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Bayesian Estimation for Inverse Weibull Distribution under Progressive Type-II Censored Data with Beta-Binomial Removals

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Abstract

This paper deals with the estimation procedure for inverse Weibull distribution under progressive type-II censored samples when removals follow Beta-binomial probability law. To estimate the unknown parameters, the maximum likelihood and Bayes estimators are obtained under progressive censoring scheme mentioned above. Bayes estimates are obtained using Markov chain Monte Carlo (MCMC) technique considering square error loss function and compared with the corresponding MLE's. Further, the expected total time on test is obtained under considered censoring scheme. Finally, a real data set has been analysed to check the validity of the study.

Keywords: inverse Weibull distribution, progressive type-II censoring, beta-binomial removals, maximum likelihood estimates Bayes estimates, MCMC technique.

1. Introduction

In the modern lifetime scenario, it is often difficult or time taking to observe all the units put on experiment due to some controlled or uncontrolled reasons, eg. time and cost constraints, accidental damage, disaster etc. The observations come from this type of situation are called censored observation. The type-I and type-II are the two very common censoring schemes which is widely used in the fields of survival and reliability studies. In type-I censoring schemes, the experimental time is fixed, say (T_0) but the number of observed failure is a random variable while in type-II censoring schemes, number of observed failure is fixed, say (m) but the experimental time is a random. Unfortunately, none of these censoring schemes have discussed the importance of removals of the live units from the test at any time before the completion of the experiment. Further, it is also possible that some units are intentionally or unintentionally removed from the experiment while they are still alive, the data arise from such type of phenomena call it as censored data. In survival/ reliability studies, we usually deals with these censored data. Therefore, there is a need of more appropriate and flexible sampling procedure for life-testing experiments. To attain this, a new censoring schemes is introduced besides the above two schemes, namely progressively type-I and type-II censoring

schemes which facilitates the removals of the units during the experiment.

Here, in this paper, we emphasize on estimation procedure under progressive type-II censoring scheme, which is developed by Cohen (1963). The detailed description of the considered scheme are as follows: Suppose, we have n experimental units are put on test at time (T_0) and going to observe m failure units/items during the experiment. The experiment proceeds in such a way that when first failure x_1 observed, R_1 of the surviving units are randomly selected from remaining $(n-1)$ surviving units and then removed i.e, we get R_1 removals from the experiment and immediate after the second failure x_2 is obtained, again R_2 of the surviving units are randomly selected from remaining $(n-R_1-2)$ surviving units and removed i.e, R_2 removals obtained. This procedures continues until the m^{th} failures. Then, at this instance, the experiment terminates and remaining $R_m = n-R_1-R_2-R_3-\cdots-R_{m-1}-m$ surviving units are randomly removed from the experiment. If theses removals $R_1 = R_2 = R_3 = \ldots R_{m-1} = R_m = 0$, then $m = n$, which correspond to complete sample situation and if $R_1 = R_2 = R_3 = \cdots = R_{m-1} = 0$, then $R_m = n-m$, which is simply conventional type-II censoring. Thus the progressive type-II censoring scheme is the generalization of type-II censoring schemes. This type of schemes generally obtained in medical/engineering fields. For example, Consider a medical experiment with n cancer patients but after the death of the first patient, some patients leave the experiment and go for treatment to other medical institution. Similarly, after the second death a few more leave and so on. Finally, the doctor stops taking observation as soon as the predetermined number of deaths (say, m) are recorded.

Statistical inferences based on estimation of parameters for different lifetime models under progressive type-II censoring scheme have been studied by Cohen (1963), Childs and Balakrishnan (2000), Balakrishnan and Sandhu (1995) and cited authors therein. Balakrishnan and Aggarwala (2000) is recommended to the readers for more detail. It may be noted here that, in this censoring scheme, the number of removals $R_1, R_2, R_3, ...$, at each stage are pre-fixed. However, in some practical situations, these removals may occur at random e.g. in the previous example, the number of patients leaves the hospital at each stage is random and can not be pre-determined. Utilizing this concept, Tse, Yang, and Yuen (2000), Wu and Chang (2003) and Yuen and Tse (1996) have considered that the number of units removed at each stage follows some specific distribution with certain probability for progressively censored samples. After that, several papers have been published on the estimation of the model parameters for various lifetime distributions under this procedure, see Singh, Singh, and Sharma (2014); Kaushik, Singh, and Singh (2017); Singh, Singh, and Kumar (2013b) and references cited therein.

It is assumed that the probability of removals remains same for all surviving units as well as it remains same at all stages in the case of progressive type-II censored sample with binomial removals. But this assumption seems to be too restrictive and unrealistic to be true in practical situations. For example, in the case of clinical study, if the deaths are recorded in the early stages of the test then definitely the probability of a removals will be high in the beginning and may decrease as the time passes. On other hand, if all the patients in the study are surviving for a longer period i.e. even the first death takes place after a long time, the chance of a drop-out of patients will be relatively small in the beginning and may increase at later stage. This confirms that, the probability of a drop-out at each stage of the experiment may not be remain constant through out the entire experiment. However, this drop out rate can not be observed although it effects the number of drop outs. Hence, one must think that the number of removals is random in nature and it follows a binomial distribution at each stage of removal with random probability of removal following some probability distribution. Due to flexible nature of the beta distribution, capable of having wide range of shapes, we have modelled the uncertainty about in the probability of a removal at various stages of the experiment as random realization of beta variables. Compounding the distribution of

number of removals with the probability of removals, results into the distribution of R_i to be beta-binomial and thus, the scheme named as progressive type-II censoring scheme with beta-binomial removals, which is abbreviated as *PT-II CBBR*. In the present piece of work, we have considered that the lifetime of the experimental units follow inverse Weibull (IW) distribution. The probability density function of inverse Weibull distribution is given as,

$$f(x) = \alpha\lambda x^{-\alpha-1} e^{-\lambda x^{-\alpha}}, \quad x > 0, \alpha, \lambda > 0. \tag{1}$$

The corresponding cdf and hazard function are given by

$$F(x) = e^{-\lambda x^{-\alpha}}, \quad x > 0, \alpha, \lambda > 0 \tag{2}$$

and

$$h(t) = \frac{\alpha\lambda t^{-\alpha-1}}{e^{\lambda t^{-\alpha}} - 1}, \tag{3}$$

respectively, where $\alpha > 0$ and $\lambda > 0$ are the shape and scale parameters, respectively. The plots of pdf and hazard function for different values of shape parameters are given in Figure 1. As we can see that it is heavy tail distribution and as $\alpha \to \infty$, the tail probability decreases. For $0 < \alpha \leq 1$, the mean does not exist and for $1 < \alpha \leq 2$, the mean exists but the variance does not exist.

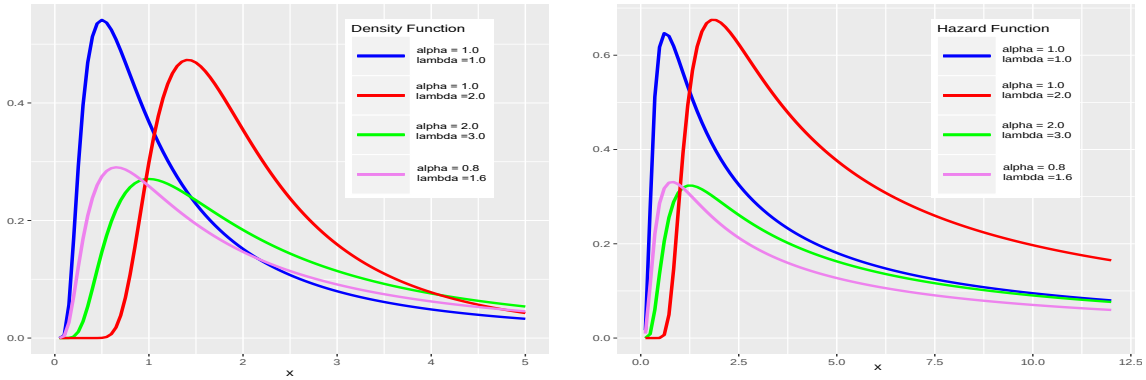

Figure 1: Probability density function and hazard rate for different values of shape and scale parameters

The inverse Weibull distribution is more useful in those situations where data indicates the non-monotone hazard rate characteristics. There are many real life examples where data don't shows the monotone hazard rate. For example, Langlands, Pocock, Kerr, and Gore (1997) have studied breast cancer data and observed that the mortality increases initially, reaches to a peak after some time and then declines slowly i.e., associated hazard rate is modified bathtub or particularly uni-modal. Such types of data can be modelled through the inverse Weibull (IW) distribution. Also, Nelson (1982) showed that, this distribution play an important role in many applications including the dynamic components of diesel engines the time to the breakdown of an insulating fluid subject to the action of a constant tension. Calabria and Pulcini (1990) provide an interpretation of inverse Weibull distribution in the context of the load-strength relationship for a component. Recently, Maswadah (2003) has fitted the inverse Weibull distribution to the flood data reported by Dumonceaux and Antle (1973)

In survival studies, the inverse Weibull distribution has been considered by several authors. Khan, Pasha, and Pasha (2008) have discussed the classical statistical properties of IW distribution. Kim, Lee, and Kang (2014) have derived the non-informative prior for the parameters of IW distribution. Noor and Aslam (2013) have proposed the Bayes estimators of the parameters of inverse Weibull mixture distribution using type-I censored samples. Recently,

Singh, Singh, and Kumar (2013a) have discussed the classical as well as Bayesian estimation procedures for unknown parameters of inverse Weibull distribution under conventional type-I and type-II censoring schemes. Thus, there is a need to developed a estimation procedure (classical as well as Bayesian) for IW distribution under more realistic and advanced censoring scheme such as progressive censoring schemes, where removals have also some probability distributions.

The main objective of this chapter is to provide the maximum likelihood estimates and Bayes estimates of unknown parameters under progressive type-II censoring scheme, where removals follow the beta-binomial probability law. The Bayes estimates are obtained using informative and non-informative prior under squared error loss function. It is noted here that the estimates obtained are not in explicit forms and they can be analysed by some suitable numerical integration technique. Therefore, we use Newton-Raphson method to find MLEs. MCMC technique has been used to solve the integration involve in posterior distribution. Also, we compare the MLEs with corresponding Bayes estimates of the unknown parameters by Monte-Carlo simulations.

The rest of the chapter is organized as follows: the maximum likelihood estimators (MLEs) of the parameter are obtained under the progressive type-II censored data with beta-binomial removals in section 2. In section 3, we have obtained Bayes estimators for unknown parameters of the IW distribution under progressive type-II censoring scheme with beta-binomial removals. The risk of estimates has been obtained. The comparison of MLE and correspond Bayes estimator under squared error loss function in term of their risks have been studied in section 4. A real data study to illustrate the application of the results in section 5. Finally, conclusions are presented in section 6.

2. The likelihood function

The likelihood function under progressive type-II censoring with pre-determined number of removals $\mathbf{R} = (R_1 = r_1, R_2 = r_2, \cdots, R_m = r_m)$ is given by

$$L(\underset{\sim}{\boldsymbol{x}}|\alpha,\lambda) = C\prod_{i=1}^{m} f(x_i,\alpha)\left\{1-F(x_i,\alpha,\lambda)\right\}^{r_i}, \tag{4}$$

where, $C = n(n-m-r_1)...(n-m-\sum_{i=1}^{m} r_i + 1)$. Substituting (1) and (2) into (4), we get

$$L(\alpha,\lambda;\underset{\sim}{\boldsymbol{x}}|R=r) = C\alpha^m\lambda^m e^{-\lambda\sum_{i=1}^{m} x_i^{-\alpha}}\prod_{i=1}^{m}\left[x_i^{-\alpha-1}\left(1-e^{-\lambda x_i^{-\alpha}}\right)^{r_i}\right]. \tag{5}$$

It has been discussed above that in the progressive censoring scheme if the experimental units are removed from the test with given probability, say p, then following Tse *et al.* (2000), the number of removals at i^{th} stage is given by,

$$Pr(R_i = r_i|p) = \binom{n-m-\sum_{j=1}^{i-1} r_j}{r_i} p^{r_i}(1-p)^{n-m-\sum_{j=1}^{i} r_j}; i = 1,2,...,m-1 \tag{6}$$

Further, it is assumed in the considered form of the censoring scheme that the probability of removals is not fixed over whole of the experimentation period and assumed to be a random variable following the probability density function

$$g(p|\xi,\zeta) = \frac{1}{B(\xi,\zeta)}p^{\xi-1}(1-p)^{\zeta-1}; \xi>0, \zeta>0, 0<p<1. \tag{7}$$

Thus, the unconditional distribution of $R_i's$ can be derived as follows

$$Pr(R_i = r_i, \xi, \zeta) = \int_0^1 Pr(R_i = r_i|p)g(p|\xi, \zeta)dp \tag{8}$$

Substituting (6) and (7) in (8), we get,

$$Pr(R_i = r_i, \xi, \zeta) = \frac{1}{B(\xi, \zeta)} \binom{n - m - \sum_{j=1}^{i-1} r_j}{r_i} \int_0^1 p^{r_i+\xi-1}(1-p)^{n-m-\sum_{j=1}^{i} r_j+\zeta-1} dp$$

After simplification, which can be written as

$$Pr(R_i = r_i, \xi, \zeta) = \binom{n - m - \sum_{j=1}^{i-1} r_j}{r_i} \frac{B\left(\xi + r_i,\ \zeta + n - m - \sum_{j=1}^{i} r_j\right)}{B(\xi, \zeta)} \tag{9}$$

where, $B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}, \xi, \zeta > 0, r_i = 0, ..., n - m - \sum_{j=1}^{i-1} r_j; i = 1, ..., (m-1)$.
It is the probability mass function (pmf) of beta-binomial distribution and it is denoted by $BB(n', \xi, \zeta)$.
Thus, the joint probability of $R_1 = r_1, R_2 = r_2, ..., R_m = r_m$ is given by

$$\begin{aligned} P(\mathbf{R}{=}\mathbf{r}, \xi, \zeta) = P[R_1 = r_1] \times P[R_2 = r_2|R_1 = r_1] \times \cdots \times \\ P[R_{m-1} = r_{m-1}|R_{m-2} = r_{m-2}, ..., R_1 = r_1] \end{aligned} \tag{10}$$

Now, we further assume that $R_i's$ is independent of $x_i's$ for all i. Then, the full likelihood function takes the following form

$$\begin{aligned} L(\underset{\sim}{x}, \Theta, \mathbf{R}, \xi, \zeta) &= L(\underset{\sim}{x}, \Theta|\mathbf{R})P(\mathbf{R}{=}\mathbf{r}, \xi, \zeta) \\ &= C\alpha^m \lambda^m e^{-\lambda \sum_{i=1}^m x_i^{-\alpha}} \prod_{i=1}^m \left[x_i^{-\alpha-1}\left(1 - e^{-\lambda x_i^{-\alpha}}\right)^{r_i}\right] \\ &\prod_{i=1}^{m-1} \binom{n - m - \sum_{j=1}^{i-1} r_j}{r_i} \frac{B\left(\xi + r_i,\ \zeta + n - m - \sum_{j=1}^{i} r_j\right)}{B(\xi, \zeta)}. \end{aligned} \tag{11}$$

Maximum likelihood estimation for α and λ

The likelihood function for progressively type-II censored sample with beta-binomial removals from (11) can be re-written as,

$$L(\underset{\sim}{\boldsymbol{x}}, \alpha, \lambda, R, \xi, \zeta) \propto L_1(\alpha, \lambda|R)L_2(\xi, \zeta|R), \tag{12}$$

where

$$L_1(\alpha, \lambda|R) = \alpha^m \lambda^m e^{-\lambda \sum_{i=1}^m x_i^{-\alpha}} \prod_{i=1}^m \left[x_i^{-\alpha-1}\left(1 - e^{-\lambda x_i^{-\alpha}}\right)^{r_i}\right] \tag{13}$$

and

$$L_2(\xi, \zeta|R) = \prod_{i=1}^{m-1} \binom{n - m - \sum_{j=1}^{i-1} r_j}{r_i} \frac{B\left(\xi + r_i,\ \zeta + n - m - \sum_{j=1}^{i} r_j\right)}{B(\xi, \zeta)}, \tag{14}$$

It may be noted here that $L_1(\alpha, \lambda|R)$ involves the parameters of the considered model but it is independent of the parameters of the distributions considered for removal probability. On the other hand $L_2(\xi, \zeta|R)$ is free from the parameters of the considered model and includes only the parameters of the distribution of the probability of removals. Therefore for finding the MLE of α and λ, we must use $L_1(\alpha, \lambda|R)$ only. On maximizing the likelihood function given by $L_1(\alpha, \lambda|R)$, we obtain the maximum likelihood estimates for model parameters α and λ. Hence, the corresponding log-likelihood function of (13) is given by,

$$\ln L_1 = \ln C + m \ln \alpha + m \ln \lambda - \lambda \sum_{i=1}^{m} x_i^{-\alpha} - (\alpha + 1) \sum_{i=1}^{m} \ln x_i + \sum_{i=1}^{m} r_i \ln(1 - e^{-\lambda x_i^{-\lambda}}) \quad (15)$$

Then, the MLEs $\hat{\alpha}$ and $\hat{\lambda}$ of α and λ, respectively can be obtained as the simultaneous solution of the following two non-linear equations:

$$\frac{m}{\alpha} - \lambda \sum_{i=1}^{m} (x_i^{-\alpha} \ln x_i) - \sum_{i=1}^{m} \ln x_i + \lambda \sum_{i=1}^{m} \left[\frac{r_i x_i^{-\alpha} e^{-\lambda x_i^{-\alpha}} \ln x_i}{(1 - e^{-\lambda x_i^{-\alpha}})} \right] \quad (16)$$

and

$$\frac{m}{\lambda} - \sum_{i=1}^{m} x_i^{-\alpha} + \sum_{i=1}^{m} \left[\frac{r_i x_i^{-\alpha} e^{-\lambda x_i^{-\alpha}}}{(1 - e^{-\lambda x_i^{-\alpha}})} \right]. \quad (17)$$

As the above equations cannot be evaluated analytically, one can use numerical technique such as Newton-Raphson method to solve these and find the MLEs. While attempting to obtain interval estimates, we note that, the exact distribution of MLEs are not easy to obtain, we suggest to use the concept of large sample theory to obtain the confidence intervals based on $I(\alpha, \lambda)$, the Fisher's information matrix, which can be estimated by

$$I\left(\hat{\alpha}, \hat{\lambda}\right) = \begin{bmatrix} -\frac{d \ln L}{d\alpha^2} & -\frac{d \ln L}{d\lambda d\alpha} \\ -\frac{d \ln L}{d\alpha d\lambda} & -\frac{d \ln L}{d\lambda^2} \end{bmatrix}_{(\hat{\alpha}, \hat{\lambda})} \quad (18)$$

The diagonal elements of $I^{-1}\left(\hat{\alpha}, \hat{\lambda}\right)$ provides the asymptotic variances for the parameters α and λ, respectively. Thus, two-sided $100(1-\gamma)\%$ normal approximation confidence interval of α and λ can be obtained as $\left\{\hat{\alpha} \mp Z_{\gamma/2}\sqrt{var(\hat{\alpha})}\right\}$ and $\left\{\hat{\lambda} \mp Z_{\gamma/2}\sqrt{var(\hat{\lambda})}\right\}$, respectively.

Maximum likelihood estimation for ξ and ζ:

From equation (14), the joint probability of **R=r** is given by

$$P(\mathbf{R}{=}\mathbf{r}, \xi, \zeta) = C^{\star} \frac{1}{B(\xi, \zeta)^{m-1}} \prod_{i=1}^{m-1} B(\xi + r_i,\ n - m - \sum_{j=1}^{i} r_j + \zeta) \quad (19)$$

Where, $C^{\star} = \frac{(n-m)!}{\prod_{i=1}^{m-1} r_i!(n-m-\sum_{j=1}^{m-1} r_j)!}$.

Taking logarithm on both sides of equation (19), we get

$$\begin{aligned} LogP = {} & \ln(C^{\star}) - (m-1)[\ln \Gamma(\xi) + \ln \Gamma(\zeta) - \ln \Gamma(\xi + \zeta)] + \sum_{i=1}^{m-1} \ln \Gamma(\xi + r_i) \\ & + \sum_{i=1}^{m-1} \ln \Gamma(n - m - \sum_{j=1}^{i} r_j + \zeta) - \sum_{i=1}^{m-1} \ln \Gamma(n - m - \sum_{j=1}^{i-1} r_j + \xi + \zeta) \end{aligned} \quad (20)$$

The MLEs $\hat{\xi}$ and $\hat{\zeta}$ of ξ and ζ, respectively can be obtained as the simultaneous solution of the following two normal non-linear equations:

$$-\frac{(m-1)}{\Gamma(\xi)}D_{\xi}(\xi)+\frac{(m-1)}{\Gamma(\xi+\zeta)}D_{\xi}(\xi+\zeta)+\sum_{i=1}^{m-1}\frac{D_{\xi}(\xi+r_i)}{\Gamma(\xi+r_i)}-\sum_{i=1}^{m-1}\frac{D_{\xi}(n-m-\sum_{j=1}^{i-1}r_j+\xi+\zeta)}{\Gamma(n-m-\sum_{j=1}^{i-1}r_j+\xi+\zeta)}=0 \tag{21}$$

$$\begin{aligned}-\frac{(m-1)}{\Gamma(\zeta)}D_{\zeta}(\zeta)+\frac{(m-1)}{\Gamma(\xi+\zeta)}D_{\zeta}(\xi+\zeta)+\sum_{i=1}^{m-1}\frac{D_{\zeta}(n-m-\sum_{j=1}^{i}r_j+\zeta)}{\Gamma(n-m-\sum_{j=1}^{i}r_j+\zeta)}\\ -\sum_{i=1}^{m-1}\frac{D_{\zeta}(n-m-\sum_{j=1}^{i-1}r_j+\xi+\zeta)}{\Gamma(n-m-\sum_{j=1}^{i-1}r_j+\xi+\zeta)}=0\end{aligned} \tag{22}$$

Where $D_a(\phi(a))=\frac{d}{da}\Gamma(\phi(a))$ is dia-gamma function. The equation's are not solvable in nice closed form therefore, we suggest the use of iterative methods.

3. Bayes estimation

In the Bayesian paradigm, we need to assume the prior distributions for unknown model parameters. The prior probability density for the parameters α and λ are assumed to be of the following forms

$$g_1(\alpha)=\frac{\nu_1^{\mu_1}}{\Gamma\mu_1}e^{-\nu_1\alpha}\alpha^{\mu_1-1};\ \alpha>0,\ \mu_1>0,\ \nu_1\ >0 \tag{23}$$

$$g_2(\lambda)=\frac{\lambda_2^{\mu_2}}{\Gamma\mu_2}e^{-\nu_2\lambda}\lambda^{\mu_2-1};\ \lambda>0,\ \mu_2>0,\ \nu_2\ >0, \tag{24}$$

where, $\mu_1,\ \nu_1,\ \mu_2$ and ν_2 are the hyper-parameters. The joint prior pdf of α and λ may be obtain as,

$$g(\alpha,\lambda)=g_1(\alpha)\,g_2(\lambda)\quad;\quad\alpha>0,\quad\lambda>0 \tag{25}$$

Thus, the joint posterior of α and λ is given by

$$\pi(\alpha,\lambda|\underset{\sim}{\boldsymbol{x}})=\frac{J}{\iint_0^{\infty}J d\alpha d\lambda} \tag{26}$$

where,

$$J=J(\alpha,\lambda)\ =\ \alpha^{m+\mu_1-1}\lambda^{m+\mu_2-1}e^{-\left(\nu_1\alpha+\nu_2\lambda+\lambda\sum_{i=1}^{m}x_i^{-\alpha}\right)}\prod_{i=1}^{m}\left[x_i^{-\alpha-1}\left(1-e^{-\lambda x_i^{-\alpha}}\right)^{r_i}\right]$$

Let $h(\cdot)$ be a function of α and λ. Then, the Bayes estimator of $h(\cdot)$ under the squared error loss function is given by,

$$\begin{aligned}\hat{h}_B(\alpha,\lambda)\ &=\ E_{\pi}(h(\alpha,\lambda))\\ &=\ \frac{\iint_0^{\infty}h(\alpha,\lambda)\,J\,d\alpha\,d\lambda}{\iint_0^{\infty}J\,d\alpha\,d\lambda}\end{aligned} \tag{27}$$

It is clear from the expression (26) that there is no closed form for the estimators, so we suggest using an MCMC procedure to compute the Bayes estimates, see (Smith and Roberts 1993; Brooks 1998; Hastings 1970) for more detail. After getting MCMC samples from the posterior distribution, we can find the Bayes estimate for the parameters in the following way

$$[E(\Theta|data)] = \left[\frac{1}{N-N_0}\sum_{i=N_0+1}^{N}\Theta_i\right],$$

where N_0 is burn-in period of the Markov chain and $\Theta_i = [\alpha_i, \lambda_i]'$. For computation of the highest posterior density (HPD) interval of Θ, order the MCMC sample of Θ as $\Theta_{(1)}$, $\Theta_{(2)}$, $\Theta_{(3)}, \cdots, \Theta_{(N)}$. Then construct all the 100(1-γ)% credible intervals of Θ say $(\Theta_{(1)}, \Theta_{(N\lfloor 1-\gamma\rfloor+1)})$, $(\Theta_{(2)}, \Theta_{(N\lfloor 1-\gamma\rfloor+2)}) \cdots, (\Theta_{(\lfloor N\gamma\rfloor)}, \Theta_{(N)})$. Finally, the HPD credible interval of α and β is that interval which has the shortest length.

In order to obtain the MCMC samples from the joint posterior density of α and λ, we use the Metropolis-Hastings (M-H) algorithm. We consider a bivariate normal distribution as the proposal density i.e. $N_2(\mu, \Sigma)$ where Σ is the variance-covariance matrix. It may be noted here that if we generate observations from the bivariate normal distribution, we may get negative values also, which are not possible as the parameters under consideration are positive valued. Therefore, we take the absolute value of the generated observations. Following this, the Metropolis-Hastings algorithm associated with the target density $\pi(\cdot)$ and the proposal density $N_2(\mu, \Sigma)$ produces a Markov chain Θ^i through the following steps.

① Set initial values $\Theta_0 = [\alpha_0, \lambda_0]'$.

② Generate new candidate parameter values $\Theta_* = [\alpha_*, \lambda_*]'$ from $N_2(\mu, \Sigma)$.

③ Calculate the ratio

$$\rho\left(\Theta_*, \Theta_{i-1}\right) = \min\left\{\frac{\pi(\Theta_*)}{\pi(\Theta_{i-1})}, 1\right\}.$$

④ Draw u from uniform(0,1).

$$\begin{cases}\text{Accept } \Theta_* \text{ as } \Theta_i \text{ if } u < \rho\left(\Theta_*, \Theta_{i-1}\right).\\ \text{If } \Theta_* \text{ is not accepted, then } \Theta_i = \Theta_{i-1}.\end{cases}$$

In using the above algorithm, the problem arises as to how to choose the initial guess. Here, we propose the use of the MLEs of (α, λ), obtained by using the method described in section 2, as initial values for the MCMC process. The choice of covariance matrix Σ is also an important issue; see Natzoufras (2009) and Kaushik *et al.* (2017) for details. One choice for Σ would be the asymptotic variance-covariance matrix $I^{-1}(\hat{\alpha}, \hat{\lambda})$. While generating M-H samples by taking $\Sigma = I^{-1}(\hat{\alpha}, \hat{\lambda})$, we noted that the acceptance rate for such a choice of Σ is about 15%. By acceptance rate, we mean the proportion of times a new set of values is generated at the iteration stages. It is well known that if the acceptance rate is low, a good strategy is to run a small pilot run using a diagonal Σ as a rough estimate of the correlation structure for the target posterior distribution and then re-run the algorithm using the corresponding estimated variance-covariance matrix; for more details see Gelmen, Carlin, Stern, and Rubin (1995). Therefore, we have also used the latter described strategy for the calculations in the following sections.

4. Expected total time on test

In practice, it is also desired to have an idea of the duration of a life test since the experiment termination time is directly associated with the cost of the experiment. For progressive

type-II censoring scheme, the termination time is given by the expectation of the m^{th} order statistics. From Balakrishnan and Aggarwala (2000), the conditional expectation of X_m given $\mathbf{R} = (R_1 = r_1, R_2 = r_2, ..., R_m = r_m)$ can be defined as

$$E\left(X_m|\ \mathbf{R}\right) = C(r) \sum_{l_1=1}^{r_1} ... \sum_{l_m=1}^{r_m} (-1)^{\sum_{i=1}^{m} l_i} \frac{\binom{r_1}{l_1}...\binom{r_m}{l_m}}{\prod_{i=1}^{m-1} h(l_i)} \int_0^{\infty} x f(x) F^{h(l_m)-1}(x) dx \tag{28}$$

Where, $h(l_i) = \sum_{j=1}^{i} l_j + i$. Putting (1), (2) in (28) and then simplifying, we have

$$E\left(X_m|\ \mathbf{R}\right) = C(r)\alpha\lambda \sum_{l_1=1}^{r_1} ... \sum_{l_m=1}^{r_m} (-1)^{\sum_{i=1}^{m} l_i} \frac{\binom{r_1}{l_1}...\binom{r_m}{l_m}}{\prod_{i=1}^{m-1} h(l_i)} \int_0^{\infty} x^{-\alpha} e^{-\lambda h(l_m) x^{-\alpha}} dx \tag{29}$$

The expected termination time PT-II CBBR is evaluated by taking the expectation on both sides of (29) with respect to the $\mathbf{R}$. That is,

$$\begin{aligned} E(X_m) &= E_R[E\left(X_m|\ \mathbf{R}\right)] \\ &= \sum_{r_1=0}^{g(r_1)} \sum_{r_2=0}^{g(r_2)} ... \sum_{r_{m-1}=0}^{g(r_{m-1})} P(\mathbf{R}{=}\mathbf{r}, \xi, \zeta) E\left(X_m|\ \mathbf{R}\right) \end{aligned} \tag{30}$$

Where, $g(r_i) = n - m - r_1 - ... - r_{i-1}$.

5. Simulation study

In this section, we have compared the performances of the various estimators on the basis of their mean square errors (MSEs). It may be mentioned here that the exact expression for the mean square errors cannot be obtained, because the estimators are not in explicit form. Therefore, MSEs are estimated on the basis of a Monte-Carlo simulation study of 2000 samples. For this purpose, we generated a specified number of observations from the distribution given in equation (1) for fixed values of the parameters under the specified censoring schemes and calculated different estimates of α and λ following the procedure described in the previous sections. This process was repeated 2000 times to obtain the simulated biases and MSEs. We have computed the MLEs by using the Newton-Raphson algorithm and Bayes estimates using MCMC method. The ML and Bayes estimates of (α, λ) are denoted as $(\alpha_{ML}, \lambda_{ML})$ and (α_B, λ_B), respectively. It is noted that Newton-Raphson algorithm has a convergence rate of 90%-95%. We have reported the results omitting the cases where algorithm do not converge. To simulate a progressive type-II censored sample from the considered distribution, we have used the algorithm given by Balakrishnan and Cramer (2014).

It may be noted here that the MSE of these estimators will depend on the sample size n, m, values of α, λ and hyper-parameters μ_1, μ_2, ν_1 and ν_2. We considered a number of values for the sample size n; namely n = 10, 20, 30, 40 and 50 and m is taken as 50%, 60%, 70%, 80%, 90% and 100% of the n. For an informative prior, the hyper parameters are chosen on the basis of the information possessed by the experimenter, denoted as Bayes1. Also, we have considered the choice of hyper-parameters as $\mu_1 = \mu_2 = \nu_1 = \nu_2 = 0$ which reduces the prior to a non-informative prior, denoted as Bayes2. In most of the cases, the experimenter can have a notion of what are the expected value of the parameter and can always associate a degree of belief to this value. In other words, the experimenter can specify the prior mean and prior variance for the parameters. The prior mean reflects the experimenter's belief about the parameter in the form of its expected value and the prior variance reflects his confidence in this expected value. Keeping this point in mind, we have chosen the hyper-parameters in such

a way that the prior mean is equal to the true value of the parameter and the belief in the prior mean is either strong or weak, i.e. the prior variance is small or large, respectively; for details see Singh *et al.* (2013a). The MSE's of the estimates of parameters with corresponding confidence interval have been calculated and the results are summarized in Tables 1, 2 and (3). From Tables 1-3, we have observed that the both MLEs and Bayes estimates provides more accurate estimates with increasing sample information m although, the Bayes estimates of the model parameters are more nearer to the true values of the model parameters. The HPD intervals have shorter width than the asymptotic confidence intervals.

Table 1 provides the MSE of estimates of the parameters for $\alpha = 2$, $\lambda = 3$, $\xi = 0.1$, $\zeta = 3$ and hyper-parameters $\mu_1 = 4, \mu_2 = 6, \nu_1 = 2, \nu_2 = 3$. It can be seen from the Table that in general, the MSE's decrease as n increases in all the considered cases. It can also be seen that the MSE of the MLE is more than that of the corresponding Bayes estimate in all cases but the difference between the MSEs of the Bayes and ML estimates decreases for increases in the value of n. It is also noted here that MSEs decreases as m increases for fix value of n. Similar trend found in the MSE of the Bayes estimates and Bayes estimates with informative prior having least MSE. The 95% asymptotic interval estimates are wider than HPD intervals. In Table 2, we have shown the effect of variation of ξ and ζ. Here, we noticed that increment in the values of ξ reflect the negative effect on the performance of all considered point and interval estimates, however as ζ increases the estimators performance becomes better. Table 3 provide the effect of magnitude of parameters α and λ on the considered estimators. As α increases or λ increases, in both of the situation the estimated MSEs of all the estimators increases, along with the width of the asymptotic and HPD interval estimators are increases.

For investigating the expected total test time (TTT), we have considered the different combination of the model parameters which are given below:

$$\begin{array}{llll}
\alpha = 2, & \lambda = 3, & \xi(= 0.5, 1.0), & \zeta\{= (15, 9, 3, 1, 0.5, 0.25), (15, 9, 3, 1, 0.5, 0.25)\}; \\
\alpha = 0.8, & \lambda = 1.6, & \zeta(= 0.5, 1.0), & \xi\{= (15, 9, 3, 1, 0.5, 0.25), (15, 9, 3, 1, 0.5, 0.25)\}; \\
\alpha = 2, & \lambda = 3, & \xi(= 1, 3), & \zeta\{= (15, 9, 3, 1, 0.5, 0.25), (15, 9, 3, 1, 0.5, 0.25)\}; \\
\alpha = 0.8, & \lambda = 1.6, & \zeta(= 1, 3), & \xi\{= (15, 9, 3, 1, 0.5, 0.25), (15, 9, 3, 1, 0.5, 0.25)\}.
\end{array}$$

For each combination, we have taken $n = (10, 20, 30, 40$ and $50)$ and for each n, m is chosen so that the sample contains the 100%, 90%, $\cdots$, 50% units of the available sample units, respectively. All results are summarised in Tables 4 - 7. From this, we have observed that the expected TTT is an increasing function of n and m as it is expected. It is interesting to know that the expected TTT decreases as ζ increases while ξ is fixed and on other hand expected TTT increases as ξ increases for given fixed value of ζ. It can also be observed that the expected TTT increases as α increases and decreases as λ decreases.

6. Real data illustration

In this section, we illustrate our proposed methodology with the four real examples. The first data set considered by us, represents the times between successive failures of air conditioning equipment in a *Boeing 720* airplane, reported by Proschan (1963):

75	57	48	29	502	12	70	21	29
386	59	27	153	26	326			

Second data set used by Bhaumik, Kapur, and Gibbons (2009), is vinyl chloride data obtained from clean upgradient monitoring wells in mg/litre:

5.1	1.2	1.3	0.6	0.5	2.4	0.5	1.1	8.0
0.8	0.4	0.6	0.9	0.4	2.0	0.5	5.3	3.2
2.7	2.9	2.5	2.3	1.0	0.2	0.1	0.1	1.8
0.9	2	4	6.8	1.2	0.4	0.2		

Table 1: MLEs and Bayes estimators (MSE in brackets) with corresponding asymptotic and HPD intervals for the paramaters for fixed values of $\alpha = 2$, $\lambda = 3$, $\xi = 0.1$, $\zeta = 3$

n	m	MLEs	Asymptotic CI	Bayes1	HPD CI	Bayes2	HPD CI
20	10	2.0951(0.1175)	(1.8206,2.2811)	1.9836(0.0775)	(1.9036,2.1983)	2.0675(0.1037)	(1.8501,2.2509)
		3.2994(0.6705)	(1.9215,4.5489)	3.1328(0.4308)	(2.4173,4.0567)	3.2707(0.5914)	(2.0943,4.3746)
	12	2.0848(0.1144)	(1.8265,2.2749)	1.9932(0.0744)	(1.9094,2.1930)	2.0778(0.0994)	(1.8582,2.2426)
		3.2889(0.6389)	(1.9833,4.4865)	3.1224(0.4144)	(2.4485,4.0252)	3.2612(0.5559)	(2.1631,4.3060)
	14	2.0812(0.1119)	(1.8312,2.2699)	2.0008(0.0720)	(1.9135,2.1886)	2.0595(0.0993)	(1.8587,2.2430)
		3.2751(0.6153)	(2.0296,4.4408)	3.2048(0.3954)	(2.4850,3.9887)	3.2219(0.5419)	(2.1901,4.2790)
	16	2.0708(0.1121)	(1.8305,2.2704)	2.0102(0.0719)	(1.9137,2.1876)	2.0693(0.0956)	(1.8662,2.2356)
		3.2651(0.6127)	(2.0354,4.4355)	3.1950(0.3909)	(2.4936,3.9802)	3.2118(0.5220)	(2.2286,4.2410)
	18	2.0607(0.1086)	(1.8376,2.2634)	2.0002(0.0682)	(1.9206,2.1814)	2.0594(0.0927)	(1.8718,2.2297)
		3.2544(0.5962)	(2.0669,4.4038)	3.1853(0.3744)	(2.5243,3.9491)	3.2016(0.5056)	(2.2600,4.2089)
	20	2.0511(0.1068)	(1.8413,2.2599)	1.9902(0.0660)	(1.9252,2.1763)	2.0496(0.0908)	(1.8759,2.2261)
		3.2365(0.5746)	(2.1097,4.3606)	3.1670(0.3573)	(2.5569,3.9164)	3.1834(0.4851)	(2.2996,4.1694)
30	15	2.0449(0.1048)	(1.8223,2.2327)	1.9333(0.0645)	(1.9047,2.1502)	2.0177(0.0882)	(1.8576,2.1980)
		3.2279(0.5487)	(2.0261,4.1739)	3.1693(0.3361)	(2.4624,3.7405)	3.0871(0.4616)	(2.2103,3.9909)
	18	2.0383(0.0722)	(1.8859,2.1692)	1.9867(0.0564)	(1.9204,2.1346)	1.9778(0.0675)	(1.8978,2.1577)
		3.1577(0.3589)	(2.3980,3.8031)	3.0992(0.2815)	(2.5665,3.6360)	3.0170(0.3369)	(2.4516,3.7513)
	21	2.0311(0.0703)	(1.8896,2.1650)	1.9866(0.0552)	(1.9223,2.1327)	2.0099(0.0655)	(1.9009,2.1539)
		3.1553(0.3159)	(2.4820,3.7185)	3.0782(0.2449)	(2.6362,3.5664)	3.0716(0.2931)	(2.5354,3.6670)
	24	2.0479(0.0689)	(1.8931,2.1621)	1.9996(0.0524)	(1.9279,2.1268)	1.9861(0.0638)	(1.9045,2.1503)
		3.1520(0.3054)	(2.5027,3.6987)	3.0577(0.1660)	(2.7858,3.4167)	3.1264(0.2480)	(2.6230,3.5796)
	27	2.0378(0.0657)	(1.8986,2.1560)	2.0010(0.0502)	(1.9325,2.1234)	1.9755(0.0599)	(1.9119,2.1429)
		3.1219(0.2948)	(2.5240,3.6780)	3.0438(0.2225)	(2.6781,3.5244)	3.0960(0.2701)	(2.5800,3.6225)
	30	2.0281(0.0557)	(1.9186,2.1361)	2.0009(0.0400)	(1.9518,2.1037)	1.9655(0.0502)	(1.9302,2.1242)
		3.1018(0.2495)	(2.6121,3.5893)	3.0240(0.1768)	(2.7653,3.4372)	3.0764(0.2256)	(2.6664,3.5360)
50	25	2.0460(0.0619)	(1.8894,2.1314)	2.0122(0.0466)	(1.9218,2.0986)	2.0136(0.0577)	(1.8994,2.1216)
		3.1277(0.4023)	(2.2880,3.8648)	3.0696(0.3005)	(2.5047,3.6442)	3.0975(0.3718)	(2.3588,3.7921)
	30	2.0360(0.0604)	(1.8915,2.1280)	2.0016(0.0448)	(1.9248,2.0952)	2.0035(0.0559)	(1.9020,2.1175)
		3.1158(0.3608)	(2.3686,3.7836)	3.0409(0.2657)	(2.5714,3.5785)	3.1153(0.3314)	(2.4362,3.7136)
	35	2.0336(0.0609)	(1.8908,2.1288)	2.0102(0.0447)	(1.9250,2.0948)	2.0168(0.0557)	(1.9033,2.1173)
		3.1127(0.3005)	(2.4868,3.6650)	3.0420(0.2221)	(2.6537,3.4954)	3.1108(0.2756)	(2.5444,3.6067)
	40	2.0306(0.0485)	(1.9150,2.1050)	2.0181(0.0329)	(1.9479,2.0727)	2.0301(0.0434)	(1.9267,2.0936)
		3.0915(0.2328)	(2.6194,3.5321)	3.0407(0.1552)	(2.7809,3.3697)	3.0830(0.2064)	(2.6776,3.4732)
	45	2.0210(0.0442)	(1.9241,2.0960)	2.0078(0.0282)	(1.9564,2.0632)	2.0208(0.0389)	(1.9352,2.0847)
		3.0819(0.2003)	(2.6834,3.4689)	3.0304(0.1261)	(2.8363,3.3148)	3.0727(0.1747)	(2.7390,3.4128)
	50	2.0109(0.0408)	(1.9306,2.0901)	1.9976(0.0247)	(1.9637,2.0572)	2.0105(0.0357)	(1.9420,2.0789)
		3.0765(0.1744)	(2.7337,3.4175)	3.0254(0.1052)	(2.8760,3.2750)	3.0672(0.1508)	(2.7848,3.3663)

Table 2: MLEs and Bayes estimators (MSE in brackets) with corresponding asymptotic and HPD intervals for the paramaters for fixed value of $\alpha = 2$, $\lambda = 3$, $n = 50$, $n = 35$

ξ	ζ	MLE	Asymptotic CI	Bayes1	HPD CI	Bayes2	HPD CI
0.1	0.1	2.0135(0.0607)	(1.9315,2.2814)	2.0339(0.0448)	(1.9657,2.1985)	2.0777(0.0555)	(1.9431,2.2518)
		3.0823(0.2975)	(2.6525,4.5494)	3.0120(0.2199)	(2.8180,4.0571)	3.0804(0.2737)	(2.7078,4.3751)
	0.5	2.0117(0.0606)	(1.9315,2.2752)	2.0320(0.0451)	(1.9645,2.1939)	2.0758(0.0551)	(1.9447,2.2434)
		3.0793(0.2981)	(2.6513,4.4870)	3.0092(0.2205)	(2.8173,4.0254)	3.0769(0.2728)	(2.7092,4.3062)
	3	2.0094(0.0608)	(1.9313,2.2701)	2.0302(0.0451)	(1.9645,2.1890)	2.0738(0.0552)	(1.9436,2.2435)
		3.0759(0.2979)	(2.6519,4.4413)	3.0060(0.2195)	(2.8189,3.9889)	3.0738(0.2732)	(2.7085,4.2791)
	15	2.0076(0.0603)	(1.9323,2.2706)	2.0279(0.0443)	(1.9663,2.1885)	2.0718(0.0557)	(1.9426,2.2356)
		3.0726(0.2972)	(2.6538,4.4362)	3.0029(0.2196)	(2.8189,3.9802)	3.0707(0.2722)	(2.7106,4.2418)
0.5	0.1	2.0296(0.0615)	(1.9305,2.2643)	2.0502(0.0453)	(1.9646,2.1821)	2.0131(0.0557)	(1.9428,2.2297)
		3.1066(0.3000)	(2.6480,4.4044)	3.0364(0.2226)	(2.8130,3.9495)	3.1052(0.2753)	(2.7048,4.2092)
	0.5	2.0278(0.0613)	(1.9307,2.2605)	2.0489(0.0448)	(1.9654,2.1767)	2.0113(0.0560)	(1.9427,2.2266)
		3.1042(0.3005)	(2.6469,4.3610)	3.0332(0.2216)	(2.8150,3.9164)	3.1022(0.2752)	(2.7053,4.1696)
	3	2.0260(0.0612)	(1.9073,2.2332)	2.0464(0.0455)	(1.9412,2.1508)	2.0908(0.0555)	(1.9209,2.1987)
		3.1005(0.2997)	(2.5140,4.1747)	3.0308(0.2212)	(2.6807,3.7407)	3.0986(0.2747)	(2.5712,3.9915)
	15	2.0244(0.0616)	(1.9074,2.1698)	2.0446(0.0448)	(1.9427,2.1348)	2.0885(0.0562)	(1.9193,2.1582)
		3.0975(0.2993)	(2.5150,3.8031)	3.0273(0.2215)	(2.6801,3.6365)	3.0954(0.2743)	(2.5716,3.7516)
3	0.1	2.0220(0.0609)	(1.9079,2.1653)	2.0420(0.0449)	(1.9421,2.1336)	2.0862(0.0557)	(1.9199,2.1546)
		3.0946(0.2994)	(2.5148,3.7186)	3.0243(0.2212)	(2.6810,3.5674)	3.0926(0.2742)	(2.5722,3.6672)
	0.5	2.0196(0.0614)	(1.9069,2.1629)	2.0401(0.0451)	(1.9419,2.1269)	2.0847(0.0556)	(1.9207,2.1506)
		3.0913(0.2992)	(2.5147,3.6990)	3.0211(0.2209)	(2.6816,3.4168)	3.0897(0.2741)	(2.5722,3.5800)
	3	2.0177(0.0609)	(1.9082,2.1563)	2.0379(0.0450)	(1.9420,2.1237)	2.0817(0.0561)	(1.9192,2.1432)
		3.0886(0.2988)	(2.5160,3.6782)	3.0184(0.2208)	(2.6822,3.5244)	3.0869(0.2743)	(2.5719,3.6233)
	15	2.0155(0.0612)	(1.9079,2.1363)	2.0359(0.0448)	(1.9425,2.1038)	2.0804(0.0554)	(1.9206,2.1245)
		3.0856(0.2984)	(2.5160,3.5897)	3.0155(0.2208)	(2.6816,3.4380)	3.0837(0.2736)	(2.5736,3.5367)
15	0.1	2.1158(0.0640)	(1.8848,2.1323)	2.1226(0.0472)	(1.9202,2.0996)	2.0991(0.0588)	(1.8970,2.1219)
		3.2392(0.3131)	(2.4625,3.8649)	3.1651(0.2320)	(2.6354,3.6446)	3.2371(0.2872)	(2.5214,3.7930)
	0.5	2.0744(0.0627)	(1.8872,2.1283)	2.0803(0.0459)	(1.9231,2.0958)	2.0579(0.0577)	(1.8991,2.1184)
		3.1754(0.3074)	(2.4739,3.7840)	3.1029(0.2265)	(2.6458,3.5791)	3.1732(0.2816)	(2.5331,3.7142)
	3	2.0340(0.0615)	(1.8900,2.1289)	2.0702(0.0455)	(1.9237,2.0953)	2.0176(0.0564)	(1.9020,2.1183)
		3.1131(0.3007)	(2.4862,3.6656)	3.0427(0.2222)	(2.6543,3.4964)	3.1112(0.2764)	(2.5425,3.6071)
	15	2.0320(0.0618)	(1.8888,2.1057)	2.0687(0.0457)	(1.9233,2.0737)	2.0157(0.0560)	(1.9022,2.0942)
		3.1103(0.3003)	(2.4878,3.5327)	3.0394(0.2220)	(2.6539,3.3698)	3.1083(0.2761)	(2.5430,3.4740)

Table 3: Average ML estimators and Bayes Estimators (Average MSE in brackets) with corresponding asymptotic/HPD Confidence interval for fixed value of $\xi = 0.1$, $\zeta = 3$, n=50 and m=35

α	λ	Par	ML		Bayes1		Bayes2	
			Estimates(MSE)	Asymptotic CI	Estimates(MSE)	HPD CI	Estimates(MSE)	HPD CI
0.8	0.5	α	0.8025(0.0280)	(0.7472,0.8558)	0.8015(0.0124)	(0.7770,0.8244)	0.8016(0.0223)	(0.7575,0.8443)
		λ	0.5293(0.1661)	(0.2081,0.8495)	0.5240(0.0875)	(0.3546,0.6937)	0.5295(0.1413)	(0.2561,0.8021)
	1.6	α	0.8051(0.0309)	(0.7452,0.8648)	0.8018(0.0152)	(0.7723,0.8314)	0.8043(0.0253)	(0.7546,0.8534)
		λ	1.6290(0.1781)	(1.2849,1.9731)	1.6247(0.0996)	(1.4310,1.8164)	1.6297(0.1534)	(1.3328,1.9251)
	2	α	0.8066(0.0340)	(0.7401,0.8714)	0.8030(0.0182)	(0.7661,0.8375)	0.8056(0.0283)	(0.7494,0.8596)
		λ	2.0291(0.1901)	(1.6619,2.3953)	2.0248(0.1117)	(1.8080,2.2393)	2.0298(0.1652)	(1.7096,2.3488)
	3	α	0.8098(0.0369)	(0.7377,0.8809)	0.8037(0.0212)	(0.7619,0.8441)	0.8080(0.0314)	(0.7467,0.8679)
		λ	3.0295(0.2019)	(2.6385,3.4186)	3.0249(0.1236)	(2.7854,3.2630)	3.0297(0.1773)	(2.6868,3.3717)
1	0.5	α	1.0022(0.0400)	(0.9236,1.0792)	1.0009(0.0243)	(0.9531,1.0475)	1.0019(0.0345)	(0.9340,1.0686)
		λ	0.5363(0.2139)	(0.1205,0.9510)	0.5306(0.1357)	(0.2668,0.7936)	0.5366(0.1892)	(0.1689,0.9042)
	1.6	α	1.0051(0.0431)	(0.9215,1.0881)	1.0017(0.0273)	(0.9480,1.0550)	1.0040(0.0375)	(0.9311,1.0767)
		λ	1.6366(0.2261)	(1.1977,2.0748)	1.6302(0.1476)	(1.3435,1.9163)	1.6368(0.2013)	(1.2458,2.0268)
	2	α	1.0065(0.0460)	(0.9164,1.0954)	1.0023(0.0303)	(0.9434,1.0613)	1.0057(0.0404)	(0.9264,1.0833)
		λ	2.0369(0.2380)	(1.5743,2.4977)	2.0302(0.1597)	(1.7203,2.3400)	2.0367(0.2133)	(1.6228,2.4502)
	3	α	1.0092(0.0488)	(0.9138,1.1040)	1.0035(0.0333)	(0.9378,1.0680)	1.0075(0.0435)	(0.9233,1.0916)
		λ	3.0367(0.2500)	(2.5505,3.5216)	3.0300(0.1717)	(2.6969,3.3631)	3.0371(0.2253)	(2.5993,3.4734)
2	0.5	α	2.0021(0.0521)	(1.9001,2.1032)	2.0014(0.0362)	(1.9294,2.0710)	2.0018(0.0464)	(1.9109,2.0923)
		λ	0.5721(0.2619)	(0.0608,1.0832)	0.5608(0.1836)	(0.2015,0.9181)	0.5738(0.2374)	(0.1096,1.0364)
	1.6	α	2.0056(0.0549)	(1.8972,2.1122)	2.0017(0.0393)	(1.9245,2.0786)	2.0040(0.0493)	(1.9073,2.1005)
		λ	1.6725(0.2739)	(1.1372,2.2061)	1.6604(0.1955)	(1.2782,2.0422)	1.6735(0.2494)	(1.1867,2.1595)
	2	α	2.0069(0.0580)	(1.8926,2.1192)	2.0021(0.0422)	(1.9198,2.0846)	2.0058(0.0524)	(1.9028,2.1071)
		λ	2.0730(0.2861)	(1.5136,2.6302)	2.0603(0.2076)	(1.6550,2.4649)	2.0735(0.2613)	(1.5637,2.5829)
	3	α	2.0091(0.0610)	(1.8896,2.1286)	2.0031(0.0452)	(1.9145,2.0918)	2.0076(0.0554)	(1.8987,2.1156)
		λ	3.0721(0.2980)	(2.4911,3.6531)	3.0602(0.2196)	(2.6316,3.4883)	3.0733(0.2732)	(2.5395,3.6060)
3	0.5	α	3.0022(0.0539)	(2.8950,3.1078)	3.0005(0.0383)	(2.9255,3.0759)	3.0017(0.0483)	(2.9057,3.0966)
		λ	0.6080(0.2919)	(0.0354,1.1810)	0.5907(0.2137)	(0.1711,1.0093)	0.6103(0.2674)	(0.0855,1.1339)
	1.6	α	3.0050(0.0570)	(2.8930,3.1169)	3.0018(0.0413)	(2.9201,3.0825)	3.0046(0.0513)	(2.9034,3.1050)
		λ	1.7087(0.3041)	(1.1117,2.3043)	1.6909(0.2257)	(1.2478,2.1322)	1.7099(0.2793)	(1.1621,2.2576)
	2	α	3.0062(0.0600)	(2.8880,3.1234)	3.0029(0.0443)	(2.9149,3.0888)	3.0051(0.0544)	(2.8977,3.1114)
		λ	2.1089(0.3160)	(1.4880,2.7277)	2.0900(0.2377)	(1.6238,2.5564)	2.1101(0.2913)	(1.5384,2.6805)
	3	α	3.0100(0.0630)	(2.8855,3.1327)	3.0032(0.0473)	(2.9106,3.0961)	3.0079(0.0574)	(2.8948,3.1196)
		λ	3.1090(0.3280)	(2.4653,3.7512)	3.0904(0.2497)	(2.6010,3.5794)	3.1098(0.3034)	(2.5150,3.7038)

Table 4: Expected total time on test $E[x_m]$ for $\alpha = 2$, $\lambda = 3$

n	m		$\xi = 0.5$							$\xi = 1$						
		$\zeta \longrightarrow$	15	9	3	1	0.5	0.25	0.1	15	9	3	1	0.5	0.25	0.1
10	5		4.14	4.29	5.34	9.92	16.31	20.21	22.37	4.37	4.76	7.18	15.24	19.69	21.95	23.14
10	6		5.62	5.87	8.07	16.24	23.22	26.31	28.49	5.93	6.81	11.72	22.61	25.94	27.96	29.27
10	7		7.93	8.58	12.78	25.73	30.81	32.61	34.54	8.85	10.42	19.4	30.96	32.9	34.98	34.97
10	8		12.04	13.46	19.66	32.86	37.81	38.81	39.06	13.48	16.35	28.34	37.7	39.62	39.61	40.66
10	9		21.43	22.97	31.68	42.81	45.37	46.36	47.26	25.23	29.19	39.97	45.63	45.62	47.16	47.15
10	10		53.76	53.76	53.76	53.76	53.76	53.76	53.76	51.84	51.84	51.84	51.84	51.84	51.84	51.84
20	10		4.68	5.14	9.52	32.75	45.34	46.49	51.25	5.51	6.8	20.63	45.17	47.17	48.22	49.46
20	12		6.64	7.66	17.31	49.51	58.42	58.41	62.69	8.21	11.03	34.93	55.08	60.21	60.2	60.19
20	14		9.97	11.89	30.98	63.35	68.04	70.19	73.42	13.16	19.86	55.02	70.98	73.88	73.87	73.86
20	16		16.68	20.77	49.51	79.26	79.25	81.41	83.03	23.14	35.77	70.44	82.13	82.12	82.11	82.66
20	18		33.79	42.75	72.53	92.3	93.73	93.72	94.31	46.4	60.01	89.78	94.42	94.41	97.04	97.03
20	20		106.53	106.53	106.53	106.53	106.53	106.53	106.53	105.95	105.95	105.95	105.95	105.95	105.95	105.95
30	15		5.19	6.12	17.91	62.48	71.96	75.99	75.98	6.68	9.99	45.41	72.23	75.31	76.48	76.47
30	18		7.57	9.65	35.8	85.2	87.13	91.4	91.39	10.9	18.38	71.34	89.97	89.96	94.09	94.08
30	21		12.05	16.76	64.7	99.88	103.86	103.97	106.34	18.95	35.51	96.32	107.6	112.06	112.57	112.56
30	24		21.1	31.33	95.65	121.96	124.63	124.62	127.81	37.2	65.97	119.8	122.34	123.86	128.91	128.9
30	27		46.95	67.99	132.31	138	142.09	142.08	142.07	77.66	108.58	139.44	141.42	141.41	141.4	141.54
30	30		155.65	155.65	155.65	155.65	155.65	155.65	155.65	160.83	160.83	160.83	160.83	160.83	160.83	160.83
40	20		5.7	7.39	34.6	91.22	100.19	103.79	103.78	8.26	15.2	80.56	98.5	99.27	102.84	104.36
40	24		8.77	12.76	66.61	112.59	119.96	119.95	121.01	14.97	31.98	111.78	120.89	123.22	123.21	126.38
40	28		14.6	22.18	98.89	144.27	144.25	144.24	150.73	27.93	64.99	140.41	140.4	151.7	151.69	151.68
40	32		26.37	46.63	143.81	166.69	167.99	167.98	167.97	56.84	107.87	158.83	160.86	166.28	166.27	166.26
40	36		62.58	97.06	183.99	191.68	191.66	197.22	197.21	112.85	159.9	179.14	189.35	189.34	189.33	194.26
40	40		219.46	219.46	219.46	219.46	219.46	219.46	219.46	212.09	212.09	212.09	212.09	212.09	212.09	212.09
50	25		6.3	9.16	56.96	121.19	132.62	132.61	132.6	10.46	24.17	112.63	132.04	132.03	132.02	132.01
50	30		10.14	16.82	103.61	144.1	155.32	155.46	156.27	20.67	53.48	145.24	149.07	153.81	153.8	153.79
50	35		17.41	32.14	152.99	179.2	179.19	180.86	180.85	43.59	103.26	175.07	175.06	182.02	182.01	185.33
50	40		33.32	66.71	190.01	200.61	209.79	209.78	213.51	84.51	156.67	199.53	203.22	210.74	210.73	210.72
50	45		79.76	137.76	250.87	250.86	250.85	250.84	250.83	164.01	217.16	222.59	238.58	238.57	238.56	238.55
50	50		259.99	259.99	259.99	259.99	259.99	259.99	259.99	261.62	261.62	261.62	261.62	261.62	261.62	261.62

Table 5: Expected total time on test $E[x_m]$ for $\alpha = 0.8$, $\lambda = 1.6$

n	m		$\xi = 0.5$							$\xi = 1$						
		$\zeta \longrightarrow$	15	9	3	1	0.5	0.25	0.1	15	9	3	1	0.5	0.25	0.1
10	5		2.65	2.8	3.68	8.31	15.77	20.44	23.39	2.87	3.23	5.46	14.03	19.95	22.79	24.04
10	6		3.95	4.22	6.19	15.62	25.25	29.11	30.93	4.43	5.11	10.08	23.86	28.04	30.14	31.5
10	7		6.12	6.76	10.93	24.95	35.22	36.5	40.08	6.98	8.43	18.77	33.89	39.76	39.75	39.73
10	8		10.53	11.94	19.74	37.02	47.53	47.52	50.13	12.79	16.14	30.15	47.71	47.7	49.23	49.22
10	9		20.71	24.22	36.93	54.77	55.73	57.23	57.22	25.28	31.11	47.8	57.36	57.57	57.56	57.55
10	10		68.86	68.86	68.86	68.86	68.86	68.86	68.86	64.88	64.88	64.88	64.88	64.88	64.88	64.88
20	10		3.18	3.56	8.07	35.95	53.75	65.49	65.48	3.83	5.05	21.64	54	57.63	60.44	60.43
20	12		4.82	5.88	17.11	59.37	77.21	77.2	82.36	6.34	9.4	40.99	76.88	78.28	80.41	82.24
20	14		8.24	10.61	35.08	83.64	92.71	101.15	101.14	11.51	18.83	71.06	94.38	95.5	99.96	104.02
20	16		15.52	20.03	64.41	108.47	113.31	120.45	121.73	23.36	37.1	101.87	118.23	118.22	119.19	119.18
20	18		36.47	50.27	111.99	133.38	133.37	133.36	140.27	54.09	80.43	135.15	139.35	144.84	144.83	144.82
20	20		163.32	163.32	163.32	163.32	163.32	163.32	163.32	163.6	163.6	163.6	163.6	163.6	163.6	163.6
30	15		3.60	4.44	18.59	85.87	95.28	103.16	109.53	4.96	8.27	57.2	98	112.49	112.48	112.47
30	18		5.94	7.91	43.66	121.49	136.58	136.57	137.03	9.19	17.79	100.58	128.76	136.28	136.26	136.25
30	21		10.23	15.57	87.09	159.28	171.41	171.4	171.38	18.05	42.45	148.58	162.97	162.96	167.86	167.85
30	24		20.87	34.92	142.66	194.16	198.59	203.7	203.69	44.03	89.39	180.83	199.15	199.13	199.12	201.29
30	27		55.82	89.45	198.19	228.69	231.09	231.08	233.05	105.82	164.53	226.63	228.4	228.39	228.38	233.93
30	30		276.39	276.39	276.39	276.39	276.39	276.39	276.39	270.2	270.2	270.2	270.2	270.2	270.2	270.2
40	20		4.06	5.54	39.16	134.45	153.45	153.44	157.5	6.48	14.14	109.25	151.35	151.34	154.3	154.29
40	24		7.01	10.66	90.32	180.53	189.86	197.83	197.81	13.58	35.69	172.15	189.22	193.74	193.73	198.72
40	28		13.11	23.44	155.95	235.69	242.85	242.84	249.26	31.45	88.41	223.43	244.67	244.66	244.65	245.11
40	32		27.54	56.58	224.23	284.21	284.2	284.19	284.18	79.07	164.19	272.27	289.49	289.47	289.46	289.45
40	36		80.51	146.86	314.85	349.19	349.18	349.17	349.16	185.82	275.36	314.88	326.04	333.13	333.12	333.11
40	40		388.78	388.78	388.78	388.78	388.78	388.78	388.78	373.99	373.99	373.99	373.99	373.99	373.99	373.99
50	25		4.56	7.07	76.98	190.46	202.82	202.8	213.33	8.59	23.95	167.62	197.49	200.65	202.65	210.87
50	30		8.23	15.38	157.51	236.9	250.73	250.72	257.53	20.79	66.23	239.33	262.75	262.74	262.73	262.72
50	35		16.44	36.33	255.71	309.86	314.72	323.41	323.39	51.49	153.8	313.71	336.49	336.48	336.47	336.45
50	40		38.52	91.82	358.63	363.76	363.79	376.93	384.22	131.05	289.07	368.52	388.16	388.15	394.25	394.24
50	45		116.4	215.53	423.94	428.49	459.53	459.52	460.2	285.76	386.26	434.95	441.18	441.17	457.89	457.88
50	50		513.71	513.71	513.71	513.71	513.71	513.71	513.71	510.02	510.02	510.02	510.02	510.02	510.02	510.02

Table 6: Expected total time on test $E[x_m]$ for $\alpha = 2$, $\lambda = 3$

n	m		$\zeta = 1$							$\zeta = 3$						
		$\xi \longrightarrow$	15	9	3	1	0.5	0.25	0.1	15	9	3	1	0.5	0.25	0.1
10	5		21.91	21.9	21.43	15.65	10.06	6.73	4.86	21.76	20.88	14.38	7.2	5.3	4.57	4.14
10	6		29.11	28.31	28.3	22.19	16.7	10.23	6.84	28.51	27.56	22.8	11.93	8.06	6.34	5.58
10	7		34.12	34.11	34.1	29.77	24.41	16.55	10.43	35.43	35.42	29.62	19.9	12.62	9.49	7.94
10	8		41.69	39.29	39.28	38.23	33.2	25.43	17.02	40.73	40.67	36.24	29.11	19.71	14.76	12.41
10	9		45.56	45.55	45.26	44.77	43.84	36.89	26.91	48.05	46.16	46.15	39.59	32.54	25.33	21.6
10	10		51.63	51.63	51.63	51.63	51.63	51.63	51.63	53.76	53.76	53.76	53.76	53.76	53.76	53.76
20	10		50.87	47.78	46.85	41.38	32.14	17.56	7.41	49.82	47.49	43.43	20.99	9.74	6.04	4.73
20	12		61.76	60.92	60.43	57	49.38	30.82	12.12	58.44	58.43	56.38	35.49	17.27	9.54	6.7
20	14		72.45	71.1	71.06	69.4	62.26	45.47	20.6	72.19	69.6	68.66	54.37	30.51	15.63	10.11
20	16		86.63	83.73	81.35	79.76	76.66	64.67	35.24	81.19	81.18	80.85	72.99	49.68	28.45	16.83
20	18		91.39	91.38	91.37	91.36	88.21	83.67	61.2	93.35	93.34	93.33	89.93	76.03	51.75	32.85
20	20		106	106	106	106	106	106	106	106.53	106.53	106.53	106.53	106.53	106.53	106.53
30	15		74.15	74.14	74.13	71.72	63.56	40.46	11.89	73.93	72	70.99	44.72	19.23	8.15	5.34
30	18		94.78	93.12	92.22	85.39	81.98	63.06	22.87	93.29	87.15	87.03	72.3	36.57	14.6	7.97
30	21		107.11	107.1	107.09	102.96	102.14	91.79	41.92	107.27	105.67	105.66	97.75	62.05	26.45	12.57
30	24		125.81	125.8	124.16	124.15	123.61	110.85	67.62	126.44	122.25	121.1	119.69	98.26	50.02	22.35
30	27		139.48	139.47	135.26	135.25	135.24	130.27	107.42	140.74	139.38	137.96	134.97	130.13	91.88	46.61
30	30		160.75	160.75	160.75	160.75	160.75	160.75	160.75	155.65	155.65	155.65	155.65	155.65	155.65	155.65
40	20		101.18	101.17	101.16	101.15	93.1	68.03	20.42	99.31	99.3	96.99	80.61	34.33	11.36	5.85
40	24		120.61	120.6	120.59	120.58	111.11	97.54	39.57	119.48	119.47	119.46	111.93	65.8	21.95	8.95
40	28		141.73	141.72	141.71	141.7	141.69	130.92	69.29	151.41	147.29	140.09	140.08	103.44	44.54	15.22
40	32		163.59	163.58	163.57	163.56	163.55	163.54	109.28	165.94	165.92	161.08	160.71	142.77	81.55	28.77
40	36		193.74	189.75	189.46	183.34	183.32	182.53	154.89	191.66	188.91	187.71	184.1	171.09	136.58	65.21
40	40		212.51	212.51	212.51	212.51	212.51	212.51	212.51	219.46	219.46	219.46	219.46	219.46	219.46	219.46
50	25		126.6	126.59	126.58	123.55	120.89	99.43	33.45	126.17	124.56	124.55	107.83	59.12	16.51	6.76
50	30		151.1	151.09	151.07	151.06	150.38	135.61	62.74	157.09	153.19	153.18	145.56	106.78	35.79	10.61
50	35		175.91	175.9	175.88	175.87	173.35	173.34	104.7	173.69	173.68	173.67	173.66	148.35	68.61	18.67
50	40		210.41	208.61	208.6	208.59	206.75	204.72	154.5	204.31	204.3	204.29	204.27	187.87	123.06	37.49
50	45		246.15	231.31	231.3	228.72	228.71	228.7	210.21	231.15	227.27	227.26	227.25	227.24	192.28	83.63
50	50		261.62	261.62	261.62	261.62	261.62	261.62	261.62	259.99	259.99	259.99	259.99	259.99	259.99	259.99

Table 7: Expected total time on test $E[x_m]$ for $\alpha = 0.8$, $\lambda = 1.6$

n	m		$\zeta = 1$							$\zeta = 3$						
		$\xi \longrightarrow$	15	9	3	1	0.5	0.25	0.1	15	9	3	1	0.5	0.25	0.1
10	5		24.7	23.62	21.48	13.71	8.66	4.75	3.29	22.91	20.83	13.28	5.41	3.73	3.01	2.7
10	6		31.27	31.26	29.61	23.74	16.01	8.72	5.13	30.97	30.75	24.28	10.12	6.14	4.63	4.04
10	7		41.86	39.94	38.62	34.36	25.15	15.53	8.46	39.52	37.68	35.21	18.76	11.2	7.63	6.22
10	8		51.42	49.26	46.73	45.44	37.16	26.57	15.47	47.81	47.77	43.26	32.27	20.15	13.6	10.69
10	9		57.87	57.86	57.12	57.11	51.1	42.5	28.4	56.95	55.14	55.13	49.11	36.01	28.37	21.13
10	10		64.88	64.88	64.88	64.88	64.88	64.88	64.88	68.86	68.86	68.86	68.86	68.86	68.86	68.86
20	10		60.41	59.28	59.27	51.92	37.2	17.87	5.76	62.01	60.89	52.86	21.5	7.57	4.45	3.16
20	12		76.9	75.75	75.74	73.44	59.84	33.88	10.47	80.34	79.99	71.72	41.51	16.71	7.98	4.99
20	14		97.02	97.01	97	94.97	88.74	60.68	21.57	93.85	93.84	93.06	70.53	33.6	14.83	8.48
20	16		123.42	121.32	120.46	109.59	109.58	89.61	42.2	119.63	119.63	119.63	119.63	119.63	119.63	119.63
20	18		133.67	133.66	133.65	128.87	128.86	120.47	81.96	142.22	137.83	131.7	131.69	102.76	69.65	36.89
20	20		163.6	163.6	163.6	163.6	163.6	163.6	163.6	163.32	163.32	163.32	163.32	163.32	163.32	163.32
30	15		106.97	106.96	104.49	98.39	81.8	44.28	10.52	103.23	103.22	94.44	59.09	17.27	6.36	3.69
30	18		137.99	137.98	135.06	135.05	119.34	80.9	23.7	131.01	131	129.83	101.29	42.07	12.85	6.01
30	21		165.34	165.33	164.46	164.45	151.73	131.8	48.86	162.89	162.88	161.04	135.98	88.6	28.74	10.71
30	24		192.69	192.1	192.09	192.08	192.07	182.78	92.67	199.23	198.09	190.15	190.14	148.14	62.01	22.25
30	27		240.11	233.36	221.5	221.49	221.48	215.97	160.35	231.29	231.28	229.57	228.22	190.41	136.69	58.26
30	30		270.2	270.2	270.2	270.2	270.2	270.2	270.2	276.39	276.39	276.39	276.39	276.39	276.39	276.39
40	20		157.2	151.26	151.25	144.24	134.25	95.36	21.22	149.95	149.94	148.7	112.44	39.91	9.78	4.17
40	24		192.91	192.9	185.49	185.48	185.46	151.89	49.92	185.39	185.38	185.37	162.71	91.31	21.69	7.36
40	28		233.05	233.04	231.92	228.38	228.37	199.66	100.75	233.47	224.61	224.6	216.69	152.02	56.29	13.82
40	32		287.34	287.33	287.32	284.45	263.35	263.33	159.76	287.7	287.68	268.38	268.37	247.82	121.28	30.94
40	36		322.06	322.05	315.43	315.42	314.51	314.5	270.89	327.21	327.2	324.56	324.55	317.18	236.51	87.07
40	40		373.99	373.99	373.99	373.99	373.99	373.99	373.99	388.78	388.78	388.78	388.78	388.78	388.78	388.78
50	25		210.96	206.74	195.87	195.86	193.44	152.43	38.63	214.3	197.84	192.46	166.54	81.04	15.86	4.88
50	30		266.94	262.3	251.68	244.63	241.88	225.92	91.04	249.75	249.74	249.72	227.43	147.81	40.85	8.77
50	35		306.25	296.78	296.77	296.76	296.75	292.66	164.97	318.49	318.48	308.98	306.7	256.79	96.77	18.26
50	40		388.53	374.21	374.08	362.64	361.01	357.37	262.62	381.86	380.28	362.88	362.87	338.05	208.41	42.96
50	45		425.17	425.16	425.15	425.14	416.05	416.03	377.57	441.93	438.81	438.8	425.8	410.41	349.73	123.17
50	50		510.02	510.02	510.02	510.02	510.02	510.02	510.02	513.71	513.71	513.71	513.71	513.71	513.71	513.71

The third data set represents the lifetime‘s data relating to relief times (in minutes) of 20 patients receiving an analgesic and reported by Gross and Clark (1975).

1.1	1.4	1.3	1.7	1.9	1.8	1.6	2.2	1.7
2.7	4.1	1.8	1.5	1.2	1.4	3.0	1.7	2.3
1.6	2.0							

Fourth data set reported by Efron (1988) represent the survival times of a group of patients suffering from Head and Neck cancer disease and treated using a combination of radiotherapy and chemotherapy (RT+CT).

12.20	23.56	23.74	25.87	31.98	37.0	41.35	47.38	55.46
58.36	63.47	68.46	78.26	74.47	81.43	84.00	92.00	94.00
110.0	112.0	119.0	127.0	130.0	133.0	140.0	146	155.0
159.0	173.0	179.0	194.0	195.0	209.0	249.0	281.0	319.0
339.0	432.0	469.0	519.0	633.0	725.0	817.0	1776	

The MLEs for the unknowns are calculated for all above data sets based on complete sample and reported in Table 8, using the procedure explained in section 2. In this Table, $\hat{\lambda}_{ML}$ and $\hat{\alpha}_{ML}$ represent the maximum likelihood estimates for the parameters α and λ, respectively. The quantity reported in brackets is the standard deviation (sd) computed based on the square root of inverse of estimated Fisher information matrix as given in equation (18). Here, we also compute the K-S statistic and corresponding p-value for the purpose of goodness-of-fit. The quantity log of likelihood, AIC and BIC are also presented.

For the illustration of our methodology, we have generated censored data for a prefixed m, ξ and ζ. It may be worthwhile to mention here that the number of drop-outs are random and we are generating the progressive type-II censored data from the complete sample data, therefore, we can study the average performance of the estimators. For this purpose, we generated 2000 censored data sets for given m and accordingly the ξ's and ζ's from the considered complete

data set. The m is chosen 80% of the complete sample size and $\xi = 3, \zeta = 0.1$ are considered. The average ML estimates, Bayes estimates with corresponding average MSE and confidence interval are reported in Table 9.

Table 8: MLEs with other statistic for considered real data sets

data	$\hat{\alpha}_{ML}(sd)$	$\hat{\lambda}_{ML}(sd)$	KS-Statistic	p-value	log-lik	AIC	BIC
Boing AC	1.1458(0.2357)	65.3018(53.3062)	0.1480	0.8970	-84.0772	172.1544	173.5705
Chloride	0.8803(0.1093)	0.6539(0.1347)	0.1134	0.7740	-58.6266	121.2532	124.3059
Relief Time	4.0173(0.6972)	6.0221(1.9636)	0.1019	0.9850	-15.4087	34.8174	36.8089
RT+CT	1.0134(0.1119)	80.7880(36.9616)	0.0926	0.8110	-279.5701	563.1403	566.7086

Table 9: Average estimates with corresponding average MSE and Assymptotic/HPD confidence Interval for fixed $\xi = 3$, $\zeta = 0.1$ and $m = 0.80 \times$ sample size, for considered real data sets

	Data	α		λ	
Boing AC	ML	1.1248(0.2650)	(0.6117, 1.6413)	65.8131(55.1362)	(-42.2472, 173.8737)
	Bayes	1.0250(0.2260)	(0.6846, 1.6378)	65.3132(53.1317)	(-36.7806, 171.4009)
Chloride	ML	0.8995(0.1385)	(0.6342, 1.1639)	0.6331(0.1643)	(0.3170, 0.9491)
	Bayes	0.8498(0.1249)	(0.6541, 1.1325)	0.1339(0.1569)	(0.3501, 0.9172)
Relief Time	ML	3.9963(0.7268)	(2.5818, 5.4195)	5.7018(1.9933)	(1.8030, 9.6023)
	Bayes	4.0066(0.7254)	(2.6356, 5.4123)	5.8021(1.9809)	(1.9252, 9.0770)
RTCT	ML	0.9928(0.1415)	(0.7258, 1.2645)	79.7676(37.8907)	(5.5043, 154.0242)
	Bayes	1.0932(0.1294)	(0.7915, 1.1693)	81.2678(35.1421)	(5.6798, 150.0634)

7. Conclusion

In this chapter, we have developed a sampling procedure for life-testing experiment called as progressive Type-II censoring scheme with beta-binomial removals (PT-II CBBR) which covers the uncertainty of the real phenomenon of life-testing procedure. The Bayesian procedure provides the more accurate and precise estimates of the parameters even if we consider the vague prior. Finally, we can conclude that the discussed methodology provides the more flexible procedure for life-testing experiment and can be recommended for their use in medical, engineering and in other areas where such type of life-tests are needed.

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