde{\rho}(i+\beta+1+\gamma)$$
$$\to \sum_{\alpha,\beta,\gamma=0}^{1} \sum_{i=-\infty}^{+\infty} \tilde{\rho}(i+\alpha-\gamma)\tilde{\rho}(i+\beta+1+\gamma),$$

as $N \to \infty$. As above, we can simplify the last expression. Then we get

$$\sum_{\alpha,\beta,\gamma=0}^{1} \sum_{i=-\infty}^{+\infty} \tilde{\rho}(i)\tilde{\rho}(i+\beta+1+2\gamma-\alpha)$$
$$= \sum_{i=-\infty}^{+\infty} \tilde{\rho}(i)\Big(\tilde{\rho}(i) + 2\tilde{\rho}(i+1) + 2\tilde{\rho}(i+2) + 2\tilde{\rho}(i+3) + \tilde{\rho}(i+4)\Big).$$

Similarly to (16)

$$\sum_{\alpha,\beta,\ell,r=0}^{1} \frac{\theta^2 h^2}{N} \sum_{j=0}^{N-1}\sum_{k=0}^{N-1} \tilde{\rho}\left(j-k-\ell+\alpha+r(\beta+2-\alpha)\right)$$
$$= \sum_{\alpha,\beta,\ell,r=0}^{1} \frac{\theta^2 h^2}{N} \sum_{j=0}^{N-1} \sum_{i=j-N+1}^{j} \tilde{\rho}\left(i-\ell+\alpha+r(\beta+2-\alpha)\right)$$
$$= \sum_{\alpha,\beta,\ell,r=0}^{1} \theta^2 h^2 \sum_{i=-N+1}^{N-1} \left(1-\frac{|i|}{N}\right) \tilde{\rho}\left(i-\ell+\alpha+r(\beta+2-\alpha)\right)$$
$$\to \sum_{\alpha,\beta,\ell,r=0}^{1} \theta^2 h^2 \sum_{i=-\infty}^{\infty} \tilde{\rho}\left(i-\ell+\alpha+r(\beta+2-\alpha)\right)$$
$$= \sum_{\alpha,\beta,\ell,r=0}^{1} \theta^2 h^2 \sum_{i=-\infty}^{\infty} \tilde{\rho}(i) = 16\theta^2 h^2 \sum_{i=-\infty}^{\infty} \tilde{\rho}(i), \quad \text{as } N \to \infty.$$

Therefore

$$\lim_{N\to\infty} N\, \mathbf{cov}(\eta_N, \zeta_N) = 16\theta^2 h^2 \sum_{i=-\infty}^{\infty} \tilde{\rho}(i)$$
$$+ \sum_{i=-\infty}^{+\infty} \tilde{\rho}(i)\Big(\tilde{\rho}(i) + 2\tilde{\rho}(i+1) + 2\tilde{\rho}(i+2) + 2\tilde{\rho}(i+3) + \tilde{\rho}(i+4)\Big).$$

1.6. Evaluation of convergence limit for $N\,\mathbf{Var}(\zeta_N)$. It is not hard to see that

$$N\,\mathbf{Var}(\zeta_N) = \frac{1}{N} \sum_{a,b,c,d=0}^{1} \sum_{k,j=0}^{N-1} \mathbf{cov}(\Delta X_{k+a}\Delta X_{k+b+2}, \Delta X_{j+c}\Delta X_{j+d+2})$$

By transformation $\Delta X_k = \theta h + \Delta Z_k$ and Isserlis' theorem, we get

$$\mathbf{cov}(\Delta X_{k+a}\Delta X_{k+b+2}, \Delta X_{j+c}\Delta X_{j+d+2})$$
$$= \mathbf{cov}(\Delta Z_{k+a}\Delta Z_{k+b+2}, \Delta Z_{j+c}\Delta Z_{j+d+2})$$
$$+ \theta^2 h^2 \sum_{\ell,r=0}^{1} \mathbf{cov}(\Delta X_{k+a+\ell(b+2-a)}, \Delta X_{j+c+r(d+2-c)})$$
$$= \sum_{\gamma=0}^{1} \tilde{\rho}(j-k-a+c-\gamma(2+b-a))\tilde{\rho}(j-k-b+d-\gamma(a-b-2))$$

$$+\theta^2 h^2 \sum_{\ell,r=0}^{1} \tilde{\rho}\left(j+c+r(d+2-c)-k-a-\ell(b+2-a)\right).$$

Therefore

$$\frac{1}{N}\sum_{a,b,c,d,\gamma=0}^{1}\sum_{k,j=0}^{N-1}\tilde{\rho}(j-k-a+c-\gamma(2+b-a))\tilde{\rho}(j-k-b+d-\gamma(a-b-2))$$
$$=\sum_{a,b,c,d,\gamma=0}^{1}\sum_{i=-(N-1)}^{N-1}\left(1-\frac{|i|}{N}\right)\tilde{\rho}(i+c-a-\gamma(2+b-a))\tilde{\rho}(i+d-b-\gamma(a-b-2))$$
$$\to\sum_{a,b,c,d,\gamma=0}^{1}\sum_{i=-\infty}^{\infty}\tilde{\rho}(i+c-a-\gamma(2+b-a))\tilde{\rho}(i+d-b-\gamma(a-b-2)),$$

as $N\to\infty$. After simplifications, we arrive at

$$\sum_{i=-\infty}^{+\infty}\tilde{\rho}(i)\Big(6\tilde{\rho}(i)+8\tilde{\rho}(i+1)+3\tilde{\rho}(i+2)+4\tilde{\rho}(i+3)+6\tilde{\rho}(i+4)+4\tilde{\rho}(i+5)+\tilde{\rho}(i+6)\Big).$$

Similarly to (16)

$$\sum_{a,b,c,d,\ell,r=0}^{1}\frac{\theta^2h^2}{N}\sum_{j=0}^{N-1}\sum_{k=0}^{N-1}\tilde{\rho}\left(j-k+c+r(d+2-c)-a-\ell(b+2-a)\right)$$
$$=\sum_{a,b,c,d,\ell,r=0}^{1}\frac{\theta^2h^2}{N}\sum_{j=0}^{N-1}\sum_{i=j-N+1}^{j}\tilde{\rho}\left(i+c+r(d+2-c)-a-\ell(b+2-a)\right)$$
$$=\sum_{a,b,c,d,\ell,r=0}^{1}\theta^2h^2\sum_{i=-N+1}^{N-1}\left(1-\frac{|i|}{N}\right)\tilde{\rho}\left(i+c+r(d+2-c)-a-\ell(b+2-a)\right)$$
$$\to\sum_{a,b,c,d,\ell,r=0}^{1}\theta^2h^2\sum_{i=-\infty}^{\infty}\tilde{\rho}\left(i+c+r(d+2-c)-a-\ell(b+2-a)\right)$$
$$=\sum_{a,b,c,d,\ell,r=0}^{1}\theta^2h^2\sum_{i=-\infty}^{\infty}\tilde{\rho}(i)=64\theta^2h^2\sum_{i=-\infty}^{\infty}\tilde{\rho}(i),\quad \text{as } N\to\infty.$$

Therefore

$$\lim_{N\to\infty}N\,\mathbf{Var}(\zeta_N)=64\theta^2h^2\sum_{i=-\infty}^{\infty}\tilde{\rho}(i)$$
$$+\sum_{i=-\infty}^{+\infty}\tilde{\rho}(i)\Big(6\tilde{\rho}(i)+8\tilde{\rho}(i+1)+3\tilde{\rho}(i+2)+4\tilde{\rho}(i+3)+6\tilde{\rho}(i+4)+4\tilde{\rho}(i+5)+\tilde{\rho}(i+6)\Big).$$

1.7. Evaluation of convergence limit for $N\,\mathbf{Var}(\phi_N)$. We have similarly to (16)

$$N\,\mathbf{Var}(\phi_N)=\frac{1}{N}\sum_{k=0}^{N-1}\sum_{j=0}^{N-1}\mathbf{cov}(\Delta X_k,\Delta X_j)=\frac{1}{N}\sum_{k=0}^{N-1}\sum_{j=0}^{N-1}\tilde{\rho}(k-j)$$
$$=\sum_{i=-N+1}^{N-1}\left(1-\frac{|i|}{N}\right)\tilde{\rho}(i)\to\sum_{i=-\infty}^{\infty}\tilde{\rho}(i),\quad \text{as } N\to\infty.$$

1.8. Evaluation of convergence limit for $N\,\mathbf{cov}(\xi_N,\phi_N)$. Arguing as above, we get

$$N\,\mathbf{cov}(\xi_N,\phi_N)=\frac{1}{N}\sum_{k=0}^{N-1}\sum_{j=0}^{N-1}\mathbf{cov}((\Delta X_k)^2,\Delta X_j)$$

$$= \frac{1}{N}\sum_{k=0}^{N-1}\sum_{j=0}^{N-1}\mathbf{cov}(2\theta h\Delta Z_k + (\Delta Z_k)^2, \Delta Z_j) = \frac{2\theta h}{N}\sum_{k=0}^{N-1}\sum_{j=0}^{N-1}\mathbf{cov}(\Delta X_k, \Delta X_j)$$
$$= \frac{2\theta h}{N}\sum_{k=0}^{N-1}\sum_{j=0}^{N-1}\tilde{\rho}(k-j) = 2\theta h \sum_{i=-N+1}^{N-1}\left(1-\frac{|i|}{N}\right)\tilde{\rho}(i)$$
$$\to 2\theta h \sum_{i=-\infty}^{\infty}\tilde{\rho}(i), \quad \text{as } N\to\infty.$$

1.9. Evaluation of convergence limit for $N\,\mathbf{cov}(\eta_N, \phi_N)$. Similarly to previous considerations,

$$N\,\mathbf{cov}(\eta_N,\phi_N) = \frac{1}{N}\sum_{k=0}^{N-1}\sum_{j=0}^{N-1}\mathbf{cov}(\Delta X_k \Delta X_{k+1}, \Delta X_j)$$
$$= \frac{1}{N}\sum_{k=0}^{N-1}\sum_{j=0}^{N-1}\mathbf{cov}(\theta h\Delta Z_k + \theta h\Delta Z_{k+1} + \Delta Z_k\Delta Z_{k+1}, \Delta Z_j)$$
$$= \sum_{\alpha=0}^{1}\frac{\theta h}{N}\sum_{k=0}^{N-1}\sum_{j=0}^{N-1}\mathbf{cov}(\Delta X_{\alpha+k}, \Delta X_j) = \sum_{\alpha=0}^{1}\frac{\theta h}{N}\sum_{k=0}^{N-1}\sum_{j=0}^{N-1}\tilde{\rho}(k+\alpha-j)$$
$$= \sum_{\alpha=0}^{1}\theta h\sum_{i=-N+1}^{N-1}\left(1-\frac{|i|}{N}\right)\tilde{\rho}(i+\alpha) \to \sum_{\alpha=0}^{1}\theta h\sum_{i=-\infty}^{\infty}\tilde{\rho}(i+\alpha)$$
$$= \sum_{\alpha=0}^{1}\theta h\sum_{i=-\infty}^{\infty}\tilde{\rho}(i) = 2\theta h\sum_{i=-\infty}^{\infty}\tilde{\rho}(i), \quad \text{as } N\to\infty.$$

1.10. Evaluation of convergence limit for $N\,\mathbf{cov}(\zeta_N, \phi_N)$. It is not hard to see that

$$N\,\mathbf{cov}(\zeta_N,\phi_N) = \frac{1}{N}\sum_{k=0}^{N-1}\sum_{j=0}^{N-1}\mathbf{cov}((\Delta X_k + \Delta X_{k+1})(\Delta X_{k+2}+\Delta X_{k+3}), \Delta X_j)$$
$$= \sum_{\alpha,\beta=0}^{1}\frac{1}{N}\sum_{k=0}^{N-1}\sum_{j=0}^{N-1}\mathbf{cov}(\Delta X_{k+\alpha}\Delta X_{k+2+\beta}, \Delta X_j)$$
$$= \sum_{\alpha,\beta=0}^{1}\frac{1}{N}\sum_{k=0}^{N-1}\sum_{j=0}^{N-1}\mathbf{cov}(\theta h\Delta Z_{k+\alpha} + \theta h\Delta Z_{k+2+\beta} + \Delta Z_{k+\alpha}\Delta Z_{k+2+\beta}, \Delta Z_j)$$
$$= \sum_{\alpha,\beta,\gamma=0}^{1}\frac{\theta h}{N}\sum_{k=0}^{N-1}\sum_{j=0}^{N-1}\mathbf{cov}(\Delta X_{k+\alpha+\gamma(2+\beta-\alpha)}, \Delta X_j)$$
$$= \sum_{\alpha,\beta,\gamma=0}^{1}\frac{1\theta h}{N}\sum_{k=0}^{N-1}\sum_{j=0}^{N-1}\tilde{\rho}(k-j+\alpha+\gamma(2+\beta-\alpha))$$
$$= \sum_{\alpha,\beta,\gamma=0}^{1}\theta h\sum_{i=-N+1}^{N-1}\left(1-\frac{|i|}{N}\right)\tilde{\rho}(i+\alpha+\gamma(2+\beta-\alpha))$$
$$\to \sum_{\alpha,\beta,\gamma=0}^{1}\theta h\sum_{i=-\infty}^{\infty}\tilde{\rho}(i+\alpha+\gamma(2+\beta-\alpha))$$
$$= \sum_{\alpha,\beta,\gamma=0}^{1}\theta h\sum_{i=-\infty}^{\infty}\tilde{\rho}(i) = 8\theta h\sum_{i=-\infty}^{\infty}\tilde{\rho}(i), \quad \text{as } N\to\infty.$$

Part 2: Proof of asymptotic normality. Let us define $Y_k = \left(Y_k^{(1)}, Y_k^{(2)}, Y_k^{(3)}\right)$ by

$$Y_k^{(1)} := \Delta Z_k, \quad Y_k^{(2)} := \Delta Z_{k+1}, \quad Y_k^{(3)} := \Delta Z_{k+2} + \Delta Z_{k+3}. \tag{17}$$

Then

$$\xi_N = \frac{1}{N}\sum_{k=0}^{N-1}\left(Y_k^{(1)} + \theta h\right)^2, \qquad \eta_N = \frac{1}{N}\sum_{k=0}^{N-1}\left(Y_k^{(1)} + \theta h\right)\left(Y_k^{(2)} + \theta h\right),$$

$$\zeta_N = \frac{1}{N}\sum_{k=0}^{N-1}\left(Y_k^{(1)} + Y_k^{(2)} + 2\theta h\right)\left(Y_k^{(3)} + \theta h\right), \quad \phi_N = \frac{1}{N}\sum_{k=0}^{N-1}\left(Y_k^{(1)} + \theta h\right).$$

We shall prove the convergence of vector $(\phi_N, \xi_N, \eta_N, \zeta_N)$ with the help of the Cramér–Wold device. Let the parameters α, β, γ, $\lambda \in \mathbb{R}$ be any fixed ones satisfying the condition $\alpha^2 + \beta^2 + \gamma^2 + \lambda^2 \neq 0$. We introduce the function

$$f(y) = \alpha(y_1 + \theta h) + \beta(y_1 + \theta h)^2 + \gamma(y_1 + \theta h)(y_2 + \theta h) + \lambda(y_1 + y_2 + 2\theta h)(y_3 + \theta h)$$

where $y = (y_1, y_2, y_3) \in \mathbb{R}^3$, so that

$$\alpha\phi_N + \beta\xi_N + \gamma\eta_N + \lambda\zeta_N = \frac{1}{N}\sum_{k=0}^{N-1} f(Y_k).$$

Thus, we need to prove that the sequence

$$\sqrt{N}\Big(\alpha(\phi_N - \mathsf{E}\,\phi_N)\Big) + \beta(\xi_N - \mathsf{E}\,\xi_N) + \gamma(\eta_N - \mathsf{E}\,\eta_N) + \lambda(\zeta_N - \mathsf{E}\,\zeta_N)$$
$$= \frac{1}{\sqrt{N}}\sum_{k=0}^{N-1}\Big(f(Y_k) - \mathsf{E}\,f(Y_k)\Big) \quad (18)$$

converges to a normal distribution. This fact can be established by application of the central limit theorem for stationary vectors (Arcones 1994, Theorem 2). In order to apply this theorem, it suffices to verify that

$$\lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1}\sum_{j=0}^{N-1} r^{(p,l)}(j-k), \quad (19)$$

and

$$\lim_{N\to\infty}\frac{1}{N}\sum_{k=0}^{N-1}\sum_{j=0}^{N-1}\left(r^{(p,l)}(j-k)\right)^2, \quad (20)$$

exist for all $p, l \in \{1, 2, 3\}$, where

$$r^{(p,l)}(k) \coloneqq \mathbf{cov}\left(Y_1^{(p)}, Y_{1+k}^{(l)}\right), \quad k \in \mathbb{Z}.$$

In turn, existence of (19)–(20) follows from Lemma 2.1, since using (17) and the definition of $\tilde{\rho}$, we can represent $r^{(p,l)}$ via $\tilde{\rho}$ as follows:

$$r^{(1,1)}(k) = r^{(2,2)}(k) = \tilde{\rho}(k), \qquad r^{(1,2)}(k) = \tilde{\rho}(k+1), \quad r^{(2,1)}(k) = \tilde{\rho}(k-1),$$
$$r^{(1,3)}(k) = \tilde{\rho}(k+2) + \tilde{\rho}(k+3), \qquad r^{(3,1)}(k) = \tilde{\rho}(k-2) + \tilde{\rho}(k-3),$$
$$r^{(2,3)}(k) = \tilde{\rho}(k+1) + \tilde{\rho}(k+2), \qquad r^{(3,2)}(k) = \tilde{\rho}(k-1) + \tilde{\rho}(k-2),$$
$$r^{(3,3)}(k) = \tilde{\rho}(k+1) + 2\tilde{\rho}(k) + \tilde{\rho}(k-1).$$

Indeed, this can be shown as follows for

$$\frac{1}{N}\sum_{k=0}^{N-1}\sum_{j=0}^{N-1}\left(r^{(1,3)}(j)\right)^2 = \frac{1}{N}\sum_{k=0}^{N-1}\sum_{j=0}^{N-1}\left(\tilde{\rho}(j-k+2) + \tilde{\rho}(j-k+3)\right)^2$$

$$
\begin{aligned}
&= \sum_{i=-N+1}^{N-1} \left(1 - \frac{|i|}{N}\right) \left(\tilde\rho(i+2) + \tilde\rho(i+3)\right)^2 = \sum_{i=-N+1}^{N-1} \left(1 - \frac{|i|}{N}\right) \tilde\rho(i+2)^2 \\
&\quad + 2 \sum_{i=-N+1}^{N-1} \left(1 - \frac{|i|}{N}\right) \tilde\rho(i+2)\tilde\rho(i+3) + \sum_{i=-N+1}^{N-1} \left(1 - \frac{|i|}{N}\right) \tilde\rho(i+3)^2 \\
&\to \sum_{i=-\infty}^{\infty} \tilde\rho(i+2)^2 + 2 \sum_{i=-\infty}^{\infty} \tilde\rho(i+2)\tilde\rho(i+3) + \sum_{i=-\infty}^{\infty} \tilde\rho(i+3)^2, \quad \text{as } N \to \infty,
\end{aligned}
$$

where the last sum converges as a finite sum of series convergent according to Lemma 2.1. Similarly, other limits in (19)–(20) are also exist and finite.

Hence, the assumptions of the central limit theorem for stationary vectors (Arcones 1994, Theorem 2) are satisfied, which implies the desired weak converge of (18) to a zero-mean normal distribution. □

Remark 2.3. The condition $H < \frac{1}{2}$ is essential, since otherwise the series in condition (19) do not converge.

3. Asymptotic normality of vector $(\hat\theta_N, \widehat{H}_N, \hat\kappa_N^2, \hat\sigma_N^2)$

In this section, we provide the main result on asymptotic properties of our estimators (3). Let us introduce the notation

$$
\vartheta = (\theta, H, \kappa^2, \sigma^2), \qquad \widehat\vartheta_N = (\hat\theta_N, \widehat{H}_N, \hat\kappa_N^2, \hat\sigma_N^2), \quad N \in \mathbb{N}. \tag{21}
$$

Theorem 3.1. *Let $H \in (0, \frac{1}{2})$. The estimator $\widehat\vartheta_N$ is asymptotically normal, namely*

$$
\sqrt{N}\left(\widehat\vartheta_N - \vartheta\right) = \sqrt{N} \begin{pmatrix} \hat\theta_N - \theta \\ \widehat{H}_N - H \\ \hat\kappa_N^2 - \kappa^2 \\ \hat\sigma_N^2 - \sigma^2 \end{pmatrix} \xrightarrow{\mathrm{d}} \mathcal{N}\left(\vec{0}, \Sigma^0\right)
$$

with the asymptotic covariance matrix Σ^0 that can be found by the formula

$$
\Sigma^0 = g'(\tau_0)\widetilde\Sigma \left(g'(\tau_0)\right)^\top,
$$

where $\widetilde\Sigma$ is defined in Theorem 2.2 and

$$
g'(\tau_0) = \begin{pmatrix}
\frac{1}{h} & 0 & 0 & 0 \\
\frac{2\theta(l-2)}{k^2 h^{2H-1} l(l+2)} & 0 & \frac{-1}{\kappa^2 h^{2H} l \log 2} & \frac{1}{\kappa^2 h^{2H} l(l+2) \log 2} \\
\frac{-4\theta}{h^{2H-1}} \frac{2(l+2)(l-1)+cl(l-2)}{l^2(l+2)} & 0 & \frac{2}{h^{2H}} \frac{(2+c)l+2}{l^2} & \frac{-2}{h^{2H}} \frac{l(c+1)+2}{l^2(l+2)} \\
\frac{4\theta(l^2+4l-4)}{l^2} & \frac{1}{h} & -\frac{4(l+1)}{hl^2} & \frac{2}{hl^2}
\end{pmatrix}.
$$

where $c = \log_2 h$ and $l = 2^{2H} - 2$.

Proof. We denote the following function of $\tau = (\phi, \xi, \eta, \zeta)$:

$$
g(\tau) = \big(g_1(\tau), g_2(\tau), g_3(\tau), g_4(\tau)\big),
$$

where

$$
\begin{aligned}
g_1(\tau) &= g_1(\phi, \xi, \eta, \zeta) = \frac{\phi}{h}, \\
g_2(\tau) &= g_2(\phi, \xi, \eta, \zeta) = \frac{1}{2} \log_2 \frac{\zeta - 4\phi^2}{\eta - \phi^2},
\end{aligned}
$$

$$g_3(\tau) = g_3(\phi, \xi, \eta, \zeta) = \frac{\eta - \phi^2}{h^{2g_2(\xi,\eta,\zeta)}(2^{2g_2(\xi,\eta,\zeta)-1} - 1)},$$
$$g_4(\tau) = g_4(\phi, \xi, \eta, \zeta) = \frac{1}{h}\left(\xi - \phi^2 - g_3(\xi, \eta, \zeta)h^{2g_2(\xi,\eta,\zeta)}\right).$$

Then the estimator (21) can be presented in the following form

$$\widehat{\vartheta}_N = (\hat{\theta}_N, \widehat{H}_N, \hat{\kappa}^2_N, \hat{\sigma}^2_N) = g(\phi_N, \xi_N, \eta_N, \zeta_N).$$

To prove the asymptotic normality of $\widehat{\vartheta}_N$ we will apply the delta-method to function $g(\tau)$ and sequence τ_N that is asymptotically normal by Theorem 2.2. For this we will need to find matrix $g'(\tau)$ and to show that it is non-singular at τ_0. Firstly, we will write partial derivatives function g_1

$$\frac{\partial g_1}{\partial \phi} = \frac{1}{h}, \qquad \frac{\partial g_1}{\partial \xi} = 0, \qquad \frac{\partial g_1}{\partial \eta} = 0, \qquad \frac{\partial g_1}{\partial \zeta} = 0.$$

Now we will proceed with evaluating partial derivatives of function g_2. Since

$$g_2(\phi, \xi, \eta, \zeta) = \frac{1}{2}\log_2 \frac{\zeta - 4\phi^2}{\eta - \phi^2} = \frac{\log(\zeta - 4\phi^2) - \log(\eta - \phi^2)}{2\log 2},$$

we observe that

$$\frac{\partial g_2}{\partial \phi} = \frac{1}{2\log 2}\left(-\frac{8\phi}{\zeta - 4\phi^2} + \frac{2\phi}{\eta - \phi^2}\right) = \frac{\phi(\zeta - 4\eta)}{(\zeta - 4\phi^2)(\eta - \phi^2)\log 2},$$
$$\frac{\partial g_2}{\partial \xi} = 0, \qquad \frac{\partial g_2}{\partial \eta} = -\frac{1}{2(\eta - \phi^2)\log 2}, \qquad \frac{\partial g_2}{\partial \zeta} = \frac{1}{2(\zeta - 4\phi^2)\log 2}.$$

In order to evaluate partial derivatives of g_3 and g_4, let us consider two intermediate functions:

$$h^{2g_2(\phi,\xi,\eta,\zeta)} = h^{\log_2 \frac{\zeta - 4\phi^2}{\eta - \phi^2}} = \left(\frac{\zeta - 4\phi^2}{\eta - \phi^2}\right)^{\log_2 h} = \left(\frac{\zeta - 4\phi^2}{\eta - \phi^2}\right)^{c},$$

where $c = \log_2 h$, and

$$2^{2g_2(\phi,\xi,\eta,\zeta)-1} - 1 = 2^{\log_2 \frac{\zeta - 4\phi^2}{\eta - \phi^2} - 1} - 1 = \frac{\zeta - 4\phi^2}{2(\eta - \phi^2)} - 1 = \frac{\zeta - 2\eta - 2\phi^2}{2(\eta - \phi^2)}.$$

Thus,

$$g_3(\phi, \xi, \eta, \zeta) = \frac{\eta - \phi^2}{\left(\frac{\zeta - 4\phi^2}{\eta - \phi^2}\right)^c \cdot \frac{\zeta - 2\eta - 2\phi^2}{2(\eta - \phi^2)}} = \frac{2(\eta - \phi^2)^{2+c}}{(\zeta - 4\phi^2)^c \cdot (\zeta - 2\eta - 2\phi^2)}.$$

Therefore, the corresponding partial derivatives are equal to

$$\begin{aligned}
\frac{\partial g_3}{\partial \phi} &= \frac{2}{(\zeta - 4\phi^2)^{2c} \cdot (\zeta - 2\eta - 2\phi^2)^2} \cdot \big(-2\phi(2 + c)(\eta - \phi^2)^{c+1}(\zeta - 2\eta - 2\phi^2) \\
&\quad - (\eta - \phi^2)^{2+c}\big(-8\phi c(\zeta - 4\phi^2)^{c-1}(\zeta - 2\eta - 2\phi^2) - 4\phi(\zeta - 4\phi^2)^c\big)\big) \\
&= -4\phi\Big(\frac{\eta - \phi^2}{\zeta - 4\phi^2}\Big)^{c+1} \cdot \frac{2(\zeta - 4\phi^2)(\zeta - 3\eta - \phi^2) + c(\zeta - 4\eta)(\zeta - 2\eta - 2\phi^2)}{(\zeta - 2\eta - 2\phi^2)^2}, \\
\frac{\partial g_3}{\partial \xi} &= 0, \\
\frac{\partial g_3}{\partial \eta} &= \frac{2}{(\zeta - 4\phi^2)^c} \cdot \frac{(c + 2)(\eta - \phi^2)^{c+1}(\zeta - 2\eta - 2\phi^2) + 2(\eta - \phi^2)^{c+2})}{(\zeta - 2\eta - 2\phi^2)^2} \\
&= 2\left(\frac{\eta - \phi^2}{\zeta - 4\phi^2}\right)^c \cdot \frac{(\eta - \phi^2)\big((c + 2)(\zeta - 2\eta - 2\phi^2) + 2(\eta - \phi^2)\big)}{(\zeta - 2\eta - 2\phi^2)^2}, \\
\frac{\partial g_3}{\partial \zeta} &= -2(\eta - \phi^2)^{2+c} \cdot \frac{c(\zeta - 4\phi^2)^{c-1}(\zeta - 2\eta - 2\phi^2) + (\zeta - 4\phi^2)}{(\zeta - 4\phi^2)^{c+1}(\zeta - 2\eta - 2\phi^2)^2}
\end{aligned}$$

$$= -2\left(\frac{\eta-\phi^2}{\zeta-4\phi^2}\right)^{c+1}\frac{(\eta-\phi^2)\left(c(\zeta-2\eta-2\phi^2)+(\zeta-4\phi^2)\right)}{(\zeta-2\eta-2\phi^2)^2}.$$

The function g_4 can be rewritten as

$$\begin{aligned} g_4(\xi,\eta,\zeta) &= \frac{1}{h}\left(\xi-\phi^2-\frac{\eta-\phi^2}{h^{2g_2(\xi,\eta,\zeta)}(2^{2g_2(\xi,\eta,\zeta)-1})} * h^{2g_2(\xi,\eta,\zeta)}\right) \\ &= \frac{1}{h}\left(\xi-\phi^2-\frac{2(\eta-\phi^2)^2}{\zeta-2\eta-2\phi^2}\right). \end{aligned}$$

Hence, we obtain that

$$\begin{aligned} \frac{\partial g_4}{\partial\phi} &= \frac{1}{h}\left(2\phi-\frac{-8\phi(\eta-\phi^2)(\zeta-2\eta-2\phi^2)+8\phi(\eta-\phi^2)^2}{(\zeta-2\eta-2\phi^2)^2}\right) \\ &= \frac{2\phi}{h}\cdot\frac{(\zeta-2\eta-2\phi^2+2\eta-2\phi^2)^2-8(\eta-\phi^2)^2}{(\zeta-2\eta-2\phi^2)^2} \\ &= \frac{2\phi\left((\zeta-4\phi^2)^2-8(\eta-\phi^2)^2\right)}{h(\zeta-2\eta-2\phi^2)^2}, \\ \frac{\partial g_4}{\partial\xi} &= \frac{1}{h}, \\ \frac{\partial g_4}{\partial\eta} &= \frac{(-1)}{h}\cdot\frac{4(\eta-\phi^2)(\zeta-2\eta-2\phi^2)-2(\eta-\phi^2)^2(-2)}{(\zeta-2\eta-2\phi^2)^2} \\ &= \frac{4(\eta-\phi^2)(\eta+3\phi^2-\zeta)}{h(\zeta-2\eta-2\phi^2)^2}, \\ \frac{\partial g_4}{\partial\zeta} &= \frac{2(\eta-\phi^2)^2}{h(\zeta-2\eta-2\phi^2)^2}. \end{aligned}$$

Therefore, we have the following derivative matrix:

$$g'(\tau) = g'(\phi,\xi,\eta,\zeta) = \begin{pmatrix} \frac{\partial g_1}{\partial\phi} & \frac{\partial g_1}{\partial\xi} & \frac{\partial g_1}{\partial\eta} & \frac{\partial g_1}{\partial\zeta} \\ \frac{\partial g_2}{\partial\phi} & \frac{\partial g_2}{\partial\xi} & \frac{\partial g_2}{\partial\eta} & \frac{\partial g_2}{\partial\zeta} \\ \frac{\partial g_3}{\partial\phi} & \frac{\partial g_3}{\partial\xi} & \frac{\partial g_3}{\partial\eta} & \frac{\partial g_3}{\partial\zeta} \\ \frac{\partial g_4}{\partial\phi} & \frac{\partial g_4}{\partial\xi} & \frac{\partial g_4}{\partial\eta} & \frac{\partial g_4}{\partial\zeta} \end{pmatrix},$$

and for $\tau=\tau_0$

$$g'(\tau_0) = \begin{pmatrix} \frac{1}{h} & 0 & 0 & 0 \\ \frac{2\theta(l-2)}{k^2h^{2H-1}l(l+2)} & 0 & \frac{-1}{\kappa^2h^{2H}l\log 2} & \frac{1}{\kappa^2h^{2H}l(l+2)\log 2} \\ \frac{-4\theta}{h^{2H-1}}\frac{2(l+2)(l-1)+cl(l-2)}{l^2(l+2)} & 0 & \frac{2}{h^{2H}}\frac{(2+c)l+2}{l^2} & \frac{-2}{h^{2H}}\frac{l(c+1)+2}{l^2(l+2)} \\ \frac{4\theta(l^2+4l-4)}{l^2} & \frac{1}{h} & -\frac{4(l+1)}{hl^2} & \frac{2}{hl^2} \end{pmatrix}.$$

where $c=\log_2 h$ and $l=2^{2H}-2$. It is not hard to see that the determinant of this matrix

$$\det(g'(\tau_0)) = -\frac{1}{2^{2H+1}\kappa^2h^{4H+2}(2^{2H-1}-1)^2\log 2}.$$

is well defined for $H\in(0,\frac{1}{2})$ and it is not equal to zero. Therefore the matrix of derivatives is non-singular at $\tau=\tau_0$ and so the delta-method can be applied, see, e. g., (Kubilius, Mishura, and Ralchenko 2017, Theorem B.6). This method implies the statement of the theorem and provides with the formula for asymptotic covariance matrix Σ^0. □

4. Simulation study

In this section we would like to illustrate estimators performance by numerical simulations. For each generated trajectory we will estimate asymptotic covariance matrices defined in Theorems 2.2 and 3.1 by using values of estimators (3). For each set of parameters we generate 1000 trajectories of the process X_t and calculate the empirical means, empirical standard deviations of the estimates, the square root of estimated asymptotic variance divided by N ($\sqrt{\hat{\sigma}/N}$) and cover probability (CP) for $\alpha = 5\%$ based on estimator of asymptotic covariance matrix. Series alike (8) are divided into sum of two convergent series

$$\sum_{i=-\infty}^{+\infty} \tilde{\rho}(i+\alpha)\tilde{\rho}(i+\beta) = \sum_{i=0}^{+\infty} \tilde{\rho}(i+\alpha)\tilde{\rho}(i+\beta) + \sum_{i=0}^{+\infty} \tilde{\rho}(i+1-\alpha)\tilde{\rho}(i+1-\beta)$$

where $\alpha, \beta \in \mathbb{Z}$. Series $\sum_{i=0}^{+\infty} \tilde{\rho}(i+\alpha)\tilde{\rho}(i+\beta)$ are estimated with precision $\delta = 10^{-4}$ by evaluating the sum $S_m = \sum_{i=0}^{m} \tilde{\rho}(i+\alpha)\tilde{\rho}(i+\beta)$ with the first m such that $|S_m - S_{m-1}| < 10^{-4}$. The series (7) is estimated using the following representation

$$\sum_{i=-\infty}^{\infty} \tilde{\rho}(i) = \tilde{\rho}(0) + 2\sum_{i=1}^{\infty} \tilde{\rho}(i),$$

where series $\sum_{i=1}^{\infty} \tilde{\rho}(i)$ is estimated with precision $\delta = 10^{-4}$ by evaluating the sum $S_m = \sum_{i=0}^{m} \tilde{\rho}(i)$ with the first m such that $|S_m - S_{m-1}| < 10^{-4}$.

For $\hat{\theta}^2_N, \widehat{H}_N, \hat{\sigma}^2_N, \hat{\kappa}^2_N, \hat{\Sigma}^0_N$ and fixed time step $h = 1$ we vary the horizon $T = h2^n$ for $n \in \{8, 10, 12, 14, 16, 18, 20\}$. For all simulations the values $\theta = 0.5$, $\sigma = 1.5$, and $\kappa = 2.5$ were used.

Table 1: The estimator $\widehat{H}_N$ with $\theta = 0.5$, $\sigma^2 = 2.25$, $\kappa^2 = 6.25$ ($h = 1$)

			N						
H			2^8	2^{10}	2^{12}	2^{14}	2^{16}	2^{18}	2^{20}
0.1	$\widehat{H}_N$	Mean	0.0935	0.0925	0.0983	0.10023	0.1001	0.1002	0.1001
		S.dev.	0.2522	0.1240	0.0628	0.03152	0.0159	0.0080	0.0039
		$\sqrt{\hat{\sigma}/N}$	0.2501	0.1242	0.0621	0.03102	0.0155	0.0078	0.0039
		CP%	100.00	100.00	97.843	94.4945	93.600	94.100	94.700
0.2	$\widehat{H}_N$	Mean	0.1883	0.1994	0.2009	0.2016	0.2008	0.2005	0.2002
		S.dev.	0.3443	0.1480	0.0709	0.0353	0.0180	0.0088	0.0045
		$\sqrt{\hat{\sigma}/N}$	0.2953	0.1468	0.0723	0.0360	0.0180	0.0090	0.0045
		CP%	100.00	100.00	98.385	96.185	93.900	95.600	95.100
0.3	$\widehat{H}_N$	Mean	0.2867	0.2964	0.2943	0.2976	0.2993	0.2998	0.3000
		S.dev.	0.4530	0.1958	0.0950	0.0472	0.0230	0.0120	0.0060
		$\sqrt{\hat{\sigma}/N}$	0.3973	0.1950	0.0952	0.0468	0.0233	0.0116	0.0058
		CP%	100.00	100.00	97.204	96.990	95.996	94.800	94.200
0.4	$\widehat{H}_N$	Mean	0.4556	0.3770	0.3977	0.4001	0.4008	0.4006	0.4002
		S.dev.	0.7684	0.4077	0.1728	0.0788	0.0406	0.0194	0.0098
		$\sqrt{\hat{\sigma}/N}$	0.6580	0.3725	0.1729	0.0821	0.0402	0.0200	0.0100
		CP%	100.00	100.00	100.00	96.815	96.597	98.709	95.983

Table 2: The estimator $\hat{\sigma}_N^2$ with $\theta = 0.5$, $\sigma^2 = 2.25$, $\kappa^2 = 6.25$ ($h = 1$)

H			N: 2^8	2^{10}	2^{12}	2^{14}	2^{16}	2^{18}	2^{20}
0.1	$\hat{\sigma}_N^2$	Mean	2.6481	2.2573	2.1847	2.2235	2.2423	2.2476	2.2494
		S.dev.	1.1740	0.9185	0.5952	0.2968	0.1449	0.0729	0.0351
		$\sqrt{\hat{\sigma}/N}$	2.9146	1.4187	0.6404	0.3003	0.1481	0.0739	0.0369
		CP%	100.00	100.00	99.569	95.596	95.200	95.700	95.700
0.2	$\hat{\sigma}_N^2$	Mean	3.2483	2.6297	2.1801	2.1760	2.2247	2.2374	2.2449
		S.dev.	1.4562	1.1533	0.8928	0.5054	0.2551	0.1244	0.0639
		$\sqrt{\hat{\sigma}/N}$	4.0741	2.0802	1.1192	0.5437	0.2639	0.1307	0.0651
		CP%	100.00	99.236	95.156	96.687	95.900	96.600	95.600
0.3	$\hat{\sigma}_N^2$	Mean	4.2080	3.3885	2.7771	2.2701	2.1988	2.2348	2.2438
		S.dev.	1.7810	1.5093	1.2168	0.9432	0.5715	0.2885	0.1451
		$\sqrt{\hat{\sigma}/N}$	6.1117	3.4183	1.9875	1.1868	0.5950	0.2909	0.1447
		CP%	100.00	90.035	88.149	92.688	96.597	95.300	94.500
0.4	$\hat{\sigma}_N^2$	Mean	5.6821	5.4706	4.5431	3.4776	2.6655	2.2155	2.1882
		S.dev.	1.8311	1.7265	1.7115	1.5328	1.2756	0.9317	0.5582
		$\sqrt{\hat{\sigma}/N}$	12.580	6.9087	4.2235	3.1358	2.0535	1.1712	0.5808
		CP%	99.495	73.316	69.288	78.662	89.136	94.624	96.285

Table 3: The estimator $\hat{\kappa}_N^2$ with $\theta = 0.5$, $\sigma^2 = 2.25$, $\kappa^2 = 6.25$ ($h = 1$)

H			N: 2^8	2^{10}	2^{12}	2^{14}	2^{16}	2^{18}	2^{20}
0.1	$\hat{\kappa}_N^2$	Mean	10.199	6.7111	6.3188	6.2798	6.2584	6.2527	6.2509
		S.dev.	31.076	2.9581	0.6486	0.2985	0.1465	0.0747	0.0355
		$\sqrt{\hat{\sigma}/N}$	2.8760	1.3996	0.6304	0.2953	0.1455	0.0725	0.0362
		CP%	100.00	99.854	98.274	94.895	95.100	94.900	94.800
0.2	$\hat{\kappa}_N^2$	Mean	14.146	9.0285	6.5452	6.3359	6.2741	6.2607	6.2550
		S.dev.	61.312	23.300	1.2291	0.5379	0.2606	0.1240	0.0644
		$\sqrt{\hat{\sigma}/N}$	3.9729	2.0298	1.0929	0.5291	0.2564	0.1269	0.0632
		CP%	100.00	98.012	95.048	95.984	94.600	95.300	94.300
0.3	$\hat{\kappa}_N^2$	Mean	14.839	17.583	8.6260	6.5135	6.3042	6.2655	6.2553
		S.dev.	120.86	201.09	16.263	1.5022	0.5756	0.2883	0.1456
		$\sqrt{\hat{\sigma}/N}$	5.9108	3.3183	1.9330	1.1579	0.5798	0.2831	0.1407
		CP%	99.700	89.511	87.084	92.258	95.596	94.700	93.800
0.4	$\hat{\kappa}_N^2$	Mean	7.9033	8.9002	16.301	20.014	8.5023	6.5251	6.3212
		S.dev.	35.401	33.767	93.602	158.18	12.442	1.2954	0.5757
		$\sqrt{\hat{\sigma}/N}$	12.123	6.6486	4.0812	3.0463	2.0029	1.1443	0.5672
		CP%	98.485	72.280	67.416	77.389	88.089	94.516	95.683

Table 4: The estimator $\hat{\theta}_N$ with $\theta = 0.5$, $\sigma^2 = 2.25$, $\kappa^2 = 6.25$ ($h = 1$)

H			N 2^8	2^{10}	2^{12}	2^{14}	2^{16}	2^{18}	2^{20}
0.1	$\hat{\theta}_N$	Mean	0.4981	0.5015	0.5013	0.5005	0.5002	0.5001	0.5000
		S.dev.	0.0933	0.0464	0.0237	0.0113	0.0058	0.0030	0.0015
		$\sqrt{\hat{\sigma}/N}$	0.0821	0.0425	0.0226	0.0117	0.0059	0.0029	0.0015
		CP%	86.773	89.781	91.909	95.996	95.400	94.800	95.000
0.2	$\hat{\theta}_N$	Mean	0.5067	0.5017	0.4997	0.5003	0.5000	0.5000	0.5000
		S.dev.	0.0971	0.0486	0.0235	0.0117	0.0060	0.0029	0.0015
		$\sqrt{\hat{\sigma}/N}$	0.0946	0.0475	0.0227	0.0116	0.0059	0.0030	0.0015
		CP%	90.541	90.367	93.003	94.177	93.800	94.700	94.500
0.3	$\hat{\theta}_N$	Mean	0.4993	0.4985	0.4994	0.5002	0.4999	0.4999	0.5000
		S.dev.	0.1054	0.0501	0.0251	0.0127	0.0058	0.0028	0.0015
		$\sqrt{\hat{\sigma}/N}$	0.1107	0.0551	0.0260	0.0120	0.0060	0.0030	0.0015
		CP%	93.114	94.755	92.943	92.043	94.895	97.000	95.400
0.4	$\hat{\theta}_N$	Mean	0.5050	0.5025	0.5015	0.5012	0.5003	0.5000	0.5001
		S.dev.	0.1305	0.0605	0.0282	0.0134	0.0066	0.0032	0.0016
		$\sqrt{\hat{\sigma}/N}$	0.1341	0.0702	0.0338	0.0155	0.0072	0.0034	0.0017
		CP%	94.950	97.409	97.378	96.975	95.026	96.559	95.582

We observe that the cover probability and asymptotic variance estimators demonstrate better results with smaller values of H in comparison with parameter H closer to $1/2$ where constructed estimators demonstrate the lower rate of convergence and this is consistent with numerical simulations results obtained in Kukush *et al.* (2022). At the same time our estimates for asymptotic variance of (2) constructed in Theorem 2.2 remain quite accurate for any values of $H \in (0, \frac{1}{2})$ and θ. Respective simulation results can be found in Table 5. Therefore, obtained simulation results show that the Σ_0 from Theorem 3.1 is more sensitive to changes in model parameters compared to the asymptotic covariance matrix from Theorem 2.2.

Finally, let us mention that for a small number N there is a percentage of iterations that result in some estimators (3) being non-evaluable (Non Ev.%) or estimated value of H being out of the interval $(0, \frac{1}{2})$ (H Out%). In such cases, asymptotic covariance matrices cannot be estimated because the required series (8) does not converge. The corresponding simulation results can be found in Table 6.

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Table 5: The cover probability for estimators $(\phi_N, \xi_N, \eta_N, \zeta_N)$ with $\theta = 0.5$, $\sigma^2 = 2.25$, $\kappa^2 = 6.25$ $(h = 1)$

H		N 2^8	2^{10}	2^{12}	2^{14}	2^{16}	2^{18}	2^{20}
0.1	ϕ_N	86.773	89.781	91.909	95.996	95.400	94.800	95.000
	ξ_N	94.444	94.161	93.851	95.095	95.000	95.300	94.400
	η_N	97.884	95.329	95.685	94.995	93.400	95.100	95.100
	ζ_N	99.471	99.124	96.332	95.395	94.100	94.300	95.500
0.2	ϕ_N	90.541	90.367	93.003	94.177	93.800	94.700	94.500
	ξ_N	96.487	94.954	94.941	94.779	95.300	94.800	94.100
	η_N	98.649	97.248	95.587	94.578	94.200	95.100	95.100
	ζ_N	100.00	98.777	97.094	95.382	94.700	94.700	93.400
0.3	ϕ_N	93.114	94.755	92.943	92.043	94.895	97.000	95.400
	ξ_N	94.910	96.154	95.473	95.376	95.496	94.600	94.800
	η_N	97.904	96.329	95.340	95.484	95.496	94.300	94.600
	ζ_N	99.102	99.126	96.671	97.097	95.696	95.600	94.900
0.4	ϕ_N	94.950	97.409	97.378	96.975	95.026	96.559	95.582
	ξ_N	95.455	93.523	93.633	95.382	95.550	94.946	95.482
	η_N	96.970	97.409	96.255	95.860	95.812	94.516	95.080
	ζ_N	100.00	99.741	97.753	96.975	97.513	96.452	94.880

Table 6: Percentage of estimated results that cannot be used for estimating asymptotic covariance matrix in previous simulations

H		N 2^8	2^{10}	2^{12}	2^{14}	2^{16}	2^{18}	2^{20}
0.1	Non Ev.%	27.0	9.1	0.5	0.0	0.0	0.0	0.0
	H Out%	35.2	22.4	6.8	0.1	0.0	0.0	0.0
0.2	Non Ev.%	37.9	24.8	6.4	0.4	0.0	0.0	0.0
	H Out%	25.1	9.8	0.7	0.0	0.0	0.0	0.0
0.3	Non Ev.%	45.2	36.5	24.7	7.0	0.1	0.0	0.0
	H Out%	21.4	6.3	0.2	0.0	0.0	0.0	0.0
0.4	Non Ev.%	60.4	47.5	44.6	37.2	23.6	7.0	0.4
	H Out%	19.8	13.9	2.0	0.0	0.0	0.0	0.0

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