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Limit Theorems for Additive Functionals of Continuous-Time Lattice Random Walks in a Stationary Random Environment

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Abstract

For a continuous-time lattice random walk $X^\Lambda = \left\{X_t^\Lambda, t \geq 0\right\}$ in a random environment Λ, we study the asymptotic behavior, as $t \to \infty$, of the normalized additive functional $c_t \int_0^t f(X_s^\Lambda)ds$, $t \geq 0$. We establish a limit theorem for it, which is similar to that in the non-lattice case, under less restrictive assumptions.

Keywords: continuous-time random walk, additive functional, domain of attraction of stable law, α-stable Lévy motion, local time, stationary random environment.

1. Introduction

Continuous-time random walks (CTRWs) are stochastic processes that model the pure-jump dynamics with random waiting times between jumps. They are widely used in physics to describe anomalous diffusion phenomena, such as transport in heterogeneous media, Hamiltonian chaos, Lévy flights, and have further applications in hydrology, biology, finance and other areas.

Limit theorems for CTRWs are important tools to understand the asymptotic behavior and scaling properties of these processes. A crucial aspect here is that, unless the distribution of waiting times is degenerate or exponential, CTRWs are non-Markov processes. This creates a challenge in deriving limit theorems for them and also leads to a plethora of possible limit processes in many cases exhibiting some "fractional" kind of dynamics: fractional diffusion processes, fractional Brownian motion, fractional stable processes etc.

A seminal result was obtained by Meerschaert and Scheffler (2004), who showed that, when the waiting time distribution has a regularly varying tail with index $\alpha \in (0, 1)$, the scaling limit of a CTRW is an operator Lévy motion subordinated to the hitting time process of a classical stable subordinator. The authors also showed that the density function of the limit process solves a fractional Cauchy problem. Since then, there was an extensive literature in this direction, we will only mention a few articles: Barczyk and Kern (2013); Jurlewicz, Kern, Meerschaert, and Scheffler (2012); Leonenko, Papić, Sikorskii, and Šuvak (2018); Baeumer and Straka (2017); Straka and Henry (2011).

Random walks in a random environment (RWRE) model the dynamics in a medium whose properties vary randomly in space or/and time. They are relevant for many applications in physics, biology, ecology, and computer science, such as diffusion in disordered media, population dynamics, percolation, and load balancing. The quenched randomness of the environment induces dependence and heterogeneity in the motion of the walker, which complicates their study and leads to some interesting limiting objects. One of the first most important contributions here is Kesten, Kozlov, and Spitzer (1975), which established the convergence of one-dimensional RWRE to diffusion processes with random drifts under some conditions on the environment. Since then, limit theorems for RWREs were studied by many authors for different kinds of environments and their effects on the walker; we cite only few of these contributions: Kesten and Spitzer (1979); Sinaĭ (1982); Guillotin-Plantard (2001); Bolthausen (1989); Avena, den Hollander, and Redig (2011); Redig and Völlering (2013).

More recently, the article Kondratiev, Mishura, and Shevchenko (2021) investigated the asymptotic behavior of the normalized additive functional $c_t \int_0^t f(X_s)ds$ for a CTRW $X = \{X_t, t \geq 0\}$, and showed that under certain conditions on the distribution of jumps and waiting times of X, the additive functional converges in distribution to the local time at zero of an alpha-stable Levy motion. The authors also considered a situation where X is delayed by a random environment given by the Poisson shot-noise potential $\Lambda(x,\gamma) = e^{-\sum_{y\in\gamma}\phi(x_y)}$, where ϕ is a bounded function decaying sufficiently fast, and γ is a homogeneous Poisson point process, independent of X, and found that in this case the weak limit has both "quenched" component depending on Λ, and a component, where Λ is "averaged".

In this article, we extend the results of Kondratiev *et al.* (2021) to the case of a lattice CTRW. This case is technically simpler, which allows us, on the one hand, to drop rather restrictive assumptions on the distribution of jumps imposed in Kondratiev *et al.* (2021), and on the other hand, to consider a very general random environment.

2. Preliminaries

2.1. Domains of attraction

We will give the basic definitions concerning the domains of attraction and stable laws, for details see (Feller 1971, Chapter XVII) and Zolotarev (1986).

Definition 2.1. A random variable ξ is said to have a stable distribution with index $\alpha \in (1,2]$ if its characteristic function has the form

$$\varphi_\xi(x) = \exp\left\{iax - c|x|^\alpha \omega(x,\alpha,\beta)\right\},$$

where $\omega(x,\alpha,\beta) = 1 + i\beta \operatorname{sign} x \tan\left(\frac{\pi\alpha}{2}\right)$; $c > 0$ is called the scale parameter, $\beta \in [-1,1]$ is called the skewness parameter, $a \in \mathbb{R}$ is the expected value.

Definition 2.2. The distribution $\mathcal{L}$ is said to belong to the domain of attraction to stable law with index $\alpha \in (1,2]$ if there exist some sequence $a_n \in \mathbb{R}$ and a slowly varying function L such that the normalized sums

$$\frac{\xi_1 + \cdots + \xi_n}{L(n)n^{1/\alpha}} - a_n$$

of iid random variables $\{\xi_n, n \geq 1\}$ with distribution $\mathcal{L}$ converge in distribution, as $n \to \infty$, to a stable distribution with index α.

Definition 2.3. If in Definition 2.2, $L(n) = \sigma$ for some constant $\sigma > 0$, we say that $\mathcal{L}$ belongs to the domain of normal attraction to stable law with index $\alpha \in (1,2]$.

2.2. Symmetric local time

Definition 2.4. For a measurable real-valued stochastic process $\{X(t), t \geq 0\}$, a symmetric local time at a point $x \in \mathbb{R}$ on an interval $[0, t]$ is defined as the limit in probability

$$\ell_X(t, x) := \mathsf{P}\text{-}\lim_{\varepsilon \to 0+} \frac{1}{2\varepsilon} \int_0^t \mathbb{1}_{[x-\varepsilon, x+\varepsilon]}(X_s) ds.$$

2.3. Notation

For any random variable X, we denote by $\varphi_X(\lambda) = \mathsf{E}\left[e^{i\lambda X}\right]$ its characteristic function. The symbols $\xrightarrow{\mathsf{P}}$ and $\xrightarrow{\mathrm{d}}$ designate the convergence in probability and in distribution, respectively.

3. Main results

Let us introduce the main object of this article. Consider a sequence ξ_n, $n \geq 1$ of iid lattice random variables; wlog let them be integer-valued with gcd $\{k \in \mathbb{Z} : \mathsf{P}(\xi_1 = k) > 0\} = 1$. The corresponding random walk $S_n = \xi_1 + \cdots + \xi_n$ is then supported by $\mathbb{Z}$. Given a sequence θ_n, $n \geq 1$, of iid positive random variables independent of ξ, a corresponding continuous-time random walk is given by $X_t = S_{N_t}$, where $N_t = \max\left\{k \geq 0 : \sum_{i=1}^k \theta_i \leq t\right\}$ is the number of jumps occurring until t. Let X be further delayed by a *random environment* $\{\Lambda(n), n \in \mathbb{Z}\}$, which is a positive process independent of $\{\theta_n, n \geq 1\}$ and $\{\xi_n, n \geq 1\}$. Specifically, we define the Λ-*delayed CTRW* by

$$X_t^\Lambda = S_{N_t^\Lambda}, \tag{1}$$

where

$$N_t^\Lambda = \max\left\{k \geq 0 : \sum_{i=1}^k \theta_i / \Lambda(S_{i-1}) \leq t\right\}. \tag{2}$$

3.1. Law of large numbers for CTRW in a random environment

A1. The jumps satisfy $\mathsf{P}(\xi_n \in \mathbb{Z}) = 1$ and $\gcd(\{k : P(\xi_n = k) > 0\}) = 1$.

We will need the following lemma from Robbins (1953).

Lemma 3.1. *Under the assumption A1, for any* $t \in \mathbb{R}$

$$\lim_{n \to \infty} \frac{1}{n} \sum_{j=1}^n e^{itS_j} = \mathbb{1}_{\{t \in 2\pi\mathbb{Z}\}}$$

almost surely.

Proposition 3.1. *Let the assumption A1 hold, and an* L^2*-process* $\Pi = \{\Pi(t), t \in \mathbb{Z}\}$ *be weakly stationary and ergodic. Then the following law of large numbers holds:*

$$\frac{1}{n} \sum_{k=1}^n \Pi(S_k) \xrightarrow{L^2(\Omega)} \mathsf{E}\left[\Pi(0)\right], \quad n \to \infty.$$

Proof. Write the spectral representation:

$$\Pi(t) = \int_{-\pi}^{\pi} e^{it\lambda} Z(\mathrm{d}\lambda),$$

where Z is an orthogonal random measure; denote its control measure by G.

Thanks to independence of S_k and Π, we may write

$$\begin{aligned}
&\mathsf{E}\left[\left(\frac{1}{n}\sum_{k=1}^{n}\Pi(S_k)-Z(\{0\})\right)^2\right]\\
&=\mathsf{E}\left[\mathsf{E}\left[\left(\frac{1}{n}\sum_{k=1}^{n}\Pi(x_k)-Z(\{0\})\right)^2\right]\Big|_{x_1=S_1,\dots,x_n=S_n}\right]\\
&=\mathsf{E}\left[\mathsf{E}\left[\left(\int_{-\pi}^{\pi}\left(\frac{1}{n}\sum_{k=1}^{n}e^{ix_k\lambda}-\mathbb{1}_{\{\lambda=0\}}\right)Z(\mathrm{d}\lambda)\right)^2\right]\Big|_{x_1=S_1,\dots,x_n=S_n}\right]\\
&=\mathsf{E}\left[\int_{-\pi}^{\pi}\left|\frac{1}{n}\sum_{k=1}^{n}e^{iS_k\lambda}-\mathbb{1}_{\{\lambda=0\}}\right|^2G(\mathrm{d}\lambda)\right].
\end{aligned}$$

The integrand here is obviously bounded, so it follows from Lemma 3.1 and the dominated convergence theorem that

$$\mathsf{E}\left[\left(\frac{1}{n}\sum_{k=1}^{n}\Pi(S_k)-Z(\{0\})\right)^2\right]\to 0,\quad n\to\infty.$$

Since Π is weakly stationary and ergodic, we have $Z(\{0\})=\mathsf{E}[\Pi(0)]$ almost surely, which implies the statement. □

Assuming that the process is just integrable, it is possible to show the convergence in $L^1(\Omega)$.

Proposition 3.2. *Let the assumption A1 hold, and an ergodic integrable process* $\Pi=\{\Pi(t),t\in\mathbb{Z}\}$ *be such that* $\Pi_n(t):=\Pi(t)\mathbb{1}_{\{|\Pi(t)|\le n\}}$ *are weakly stationary for all* $n\ge 1$*. Then,*

$$\frac{1}{n}\sum_{k=1}^{n}\Pi(S_k)\xrightarrow{L^1(\Omega)}\mathsf{E}[\Pi(0)],\quad n\to\infty.$$

Proof. From Proposition 3.1 it follows that for each $n\ge 1$

$$\frac{1}{m}\sum_{k=1}^{m}\Pi_n(S_k)\xrightarrow{L^1(\Omega)}\mathsf{E}[\Pi_n(0)],\quad m\to\infty. \tag{3}$$

Note that it follows from our assumptions that $\mathsf{E}[\Pi(x)]$ and hence $\mathsf{E}\left[\Pi(x)\mathbb{1}_{\{|\Pi(x)|>n\}}\right]$ do not depend on x. Therefore, using the independence of Π and $S_k, k\ge 1$, we have

$$\begin{aligned}
&\sup_{m\ge 1}\mathsf{E}\left[\left|\frac{1}{m}\sum_{k=1}^{m}\Pi(S_k)-\frac{1}{m}\sum_{k=1}^{m}\Pi_n(S_k)\right|\right]\\
&\le\sup_{m\ge 1}\frac{1}{m}\sum_{k=1}^{m}\mathsf{E}\left[\left|\Pi(S_k)\mathbb{1}_{\{|\Pi(S_k)|>n\}}\right|\right]\\
&=\sup_{m\ge 1}\frac{1}{m}\sum_{k=1}^{m}\mathsf{E}\left[\mathsf{E}\left[\left|\Pi(x)\mathbb{1}_{\{|\Pi(x)|>n\}}\right|\right]\Big|_{x=S_k}\right]\\
&=\sup_{m\ge 1}\frac{1}{m}\sum_{k=1}^{m}\mathsf{E}\left[\left|\Pi(0)\mathbb{1}_{\{|\Pi(0)|>n\}}\right|\right]\\
&=\mathsf{E}\left[\left|\Pi(0)\mathbb{1}_{\{|\Pi(0)|>n\}}\right|\right]\to 0,\quad n\to\infty.
\end{aligned} \tag{4}$$

Now write

$$\begin{aligned}
&\mathsf{E}\left[\left|\frac{1}{m}\sum_{k=1}^{m}\Pi(S_k)-\mathsf{E}\left[\Pi(0)\right]\right|\right]\\
&\le \mathsf{E}\left[\left|\frac{1}{m}\sum_{k=1}^{m}\Pi(S_k)-\frac{1}{m}\sum_{k=1}^{m}\Pi_n(S_k)\right|\right]\\
&+\mathsf{E}\left[\left|\frac{1}{m}\sum_{k=1}^{m}\Pi_n(S_k)-\mathsf{E}\left[\Pi_n(0)\right]\right|\right]+\left|\mathsf{E}\left[\Pi_n(0)\right]-\mathsf{E}\left[\Pi(0)\right]\right|.
\end{aligned}$$

This yields, thanks to eq. (3), that

$$\begin{aligned}
&\limsup_{m\to\infty}\mathsf{E}\left[\left|\frac{1}{m}\sum_{k=1}^{m}\Pi(S_k)-\mathsf{E}\left[\Pi(0)\right]\right|\right]\\
&\le \sup_{m\ge 1}\mathsf{E}\left[\left|\frac{1}{m}\sum_{k=1}^{m}\Pi(S_k)-\frac{1}{m}\sum_{k=1}^{m}\Pi_n(S_k)\right|\right]+\left|\mathsf{E}\left[\Pi(0)\mathbb{1}_{\{|\Pi(0)|>n\}}\right]\right|.
\end{aligned}$$

Letting $n\to\infty$ and using eq. (4), we arrive at the desired statement. □

Remark 3.1. *The assumption of Proposition 3.2 about* Π *may sound a bit artificial. A natural sufficient condition is that* Π *is strongly stationary, ergodic and integrable.*

3.2. Law of large numbers for counting process

We start by establishing a law of large numbers for the counting process. We will need two additional assumptions.

A2. The process $\Lambda^{-1}=\{\Lambda^{-1}(t),t\in\mathbb{Z}\}$ satisfies the conditions of Proposition 3.1 or Proposition 3.2.

A3. The time between jumps θ_n is integrable:

$$\mathsf{E}\left[\theta_1\right]=\mu.$$

The assumption A2 implies that $\Lambda^{-1}(t)$ is integrable; we denote $\overline{\Lambda^{-1}}:=\mathsf{E}\left[\Lambda^{-1}(0)\right]$.

Proposition 3.3. *Let assumptions A1–A3 be satisfied and the process* N_t^{Λ} *is defined by eq. (2). Then,*

$$\frac{1}{n}\sum_{k=1}^{n}\frac{\theta_k}{\Lambda(S_{k-1})}\xrightarrow{\mathsf{P}}\mu\overline{\Lambda^{-1}},\quad n\to\infty,$$

and

$$\frac{N_t^{\Lambda}}{t}\xrightarrow{\mathsf{P}}\frac{1}{\mu\overline{\Lambda^{-1}}},\quad t\to\infty.$$

Proof. For the proof, we will use the ideas from Kondratiev *et al.* (2021). Define $\gamma_n:=\frac{1}{n}\sum_{k=1}^{n}\frac{\theta_k}{\Lambda(S_{k-1})}$. First, we note that by assumption A3 we have

$$\varphi_{\theta_1}(t)=1+i\mu t+o(t),t\to 0.$$

Denoting by Log the principal branch of complex logarithm (i.e. such that $\operatorname{Log}z\in(-\pi,\pi]$ for all non-zero z on the unit circle), it is clear that

$$\operatorname{Log}(1+x)-x=o(x),x\to 0.$$

Hence, we have that

$$r(t) := \mathrm{Log}(\varphi_{\theta_1}(t)) - i\mu t = o(t), \quad t \to 0.$$

Since the random variables $\{\theta_n, n \geq 1\}$ are jointly independent and do not depend on Λ, the characteristic function of γ_n can be written as

$$\begin{aligned}
\varphi_{\gamma_n}(\lambda) &= \mathsf{E}\left[\mathsf{E}\left[\exp\left\{ \left\{ \frac{i\lambda}{n} \sum_{k=1}^{n} \theta_k x_k \right\} \right\} \quad \right] \Bigg|_{x_k = \Lambda(S_{k-1})^{-1},\ k=1,\dots,n} \right] \\
&= \mathsf{E}\left[\prod_{k=1}^{n} \varphi_{\theta_1}\Big(\frac{\lambda}{n\Lambda(S_{k-1})}\Big) \right] = \mathsf{E}\left[\exp\left\{ \sum_{k=1}^{n} \mathrm{Log}\, \varphi_{\theta_1}\Big(\frac{\lambda}{n\Lambda(S_{k-1})}\Big) \right\} \right] \\
&= \mathsf{E}\left[\exp\left\{ \frac{i\mu\lambda}{n} \sum_{k=1}^{n} \frac{1}{\Lambda(S_{k-1})} + R_n \right\} \right],
\end{aligned}$$

where

$$R_n = \sum_{k=1}^{n} r\Big(\frac{\lambda}{n\Lambda(S_{k-1})}\Big), \ r(x) = \mathrm{Log}\, \varphi_{\theta_1}(x) - i\mu x = o(x), \quad x \to 0.$$

It follows from the Assumption A2 that

$$Y_n := \frac{1}{n} \sum_{k=1}^{n} \frac{1}{\Lambda(S_{k-1})} \xrightarrow{\mathsf{P}} \overline{\Lambda^{-1}}, \quad n \to \infty. \tag{5}$$

Therefore, in order to establish the first statement, it suffices to show that $R_n \xrightarrow{\mathsf{P}} 0$.

Let us first assume that the process Λ^{-1} is bounded by a some nonrandom constant.

Fix $\varepsilon > 0$. For any $a > 0$, there exists $\delta > 0$, $|r(x)| \leq a|x|$, with $|x| < \delta$. According to our assumption, for sufficiently large n, $\Lambda(S_k)^{-1} \leq n\delta / |\lambda|$ holds, and hence,

$$|R_n| \leq \sum_{k=1}^{n} a \left| \frac{\lambda}{n\Lambda(S_{k-1})} \right| \leq a|\lambda| Y_n.$$

Therefore,

$$\varlimsup_{n\to\infty} \mathsf{P}(|R_n| \geq \varepsilon) \leq \varlimsup_{n\to\infty} \mathsf{P}\left(Y_n \geq \frac{\varepsilon}{a|\lambda|} \right).$$

Letting $a \to 0+$, we get

$$\varlimsup_{n\to\infty} \mathsf{P}(|R_n| \geq \varepsilon) \leq \lim_{C\to\infty} \varlimsup_{n\to\infty} \mathsf{P}(Y_n \geq C) = 0,$$

since the sequence $\{Y_n, n \geq 1\}$ is bounded in probability by eq. (5). Therefore, $R_n \xrightarrow{\mathsf{P}} 0$, $n \to \infty$. Consequently, we obtain the following convergence from Lévy's theorem:

$$\frac{1}{m} \sum_{k=1}^{m} \frac{\theta_k}{\Lambda(S_{k-1})} \xrightarrow{\mathsf{P}} \mu\overline{\Lambda^{-1}}, m \to \infty.$$

In the general case where Λ^{-1} is unbounded, we define $\Lambda_n(t) := \Lambda(t)\mathbb{1}_{\{|\Lambda(t)| \geq 1/n\}}$. From the above reasoning, it follows that

$$\frac{1}{m} \sum_{k=1}^{m} \frac{\theta_k}{\Lambda_n(S_{k-1})} \xrightarrow{\mathsf{P}} \mu\overline{\Lambda_n^{-1}}, \quad m \to \infty,$$

where $\overline{\Lambda_n^{-1}} = \mathsf{E}\left[\Lambda_n^{-1}(0)\right]$. Write

$$\left|\frac{1}{m}\sum_{k=1}^{m}\frac{\theta_k}{\Lambda(S_{k-1})} - \mu\overline{\Lambda^{-1}}\right|$$
$$\leq \left|\frac{1}{m}\sum_{k=1}^{m}\frac{\theta_k}{\Lambda(S_{k-1})} - \frac{1}{m}\sum_{k=1}^{m}\frac{\theta_k}{\Lambda_n(S_{k-1})}\right| + \left|\frac{1}{m}\sum_{k=1}^{m}\frac{\theta_k}{\Lambda_n(S_{k-1})} - \mu\overline{\Lambda_n^{-1}(0)}\right| + \left|\mu\overline{\Lambda_n^{-1}(0)} - \mu\overline{\Lambda^{-1}(0)}\right|.$$

For the first term here, we apply the same reasoning as in Proposition 3.2:

$$\mathsf{E}\left[\left|\frac{1}{m}\sum_{k=1}^{m}\frac{\theta_k}{\Lambda(S_{k-1})} - \frac{1}{m}\sum_{k=1}^{m}\frac{\theta_k}{\Lambda_n(S_{k-1})}\right|\right] \leq \frac{1}{m}\sum_{k=1}^{m}\mathsf{E}\left[\left|\theta_k\left(\Lambda^{-1}(S_{k-1}) - \Lambda_n^{-1}(S_{k-1})\right)\right|\right]$$
$$= \frac{1}{m}\sum_{k=1}^{m}\mathsf{E}\left[\theta_k\right]\mathsf{E}\left[\left|\Lambda^{-1}(S_{k-1})\right|\mathbb{1}_{\{|\Lambda(S_{k-1})|<1/n\}}\right] = \mu\,\mathsf{E}\left[\left|\Lambda^{-1}(0)\right|\mathbb{1}_{\{|\Lambda(0)|<1/n\}}\right] \to 0, \quad n\to\infty.$$

This also shows that the third term converges to 0. For the second term, fix arbitrary $\varepsilon > 0$ and write

$$\varlimsup_{m\to\infty}\mathsf{P}\left(\left|\frac{1}{m}\sum_{k=1}^{m}\frac{\theta_k}{\Lambda(S_{k-1})} - \mu\overline{\Lambda^{-1}(0)}\right| > \varepsilon\right)$$
$$\leq \mathsf{P}\left(\left|\frac{1}{m}\sum_{k=1}^{m}\frac{\theta_k}{\Lambda(S_{k-1})} - \frac{1}{m}\sum_{k=1}^{m}\frac{\theta_k}{\Lambda_n(S_{k-1})}\right| > \varepsilon/3\right) + \mathsf{P}\left(\left|\mu\overline{\Lambda_n^{-1}(0)} - \mu\overline{\Lambda^{-1}(0)}\right| > \varepsilon/3\right)$$
$$\leq \frac{\frac{1}{m}\sum_{k=1}^{m}\mathsf{E}\left[\left|\frac{\theta_k}{\Lambda(S_{k-1})} - \frac{\theta_k}{\Lambda_n(S_{k-1})}\right|\right]}{\varepsilon/3} + \mathsf{P}\left(\left|\mu\overline{\Lambda_n^{-1}(0)} - \mu\overline{\Lambda^{-1}(0)}\right| > \varepsilon/3\right)$$
$$\leq \frac{3\mu_\theta\mathsf{E}\left[\left|\Lambda^{-1}(0)\right|\mathbb{1}_{\{|\Lambda^{-1}(0)|<1/n\}}\right]}{\varepsilon} + \mathsf{P}\left(\left|\mu\overline{\Lambda_n^{-1}(0)} - \mu\overline{\Lambda^{-1}(0)}\right| > \varepsilon/3\right).$$

Letting $n\to\infty$, we arrive at the first statement.
To show the second statement, note that whenever $x < \left(\mu\overline{\Lambda^{-1}}\right)^{-1}$,

$$\mathsf{P}\left(N_t^{\Lambda} \leq tx\right) = \mathsf{P}\left(\sum_{i=1}^{\lfloor tx\rfloor}\frac{\theta_i}{\Lambda(S_{i-1})} \geq t\right) = \mathsf{P}\left(\gamma_{\lfloor tx\rfloor} \geq \frac{t}{\lfloor tx\rfloor}\right) \to 0, \quad t\to+\infty,$$

since $\lim_{t\to\infty}\frac{t}{\lfloor tx\rfloor} = \frac{1}{x} < \mu\overline{\Lambda^{-1}}$. Similarly, for any $x > \left(\mu\overline{\Lambda^{-1}}\right)^{-1}$, we get $\mathsf{P}\left(N_t^{\Lambda} \geq tx\right) \to 0$, $t\to\infty$, concluding the proof. □

Before formulating the basic law, let us formulate one more condition, namely the condition on the distribution of jumps and an auxiliary statement that will determine the properties of the additive functional:

A4. The jump values $\{\xi_n,\, n\geq 1\}$ are centered and belong to the domain of attraction of an α-stable distributions with $\alpha\in(1,2]$. In this case (see e.g. (Resnick 1986, Proposition 3.4), there is a weak convergence in $D[0,\infty)$ with Skorohod's topology

$$\left\{\frac{1}{L(n)n^{1/\alpha}}\sum_{k=1}^{[nt]}\xi_k, t\geq 0\right\} \xrightarrow{\mathrm{d}} \{Z_\alpha(t), t\geq 0\} \quad, n\to\infty,$$

to an α-stable Lévy process Z_α.

We denote $c_t = L(t)t^{1/\alpha-1}$ for $t>0$; observe that it is regularly varying of index $1/\alpha - 1$.

Theorem 3.1 ((Borodin and Ibragimov 1995, Theorems 2.2. and 4.2)). *Under the assumptions A1 and A4, for any function f such that $\sum_{k=-\infty}^{\infty} |f(k)| < \infty$, the convergence in law holds:*

$$c_n \sum_{k=1}^{n} f(S_k) \xrightarrow{\mathrm{d}} \ell_\alpha(1,0) \sum_{k=-\infty}^{\infty} f(k), \quad n \to \infty,$$

where $\ell_\alpha(1,0)$ is the symmetric local time at zero of the process Z_α on the interval $[0,t]$.

3.3. Limit theorem for additive functional of CTRW

The main result of this article is the following theorem.

Theorem 3.2. *Let the conditions A1-A4 be satisfied, $\sum_{n=-\infty}^{\infty} |g(n)| < \infty$. Then, the finite-dimensional distributions of the process*

$$c_t \int_0^{tu} g(X_s^\Lambda) ds, u \geq 0,$$

converge as $t \to \infty$ to those of

$$\mu^{1/\alpha} \left(\overline{\Lambda^{-1}}\right)^{1/\alpha - 1} \sum_{k=-\infty}^{\infty} \frac{g(k)}{\Lambda(k)} \ell_\alpha(u,0), u \geq 0,$$

where ℓ_α is the symmetric local time of an α-stable Lévy motion Z_α, independent of Λ.

Remark 3.2. *Since*

$$\mathbb{E}\left[\sum_{n=-\infty}^{\infty} \frac{|g(n)|}{\Lambda(n)}\right] = \mathbb{E}\left[\Lambda^{-1}(0)\right] \sum_{n=-\infty}^{\infty} |g(n)| < \infty,$$

the series $\sum_{n=-\infty}^{\infty} \frac{g(n)}{\Lambda(n)}$ converges almost surely.

Proof. To establish the claim, we use the same ideas as in Kondratiev *et al.* (2021). For technical simplicity we will show the convergence of one-dimensional distribution at $u = 1$; the proof in general case is similar. We will actually establish a slightly stronger result. Namely, denoting

$$A_t^\Lambda = c_t \int_0^t g(X_s^\Lambda) ds, B^\Lambda = \mu^{1/\alpha} \left(\overline{\Lambda^{-1}}\right)^{1/\alpha - 1} \sum_{k=-\infty}^{\infty} \frac{g(k)}{\Lambda(k)} \ell_\alpha(1,0),$$

we will show that, conditionally on Λ, $A_t^\Lambda \xrightarrow{\mathrm{d}} B^\Lambda$, $t \to \infty$, in probability. The latter convergence is the convergence in probability $\mathsf{E}\left[F(A_t^\Lambda) \mid \Lambda\right] \xrightarrow{\mathsf{P}} \mathsf{E}\left[F(B^\Lambda) \mid \Lambda\right], t \to \infty$, for any continuous bounded F (if the convergence were almost sure, we would have the so-called quenched convergence).

For $n \geq 0$, denote $\tau_n = \sum_{k=1}^{n} \frac{\theta_k}{\Lambda(S_{k-1})}$, $n \geq 0$, the time of nth jump of X_t^Λ. Then,

$$c_t \int_0^t g(X_s^\Lambda) ds = c_t \sum_{k=1}^{N_t^\Lambda} \theta_k \frac{g(S_{k-1})}{\Lambda(S_{k-1})} + c_t \big(t - \tau_{N_t^\Lambda}\big) g(S_{N_t^\Lambda}).$$

Let us deal with the first sum. We start by looking at

$$\zeta_n = \sum_{k=1}^{n} \theta_k \frac{g(S_{k-1})}{\Lambda(S_{k-1})}.$$

For a non-random function $h\colon \mathbb{Z} \to (0,\infty)$, denote by

$$\varphi_n(\lambda, h) = \mathsf{E}\left[e^{i\lambda c_n \zeta_n} \mid \Lambda = h \right]$$

the conditional characteristic function of $c_n\zeta_n$ given $\Lambda = h$; in view of Remark 3.2, we can assume that $\sum_{n=-\infty}^{\infty} \frac{|g(n)|}{h(n)} < \infty$. Thanks to independence of ξ, θ and Λ, we can write for any $\lambda \neq 0$ that

$$\begin{aligned}
\varphi_n(\lambda, h) &= \mathsf{E}\left[\mathsf{E}\left[\exp\left\{ i\lambda c_n \sum_{k=1}^{n} \theta_k x_k \right\} \right] \Big|_{x_k = \frac{g(S_{k-1})}{h(S_{k-1})},\ k=1,\dots,n} \right] \\
&= \mathsf{E}\left[\prod_{k=1}^{n} \varphi_\theta\Big(\frac{\lambda c_n g(S_{k-1})}{h(S_{k-1})} \Big) \right] \\
&= \mathsf{E}\left[\exp\left\{ \sum_{k=1}^{n} \operatorname{Log} \varphi_\theta\Big(\frac{\lambda c_n g(S_{k-1})}{h(S_{k-1})} \Big) \right\} \right] \\
&= \mathsf{E}\left[\exp\left\{ i\mu\lambda c_n \sum_{k=1}^{n} \frac{g(S_{k-1})}{h(S_{k-1})} + R_n \right\} \right],
\end{aligned}$$

where

$$R_n = \sum_{k=1}^{n} r\Big(\lambda c_n \frac{g(S_{k-1})}{h(S_{k-1})} \Big), \ r(x) = \operatorname{Log} \varphi_\theta(x) - i\mu x = o(x), x \to 0.$$

It follows from Theorem 3.1 that

$$c_n \sum_{k=1}^{n} \frac{g(S_{k-1})}{h(S_{k-1})} \xrightarrow{\mathrm{d}} \sum_{k=-\infty}^{\infty} \frac{g(k)}{h(k)} \ell_\alpha(1,0), \quad n \to \infty.$$

As in Proposition 3.3, we need to show that $R_n \xrightarrow{\mathsf{P}} 0$, $n \to \infty$. To this end, fix $\varepsilon > 0$ and let $\delta > 0$ be such that $|r(t)| < \varepsilon |t|$ for any $|t| < \delta$. Then, we can write

$$|R_n| \le \varepsilon |\lambda| c_n \sum_{k=1}^{n} \left| \frac{g(S_{k-1})}{h(S_{k-1})} \right|.$$

Hence, thanks to eq. (6), for any $\eta > 0$,

$$\varlimsup_{n\to\infty} \mathsf{P}\left(|R_n| \ge \eta \right) \le \mathsf{P}\left(\sum_{k=-\infty}^{\infty} \left| \frac{g(k)}{h(k)} \right| \ell_\alpha(1,0) \ge \frac{\eta}{\varepsilon|\lambda|} \right).$$

Letting $\varepsilon \to 0+$, we obtain that $\varlimsup_{n\to\infty} \mathsf{P}\left(|R_n| \ge \eta \right) = 0$, which gives us the required convergence.

By virtue of Lévy's theorem, we get that, conditionally on $\Lambda = h$,

$$c_n \zeta_n \xrightarrow{\mathrm{d}} \mu \sum_{k=-\infty}^{\infty} \frac{g(k)}{h(k)} \ell_\alpha(1,0), \quad n \to \infty. \tag{6}$$

The next step is to show a similar convergence for $\zeta_{N_t^\Lambda}$. Thanks to Proposition 3.3, for any $\varepsilon > 0$,

$$\mathsf{P}\left(\left| \frac{N_t^\Lambda}{t} - a \right| \ge \varepsilon \right) \to 0, \quad t \to \infty,$$

where $a = \left(\mu \overline{\Lambda^{-1}} \right)^{-1}$, whence

$$\mathsf{P}\left(\left| \frac{N_t^\Lambda}{t} - a \right| \ge \varepsilon \;\middle|\; \Lambda \right) \xrightarrow{\mathsf{P}} 0, \quad t \to \infty. \tag{7}$$

Observe that

$$c_{at}\left|\zeta_{N_t^\Lambda}-\zeta_{\lfloor at\rfloor}\right|\le c_{at}\sum_{k:|k-at|<\varepsilon t+1}\theta_k\frac{|g(S_{k-1})|}{\Lambda(S_{k-1})}$$

whenever $\left|\frac{N_t^\Lambda}{t}-a\right|<\varepsilon$. Therefore, for any $\eta>0$,

$$\begin{aligned}&\mathsf{P}\left(c_{at}\left|\zeta_{N_t^\Lambda}-\zeta_{\lfloor at\rfloor}\right|\ge\eta\mid\Lambda\right)\\&\le\mathsf{P}\left(c_{at}\sum_{k:|k-at|<\varepsilon t+1}\theta_k\frac{|g(S_{k-1})|}{\Lambda(S_{k-1})}\ge\eta\;\middle|\;\Lambda\right)+\mathsf{P}\left(\left|\frac{N_t^\Lambda}{t}-a\right|\ge\varepsilon\;\middle|\;\Lambda\right).\end{aligned}\tag{8}$$

Similarly to eq. (6), it can be shown that, conditionally on Λ,

$$c_{at}\sum_{k:|k-at|<\varepsilon t+1}\theta_k\frac{|g(S_{k-1})|}{\Lambda(S_{k-1})}\xrightarrow{\mathrm{d}}\mu\sum_{k=-\infty}^{\infty}\frac{|g(k)|}{\Lambda(k)}\big(\ell_\alpha(1+\varepsilon/a,0\big)-\ell_\alpha(1-\varepsilon/a,0)\big).\tag{9}$$

Due to eq. (7), from any positive sequence $t_n\to\infty$, we can extract a subsequence t_{n_l} such that

$$\mathsf{P}\left(\left|\frac{N^\Lambda_{t_{n_l}}}{t_{n_l}}-a\right|\ge\varepsilon\;\middle|\;\Lambda\right)\to0,\quad t\to\infty,$$

almost surely. Combining this with eq. (8), we get

$$\begin{aligned}&\limsup_{l\to\infty}\mathsf{P}\left(c_{at_{n_l}}\left|\zeta_{N^\Lambda_{t_{n_l}}}-\zeta_{\lfloor at_{n_l}\rfloor}\right|\ge\eta\mid\Lambda\right)\\&\le\limsup_{l\to\infty}\mathsf{P}\left(c_{at_{n_l}}\sum_{k:|k-at_{n_l}|<\varepsilon t_{n_l}+1}\theta_k\frac{|g(S_{k-1})|}{\Lambda(S_{k-1})}\ge\eta\;\middle|\;\Lambda\right)\end{aligned}$$

almost surely. Further, using eq. (9) and the portmanteau theorem, we obtain

$$\begin{aligned}&\limsup_{l\to\infty}\mathsf{P}\left(c_{at_{n_l}}\left|\zeta_{N^\Lambda_{t_{n_l}}}-\zeta_{\lfloor at_{n_l}\rfloor}\right|\ge\eta\mid\Lambda\right)\\&\le\mathsf{P}\left(\mu\sum_{k=-\infty}^{\infty}\frac{|g(k)|}{\Lambda(k)}\big(\ell_\alpha(1+\varepsilon/a,0\big)-\ell_\alpha(1-\varepsilon/a,0)\big)\ge\eta\;\middle|\;\Lambda\right)\\&=\mathsf{P}\left(\mu\sum_{k=-\infty}^{\infty}\frac{|g(k)|}{h(k)}\big(\ell_\alpha(1+\varepsilon/a,0\big)-\ell_\alpha(1-\varepsilon/a,0)\big)\ge\eta\right)\Bigg|_{h=\Lambda}\end{aligned}$$

almost surely. Finally, letting $\varepsilon\to0$ and noticing that the process ℓ is continuous, we obtain

$$\mathsf{P}\left(c_{at_{n_l}}\left|\zeta_{N^\Lambda_{t_{n_l}}}-\zeta_{\lfloor at_{n_l}\rfloor}\right|\ge\eta\;\middle|\;\Lambda\right)\to0,\quad l\to\infty$$

almost surely. Since the sequence t_n was arbitrary, it follows that

$$\mathsf{P}\left(c_{at}\left|\zeta_{N_t^\Lambda}-\zeta_{\lfloor at\rfloor}\right|\ge\eta\mid\Lambda\right)\xrightarrow{\mathsf{P}}0,\quad t\to\infty.$$

Combining this with eq. (6), we get that conditionally on $\Lambda=h$,

$$c_{at}\zeta_{N_t^\Lambda}\xrightarrow{\mathrm{d}}\mu\sum_{k=-\infty}^{\infty}\frac{g(k)}{h(k)}\ell_\alpha(1,0),\quad t\to\infty,$$

in probability. Since c_t is regularly varying of index $1/\alpha - 1$, then, conditionally on $\Lambda = h$,

$$c_t \zeta_{N_t^\Lambda} \xrightarrow{\mathrm{d}} \mu a^{1-1/\alpha} \sum_{k=-\infty}^{\infty} \frac{g(k)}{h(k)} \ell_\alpha(1,0)$$
$$= \mu^{1/\alpha} \big(\overline{\Lambda^{-1}}\big)^{1/\alpha - 1} \sum_{k=-\infty}^{\infty} \frac{g(k)}{h(k)} \ell_\alpha(1,0), \quad t \to \infty,$$

in probability.

To complete the proof, it remains to show that $c_t\big(t - \tau_{N_t^\Lambda}\big) g(S_{N_t^\Lambda}) \xrightarrow{\mathsf{P}} 0$, $t \to \infty$. This follows from the boundedness of $(t - \tau_{N_t^\Lambda})$ and $\Lambda(S_{N_t^\Lambda})$ in probability, boundedness of g, and the convergence $c_t \to 0$ as $t \to \infty$. □

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