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Riesz and Beta-Riesz Distributions

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Abstract

This article derives several properties of the Riesz distributions, such as their corresponding Bartlett decompositions, the inverse Riesz distributions, the distribution of the generalised variance and the density of their eigenvalues for real normed division algebras. In addition, introduce a kind of generalised beta distribution termed beta-Riesz distribution for real normed division algebras. Two versions of this distributions are proposed and some properties are studied.

Keywords: Beta-Riesz distribution, Riesz distribution, generalised beta function and distribution, real, complex quaternion and octonion random matrices.

1. Introduction

It is imminent the important role played by Wishart and beta distributions type I and II in the context of multivariate statistics. In particular, the relationship between these two distributions to obtain the beta distribution in terms of the distribution of two Wishart matrices.

In the last three decades, the family of elliptical contoured distributions has emerged as a robust alternative for dealing with non-normal samples. A number of well known distributions belong to this class, such is the case of normal, t, Bessel, Kotz type, logistic, Pearson type II and IV, among many others; but also infinitely many new distributions can be constructed by choosing a suitable kernel corresponding to a measurable function. Elliptical distributions are characterized by several properties, but for the context of sampling, their large variability of kurtosis and weight of tails, can assure a best explanation of the sample, rather than the usual forced fit to a normal model in presence of non explicable extremal points. Another important property of this set resides in the normal invariant statistics; i.e., assume that certain random matrix $\mathbf{X}$ follows a matrix multivariate elliptical distribution, then, many matrices of the form $\mathbf{Y} = f(\mathbf{X})$, for special functions $f(\cdot)$, are invariant under the complete family of elliptical distribution, in the sense that the distribution of $\mathbf{Y}$ is invariant, independently of the particular distribution of $\mathbf{X}$, in fact the distribution of $\mathbf{Y}$ coincides with the case where $\mathbf{X}$ has a matrix multivariate normal distribution, see Fang and Zhang (1990) and Gupta and Varga (1993).

However, we must note that the addressed invariance only holds under a probabilistic dependence assumption; i.e., if the matrix $\mathbf{X} = \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix}$ follows an elliptical distribution, then $\mathbf{X}_1$ and $\mathbf{X}_2$ must be probabilistically dependent (recall that $\mathbf{X}_1$ and $\mathbf{X}_2$ are probabilistically independent if $\mathbf{X}$ has a matrix multivariate normal distribution, Gupta and Varga (1993)). Now, let $\mathbf{X}^T$ denotes the transpose of $\mathbf{X}$ and consider a non negative definite matrix $\mathbf{B}$, where $\mathbf{B}^{1/2}$ denotes its non negative squared root, see Fang and Zhang (1990), then, the matrix $\mathbf{F} = (\mathbf{X}_2^T\mathbf{X}_2)^{-1/2}(\mathbf{X}_1^T\mathbf{X}_1)(\mathbf{X}_2^T\mathbf{X}_2)^{-1/2}$ is said to have a matrix multivariate Beta type II distribution, moreover, the same distribution is obtained when $\mathbf{X}$ is assumed to follow matrix multivariate distribution, see Fang and Zhang (1990) and Gupta and Varga (1993).

Now, if the random matrix $\mathbf{X}$ follows a matrix multivariate elliptical distribution and the random matrix $\mathbf{W} = \mathbf{X}^T\mathbf{X}$ is defined, for every particular elliptical distribution, $\mathbf{W}$ follows also a different distribution. The distribution of $\mathbf{W}$ is usually known as the generalised Wishart distribution, and it heritages several properties of elliptical contoured distributions.

By another hand some recent advances in probability has reached interesting general distributions, such as the case of Riesz distribution, due to Hassairi and Lajmi (2001), and named under the denomination of Riesz natural exponential family (Riesz NEF); a distribution based in an special case of the well known Riesz measure given by Faraut and Korányi (1994, p. 137). In fact, Riesz distribution includes Wishart and Gamma matrix multivariate distributions as particular cases. Now, integrating theories have also appeared in other contexts. For example, in matrix multivariate distribution theory, some extensions from real to complex or quaternion or octonion fields appeared separately with great theoretical effort in many cases, the results in each field arose unconnected each other for years; only recent a new approach in the context of real normed division algebras could integrate, with an unified theory, all the addressed dispersed results in such sets of numbers. Following this tendency, recently, Díaz-García (2015a) proposes two versions of the Riesz distribution for real normed division algebras. Alternatively, Díaz-García (2015b) shows that the two versions of Riesz distributions correspond to two generalised Wishart distributions, both derived from certain matrix multivariate elliptical distributions, which are termed matrix multivariate Kotz-Riesz distributions.

Faraut and Korányi (1994), and subsequently Hassairi, Lajmi, and Zine (2005), propose a beta-Riesz distribution, which contains as a special case to the beta distribution obtained in terms of the distribution Wishart, which shall be named beta-Wishart, all this subjects in the context of simple Euclidean Jordan algebras. Such beta-Riesz distribution is obtained analogously to the beta-Wishart distribution, but starting with a Riesz distribution.

Based in these last two versions of the Riesz distributions, it is possible to obtain two versions of the beta-Riesz distributions, which by analogy with the beta-Wishart distributions are termed beta-Riesz type I. As in classical beta-Wishart distribution, in addition it is feasible to propose two version for the beta-Riesz distribution type II. Each of the two versions for each beta-Riesz distributions of type I and II, (both versions for each) contain as particular cases to beta-Wishart distribution type I and beta-Wishart distribution type II, respectively.

Thus, there is no doubt about the theoretical and applied potential of Riesz distribution into the setting in the setting of the integrative modern multivariate analysis. In general, every problem possible ruled by a Wishart process with considerable constraints, potentially can be studied under a more robust Riesz distribution; namely, some opportunities for consideration involve estimation of covariance matrices and principal component estimation, under any of the classical or bayesian approaches. Beta-Riesz distribution, for example, arises in a natural way in bayesian inference, when the two parameter a priori distributions are probabilistically independent. To prove this fact, recall the example about matrix $\mathbf{X}$, referred in the third paragraph of this section, and note that several situations consider are governed by probabilistically independent matrices $\mathbf{X}_1$ and $\mathbf{X}_2$, both of them following a Kotz-Riesz elliptical distribution. Under these assumptions, the distribution of the corresponding random matrix $\mathbf{F}$ does not follow a beta-Wishart, but the beta-Riesz distribution. Then, if the two parameters, $\boldsymbol{\delta}_1$ and $\boldsymbol{\delta}_2$ are distributed as $(\mathbf{X}_1^T\mathbf{X}_1)$ and $(\mathbf{X}_2^T\mathbf{X}_2)$, respectively; i.e. $\boldsymbol{\delta}_1$ and $\boldsymbol{\delta}_2$ follows independent Riesz distributions, then the parameter matrix defined as $\mathbf{F} = \boldsymbol{\delta}_2^{-1/2}\boldsymbol{\delta}_1\boldsymbol{\delta}_2^{-1/2}$, has a beta-Riesz distribution.

Although during the 90's and 2000's were obtained important results in theory of random matrices distributions, the past 30 years have reached a substantial development. Essentially, these advances have been archived through two approaches based on the *theory of Jordan algebras* and the *theory of real normed division algebras*. A basic source of the mathematical tools of theory of random matrices distributions under Jordan algebras can be found in Faraut and Korányi (1994); and specifically, some works in the context of theory of random matrices distributions based on Jordan algebras are provided in Massam (1994), Casalis and Letac (1996), Hassairi and Lajmi (2001), and Hassairi *et al.* (2005), and the references therein. Parallel results on theory of random matrices distributions based on real normed division algebras have been also developed in random matrix theory and statistics, see for example Gross and Richards (1987), Dumitriu (2002), Forrester (2005), Díaz-García and Gutiérrez-Jáimez (2011, 2013), among others. In addition, from mathematical point of view, several basic properties of the matrix multivariate Riesz distribution under *the structure theory of normal j-algebras* and under *theory of Vinberg algebras* in place of Jordan algebras have been studied, see Ishi (2000) and Boutouria and Hassairi (2009), respectively.

From a applied point of view, the relevance of *the octonions* remains unclear. An excellent review of the history, construction and many other properties of octonions is given in Baez (2002), where it is stated that... **However, there is still no *proof* that the octonions are useful for understanding the real world**. We can only hope that eventually this question will be settled one way or another."

For the sake of completeness, in this article the case of octonions is considered, but the veracity of the results obtained for this case can only be conjectured. Nonetheless, Forrester (2005, Section 1.4.5, pp. 22-24) it is proved that the bi-dimensional density function of the eigenvalue, for a Gaussian ensemble of a 2×2 octonionic matrix, is obtained from the general joint density function of the eigenvalues for the Gaussian ensemble, assuming $m = 2$ and $\beta = 8$, see Section 2. Moreover, as is established in Faraut and Korányi (1994) and Sawyer (1997) the result obtained in this article are valid for the *algebra of Albert*, that is when hermitian matrices ($\mathbf{S}$) or hermitian product of matrices ($\mathbf{X}^*\mathbf{X}$) are 3×3 octonionic matrices.

This article studies two versions for beta-Riesz distributions type I and II for real normed division algebras. Section 2 reviews some definitions and notation on real normed division algebras. And also, introduces other mathematical tools as three Jacobians with respect to Lebesgue measure and some integral results for real normed division algebras. Section 3 proposes diverse properties of two versions of the Riesz distributions as their Bartlett decompositions, inverse Riesz distributions, the distribution of the generalized variance and the distribution of their eigenvalues. Section 4 introduces two generalised beta functions and, in terms of these, two beta-Riesz distributions of type I and II are obtained for real normed division algebras. Also, the relationship between the Riesz distributions and the beta-Riesz distributions are studied. This section concludes studying the eigenvalues distributions of beta-Riesz distributions type I and II in their two versions for real normed division algebras.

2. Preliminary results

A detailed discussion of real normed division algebras may be found in Baez (2002) and Neukirch, Prestel, and Remmert (1990). For convenience, we shall introduce some notation, although in general we adhere to standard notation forms.

For our purposes: Let $\mathbb{F}$ be a field. An *algebra* $\mathfrak{A}$ over $\mathbb{F}$ is a pair $(\mathfrak{A}; m)$, where $\mathfrak{A}$ is a *finite-dimensional vector space* over $\mathbb{F}$ and *multiplication* $m : \mathfrak{A}\times\mathfrak{A} \to A$ is an $\mathbb{F}$-bilinear map; that is, for all $\lambda \in \mathbb{F}$, $x, y, z \in \mathfrak{A}$,

$$\begin{aligned} m(x, \lambda y + z) &= \lambda m(x; y) + m(x; z) \\ m(\lambda x + y; z) &= \lambda m(x; z) + m(y; z). \end{aligned}$$

Two algebras $(\mathfrak{A}; m)$ and $(\mathfrak{E}; n)$ over $\mathbb{F}$ are said to be *isomorphic* if there is an invertible map $\phi : \mathfrak{A} \to \mathfrak{E}$ such that for all $x, y \in \mathfrak{A}$,

$$\phi(m(x, y)) = n(\phi(x), \phi(y)).$$

By simplicity, we write $m(x; y) = xy$ for all $x, y \in \mathfrak{A}$.

Let $\mathfrak{A}$ be an algebra over $\mathbb{F}$. Then $\mathfrak{A}$ is said to be

1. *alternative* if $x(xy) = (xx)y$ and $x(yy) = (xy)y$ for all $x, y \in \mathfrak{A}$,
2. *associative* if $x(yz) = (xy)z$ for all $x, y, z \in \mathfrak{A}$,
3. *commutative* if $xy = yx$ for all $x, y \in \mathfrak{A}$, and
4. *unital* if there is a $1 \in \mathfrak{A}$ such that $x1 = x = 1x$ for all $x \in \mathfrak{A}$.

If $\mathfrak{A}$ is unital, then the identity 1 is uniquely determined.

An algebra $\mathfrak{A}$ over $\mathbb{F}$ is said to be a *division algebra* if $\mathfrak{A}$ is nonzero and $xy = 0_{\mathfrak{A}} \Rightarrow x = 0_{\mathfrak{A}}$ or $y = 0_{\mathfrak{A}}$ for all $x, y \in \mathfrak{A}$.

The term "division algebra", comes from the following proposition, which shows that, in such an algebra, left and right division can be unambiguously performed.

Let $\mathfrak{A}$ be an algebra over $\mathbb{F}$. Then $\mathfrak{A}$ is a division algebra if, and only if, $\mathfrak{A}$ is nonzero and for all $a, b \in \mathfrak{A}$, with $b \neq 0_{\mathfrak{A}}$, the equations $bx = a$ and $yb = a$ have unique solutions $x, y \in \mathfrak{A}$.

In the sequel we assume $\mathbb{F} = \Re$ and consider classes of division algebras over $\Re$ or "*real division algebras*" for short.

We introduce the algebras of *real numbers* $\Re$, *complex numbers* $\mathfrak{C}$, *quaternions* $\mathfrak{H}$ and *octonions* $\mathfrak{O}$. Then, if $\mathfrak{A}$ is an alternative real division algebra, then $\mathfrak{A}$ is isomorphic to $\Re$, $\mathfrak{C}$, $\mathfrak{H}$ or $\mathfrak{O}$.

Let $\mathfrak{A}$ be a real division algebra with identity 1. Then $\mathfrak{A}$ is said to be *normed* if there is an inner product $(\cdot, \cdot)$ on $\mathfrak{A}$ such that

$$(xy, xy) = (x, x)(y, y) \qquad \text{for all } x, y \in \mathfrak{A}.$$

If $\mathfrak{A}$ is a *real normed division algebra*, then $\mathfrak{A}$ is isomorphic $\Re$, $\mathfrak{C}$, $\mathfrak{H}$ or $\mathfrak{O}$.

There are exactly four normed division algebras: real numbers ($\Re$), complex numbers ($\mathfrak{C}$), quaternions ($\mathfrak{H}$) and octonions ($\mathfrak{O}$), see Baez (2002).

Let $\mathfrak{A}$ be a division algebra over the real numbers. Then $\mathfrak{A}$ has dimension either 1, 2, 4 or 8. In other branches of mathematics, the parameters $\alpha = 2/\beta$ and $t = \beta/4$ are used, see Edelman and Rao (2005) and Kabe (1984), respectively.

Let $\mathcal{L}^{\beta}_{m,n}$ be the set of all $m \times n$ matrices of rank $m \leq n$ over $\mathfrak{A}$ with m distinct positive singular values, where $\mathfrak{A}$ denotes a *real finite-dimensional normed division algebra*. Let $\mathfrak{A}^{m \times n}$ be the set of all $m \times n$ matrices over $\mathfrak{A}$. The dimension of $\mathfrak{A}^{m \times n}$ over $\Re$ is βmn. Let $\mathbf{A} \in \mathfrak{A}^{m \times n}$, then $\mathbf{A}^* = \bar{\mathbf{A}}^T$ denotes the usual conjugate transpose.

We denote by $\mathfrak{S}^{\beta}_m$ the real vector space of all $\mathbf{S} \in \mathfrak{A}^{m \times m}$ such that $\mathbf{S} = \mathbf{S}^*$. In addition, let $\mathfrak{P}^{\beta}_m$ be the *cone of positive definite matrices* $\mathbf{S} \in \mathfrak{A}^{m \times m}$. Thus, $\mathfrak{P}^{\beta}_m$ consist of all matrices $\mathbf{S} = \mathbf{X}^*\mathbf{X}$, with $\mathbf{X} \in \mathfrak{L}^{\beta}_{m,n}$; then $\mathfrak{P}^{\beta}_m$ is an open subset of $\mathfrak{S}^{\beta}_m$.

Let $\mathfrak{D}^{\beta}_m$ consisting of all $\mathbf{D} \in \mathfrak{A}^{m \times m}$, $\mathbf{D} = \operatorname{diag}(d_1, \dots, d_m)$. Let $\mathfrak{T}^{\beta}_U(m)$ be the subgroup of all *upper triangular* matrices $\mathbf{T} \in \mathfrak{A}^{m \times m}$ such that $t_{ij} = 0$ for $1 < i < j \leq m$.

For any matrix $\mathbf{X} \in \mathfrak{A}^{n\times m}$, $d\mathbf{X}$ denotes the *matrix of differentials* (dx_{ij}). Finally, we define the *measure* or volume element $(d\mathbf{X})$ when $\mathbf{X} \in \mathfrak{A}^{m\times n}, \mathfrak{S}_m^\beta$, and $\mathfrak{D}_m^\beta$, see Dumitriu (2002) and Díaz-García and Gutiérrez-Jáimez (2011).

If $\mathbf{X} \in \mathfrak{A}^{m\times n}$ then $(d\mathbf{X})$ (the Lebesgue measure in $\mathfrak{A}^{m\times n}$) denotes the exterior product of the βmn functionally independent variables

$$(d\mathbf{X}) = \bigwedge_{i=1}^{m}\bigwedge_{j=1}^{n} dx_{ij} \quad \text{where} \quad dx_{ij} = \bigwedge_{k=1}^{\beta} dx_{ij}^{(k)}.$$

If $\mathbf{S} \in \mathfrak{S}_m^\beta$ (or $\mathbf{S} \in \mathfrak{T}_U^\beta(m)$ with $t_{ii} > 0$, $i = 1,\dots,m$) then $(d\mathbf{S})$ (the Lebesgue measure in $\mathfrak{S}_m^\beta$ or in $\mathfrak{T}_U^\beta(m)$) denotes the exterior product of the exterior product of the $m(m-1)\beta/2+m$ functionally independent variables,

$$(d\mathbf{S}) = \bigwedge_{i=1}^{m} ds_{ii} \bigwedge_{i<j}^{m}\bigwedge_{k=1}^{\beta} ds_{ij}^{(k)}.$$

Observe, that for the Lebesgue measure $(d\mathbf{S})$ defined thus, it is required that $\mathbf{S} \in \mathfrak{P}_m^\beta$, that is, $\mathbf{S}$ must be a non singular Hermitian matrix (Hermitian definite positive matrix).

If $\mathbf{\Lambda} \in \mathfrak{D}_m^\beta$ then $(d\mathbf{\Lambda})$ (the Legesgue measure in $\mathfrak{D}_m^\beta$) denotes the exterior product of the βm functionally independent variables

$$(d\mathbf{\Lambda}) = \bigwedge_{i=1}^{n}\bigwedge_{k=1}^{\beta} d\lambda_i^{(k)}.$$

Now, we show three Jacobians in terms of the β parameter, which are proposed as extensions of real, complex or quaternion cases, see Díaz-García and Gutiérrez-Jáimez (2011).

Lemma 2.1 *Let* $\mathbf{X}$ *and* $\mathbf{Y} \in \mathfrak{P}_m^\beta$ *matrices of functionally independent variables, and let* $\mathbf{Y} = \mathbf{A}\mathbf{X}\mathbf{A}^* + \mathbf{C}$*, where* $\mathbf{A} \in \mathcal{L}_{m,m}^\beta$ *and* $\mathbf{C} \in \mathfrak{P}_m^\beta$ *are matrices of constants. Then*

$$(d\mathbf{Y}) = |\mathbf{A}^*\mathbf{A}|^{(m-1)\beta/2+1}(d\mathbf{X}), \tag{1}$$

where $|\mathbf{B}|$ *denotes the determinant of* $\mathbf{B}$.

Lemma 2.2 (Cholesky's decomposition) *Let* $\mathbf{S} \in \mathfrak{P}_m^\beta$ *and* $\mathbf{T} \in \mathfrak{T}_U^\beta(m)$ *with* $t_{ii} > 0$, $i = 1, 2, \dots, m$. *Then*

$$(d\mathbf{S}) = \begin{cases} 2^m \prod_{i=1}^{m} t_{ii}^{(m-i)\beta+1}(d\mathbf{T}) & \text{if } \mathbf{S} = \mathbf{T}^*\mathbf{T}; \\ 2^m \prod_{i=1}^{m} t_{ii}^{(i-1)\beta+1}(d\mathbf{T}) & \text{if } \mathbf{S} = \mathbf{T}\mathbf{T}^*. \end{cases} \tag{2}$$

Lemma 2.3 *Let* $\mathbf{S} \in \mathfrak{P}_m^\beta$. *Then, ignoring the sign, if* $\mathbf{Y} = \mathbf{S}^{-1} + \mathbf{C}$, $\mathbf{C} \in \mathfrak{P}_m^\beta$ *is a matrix of constants,*

$$(d\mathbf{Y}) = |\mathbf{S}|^{-\beta(m-1)-2}(d\mathbf{S}). \tag{3}$$

Next is stated a general result, that is useful in a variety of situations, which enable us to transform the density function of a matrix $\mathbf{X} \in \mathfrak{P}_m^\beta$ to the density function of its eigenvalues, see Díaz-García (2014).

Lemma 2.4 *Let $\mathbf{X} \in \mathfrak{P}_m^\beta$ be a random matrix with a density function $f_{\mathbf{X}}(\mathbf{X})$. Then the joint density function of the eigenvalues $\lambda_1, \dots, \lambda_m$ of $\mathbf{X}$ is*

$$\frac{\pi^{m^2\beta/2+\varrho}}{\Gamma_m^\beta[m\beta/2]} \prod_{i<j}^m (\lambda_i - \lambda_j)^\beta \int_{\mathbf{H}\in\mathfrak{U}^\beta(m)} f(\mathbf{HLH}^*)(d\mathbf{H}) \tag{4}$$

where $\mathbf{L} = \operatorname{diag}(\lambda_1, \dots, \lambda_m)$, $\lambda_1 > \cdots > \lambda_m > 0$, $(d\mathbf{H})$ is the normalised Haar measure, $\Gamma_m^\beta[a]$ denotes the Gamma function for the space $\mathfrak{S}_m^\beta$ (Gross and Richards 1987) and

$$\varrho = \begin{cases} 0, & \beta = 1; \\ -m, & \beta = 2; \\ -2m, & \beta = 4; \\ -4m, & \beta = 8. \end{cases}$$

Finally, let's recall the multidimensional convolution theorem in terms of the Laplace transform. For this purpose, let's use the complexification $\mathfrak{S}_m^{\beta,\mathfrak{C}} = \mathfrak{S}_m^\beta + i\mathfrak{S}_m^\beta$ of $\mathfrak{S}_m^\beta$. That is, $\mathfrak{S}_m^{\beta,\mathfrak{C}}$ consist of all matrices $\mathbf{Z} \in (\mathfrak{F}^\mathfrak{C})^{m\times m}$ of the form $\mathbf{Z} = \mathbf{X} + i\mathbf{Y}$, with $\mathbf{X}, \mathbf{Y} \in \mathfrak{S}_m^\beta$. It comes to $\mathbf{X} = \operatorname{Re}(\mathbf{Z})$ and $\mathbf{Y} = \operatorname{Im}(\mathbf{Z})$ as the *real and imaginary parts* of $\mathbf{Z}$, respectively. The *generalised right half-plane* $\mathbf{\Phi}_m^\beta = \mathfrak{P}_m^\beta + i\mathfrak{S}_m^\beta$ in $\mathfrak{S}_m^{\beta,\mathfrak{C}}$ consists of all $\mathbf{Z} \in \mathfrak{S}_m^{\beta,\mathfrak{C}}$ such that $\operatorname{Re}(\mathbf{Z}) \in \mathfrak{P}_m^\beta$, see (Gross and Richards 1987, p. 801).

Definition 2.1 *If $f(\mathbf{X})$ is a function of $\mathbf{X} \in \mathfrak{P}_m^\beta$, the Laplace transform of $f(\mathbf{X})$ is defined to be*

$$g(\mathbf{T}) = \int_{\mathbf{X}\in\mathfrak{P}_m^\beta} \operatorname{etr}\{-\mathbf{XZ}\} f(\mathbf{X})(d\mathbf{X}). \tag{5}$$

where $\mathbf{Z} \in \mathbf{\Phi}_m^\beta$ and $\operatorname{etr}(\cdot) = \exp(\operatorname{tr}(\cdot))$.

Lemma 2.5 *If $g_1(\mathbf{Z})$ and $g_2(\mathbf{Z})$ are the respective Laplace transforms of the densities $f_{\mathbf{X}}(\mathbf{X})$ and $g_{\mathbf{Y}}(\mathbf{Y})$ then the product $g_1(\mathbf{Z})g_2(\mathbf{Z})$ is the Laplace transform of the convolution $f_{\mathbf{X}}(\mathbf{X}) * g_{\mathbf{Y}}(\mathbf{Y})$, where*

$$h_{\mathbf{\Xi}}(\mathbf{\Xi}) = f_{\mathbf{X}}(\mathbf{X}) * g_{\mathbf{Y}}(\mathbf{Y}) = \int_{\mathbf{0}<\mathbf{\Upsilon}<\mathbf{\Xi}} f_{\mathbf{X}}(\mathbf{\Upsilon}) g_{\mathbf{Y}}(\mathbf{\Xi} - \mathbf{\Upsilon})(d\mathbf{\Upsilon}), \tag{6}$$

with $\mathbf{\Xi} = \mathbf{X} + \mathbf{Y}$ and $\mathbf{\Upsilon} = \mathbf{X}$.

3. Riesz distributions

This section shows two versions of the Riesz distributions (Díaz-García 2015a) and the study of their Bartlett decompositions. Also, inverse Riesz distributions and the joint density of its eigenvalues are obtained.

Definition 3.1 *Let $\mathbf{\Sigma} \in \mathbf{\Phi}_m^\beta$ and $\kappa = (k_1, k_2, \dots, k_m) \in \Re^m$.*

1. *Then it is said that $\mathbf{X}$ has a Riesz distribution of type I if its density function is*

$$\frac{\beta^{am+\sum_{i=1}^m k_i}}{\Gamma_m^\beta[a,\kappa] |\mathbf{\Sigma}|^a q_\kappa(\mathbf{\Sigma})} \operatorname{etr}\{-\beta\mathbf{\Sigma}^{-1}\mathbf{X}\} |\mathbf{X}|^{a-(m-1)\beta/2-1} q_\kappa(\mathbf{X})(d\mathbf{X}) \tag{7}$$

for $\mathbf{X} \in \mathfrak{P}_m^\beta$ and $\operatorname{Re}(a) \geq (m-1)\beta/2 - k_m$; where $\Gamma_m^\beta[a,\kappa]$ is the generalised gamma function of weight κ and $q_\kappa(\mathbf{A})$ is the highest wight vector or generalised power of $\mathbf{A}$(see Gross and Richards 1987); denoting this fact as $\mathbf{X} \sim \mathfrak{R}_m^{\beta,I}(a, \kappa, \mathbf{\Sigma})$.

2. *Then it is said that* $\mathbf{X}$ *has a Riesz distribution of type II if its density function is*

$$\frac{\beta^{am-\sum_{i=1}^m k_i}}{\Gamma_m^\beta[a,-\kappa]|\mathbf{\Sigma}|^a q_\kappa(\mathbf{\Sigma}^{-1})} \operatorname{etr}\{-\beta\mathbf{\Sigma}^{-1}\mathbf{X}\}|\mathbf{X}|^{a-(m-1)\beta/2-1} q_\kappa(\mathbf{X}^{-1})(d\mathbf{X}) \tag{8}$$

for $\mathbf{X} \in \mathfrak{P}_m^\beta$ *and* $\operatorname{Re}(a) > (m-1)\beta/2 + k_1$*; where* $\Gamma_m^\beta[a,-\kappa]$ *is the generalised gamma function of weight* κ *proposed by Khatri (1966); denoting this fact as* $\mathbf{X} \sim \mathfrak{R}_m^{\beta,II}(a,\kappa,\mathbf{\Sigma})$.

Theorem 3.1 *Let* $\mathbf{\Sigma} \in \mathbf{\Phi}_m^\beta$ *and* $\kappa = (k_1, k_2, \dots, k_m) \in \Re^m$. *And let* $\mathbf{Y} = \mathbf{X}^{-1}$.

1. *Then if* $\mathbf{X}$ *has a Riesz distribution of type I the density of* $\mathbf{Y}$ *is*

$$\frac{\beta^{am+\sum_{i=1}^m k_i}}{\Gamma_m^\beta[a,\kappa]|\mathbf{\Sigma}|^a q_\kappa(\mathbf{\Sigma})} \operatorname{etr}\{-\beta\mathbf{\Sigma}^{-1}\mathbf{Y}^{-1}\}|\mathbf{Y}|^{-(a+(m-1)\beta/2+1)} q_\kappa(\mathbf{Y}^{-1})(d\mathbf{Y}) \tag{9}$$

for $\operatorname{Re}(a) \geq (m-1)\beta/2 - k_m$ *and is termed as* inverse Riesz distribution of type I.

2. *Then if* $\mathbf{X}$ *has a Riesz distribution of type II the density of* $\mathbf{Y}$ *is*

$$\frac{\beta^{am-\sum_{i=1}^m k_i}}{\Gamma_m^\beta[a,-\kappa]|\mathbf{\Sigma}|^a q_\kappa(\mathbf{\Sigma}^{-1})} \operatorname{etr}\{-\beta\mathbf{\Sigma}^{-1}\mathbf{Y}^{-1}\}|\mathbf{Y}|^{-(a+(m-1)\beta/2+1)} q_\kappa(\mathbf{Y})(d\mathbf{Y}) \tag{10}$$

for $\operatorname{Re}(a) > (m-1)\beta/2 + k_1$ *and it said that* $\mathbf{Y}$ *has a* inverse Riesz distribution of type II.

Proof. It is immediately noted that $(d\mathbf{X}) = |\mathbf{Y}|^{-\beta(m-1)-2}(d\mathbf{Y})$ and from (7) and (8). □

Note that, the density function (9) was studied previously by Tounsi and Zine (2012) in real case.

Observe that, if $\kappa = (0, 0, \dots, 0)$ in two densities in Definition 3.1 and Theorem 3.1 the matrix multivariate gamma and inverse gamma distributions are obtained. As consequence, in this last case if $\mathbf{\Sigma} = 2\mathbf{\Sigma}$ and $a = \beta n/2$, the Wishart and inverse Wishart distributions are obtained, too.

Theorem 3.2 *Let* $\mathbf{T} \in \mathfrak{T}_U^\beta(m)$ *with* $t_{ii} > 0$, $i = 1, 2, \dots, m$ *and define* $\mathbf{X} = \mathbf{T}^*\mathbf{T}$.

1. *If* $\mathbf{X}$ *has a Riesz distribution of type I, (7), with* $\mathbf{\Sigma} = \mathbf{I}_m$*, then the elements* t_{ij} $(1 \leq i \leq j \leq m)$ *of* $\mathbf{T}$ *are all independent. Furthermore,* $t_{ii}^2 \sim \mathcal{G}^\beta(a + k_i - (i-1)\beta/2, 1)$ *and* $\sqrt{2}t_{ij} \sim \mathcal{N}_1^\beta(0,1)$ $(1 \leq i < j \leq m)$.

2. *If* $\mathbf{X}$ *has a Riesz distribution of type II, (8), with* $\mathbf{\Sigma} = \mathbf{I}_m$*, then the elements* t_{ij} $(1 \leq i \leq j \leq m)$ *of* $\mathbf{T}$ *are all independent. Moreover,* $t_{ii}^2 \sim \mathcal{G}^\beta(a - k_i - (i-1)\beta/2, 1)$ *and* $\sqrt{2}t_{ij} \sim \mathcal{N}_1^\beta(0,1)$ $(1 \leq i < j \leq m)$.

Where $x \sim \mathcal{G}^\beta(a,\alpha)$ *denotes a gamma distribution with parameters* a *and* α *and* $y \sim \mathcal{N}_1^\beta(0,1)$ *denotes a random variable with standard normal distribution for real normed division algebras. Moreover, their respective densities are*

$$\mathcal{G}^\beta(x : a, \alpha) = \frac{1}{(\alpha/\beta)^a \Gamma[a]} \exp\{-\beta x/\alpha\} x^{a-1}(dx),$$

and

$$\mathcal{N}_1^\beta(y : 0, 1) = \frac{1}{(2\pi/\beta)^{\beta/2}} \exp\{-\beta y^2/2\}(dy)$$

where $x \in \mathfrak{L}_{1,1}^\beta$, $y \in \mathfrak{P}_1^\beta$, *Re*$(a) > 0$ *and* $\alpha \in \mathbf{\Phi}_1^\beta$*, see Díaz-García and Gutiérrez-Jáimez (2011).*

Proof. This is given for the case of Riesz distribution type I. The proof for Riesz distribution type II is the same thing. The density of $\mathbf{X}$ is

$$\frac{\beta^{am+\sum_{i=1}^{m} k_i}}{\Gamma_m^\beta[a,\kappa]} \operatorname{etr}\{-\beta\mathbf{X}\}|\mathbf{X}|^{a-(m-1)\beta/2-1} q_\kappa(\mathbf{X})(d\mathbf{X}). \tag{11}$$

Since $\mathbf{X} = \mathbf{T}^*\mathbf{T}$ we have

$$\begin{aligned}
\operatorname{tr}\mathbf{X} &= \operatorname{tr}\mathbf{T}^*\mathbf{T} = \sum_{i\leq j}^{m} t_{ij}^2,\\
|\mathbf{X}| &= |\mathbf{T}^*\mathbf{T}| = |\mathbf{T}|^2 = \prod_{i=1}^{m} t_{ii}^2,\\
q_\kappa(\mathbf{X}) &= q_\kappa(\mathbf{T}^*\mathbf{T}) = |\mathbf{T}^*\mathbf{T}|^{k_m}\prod_{i=1}^{m-1}|\mathbf{T}_i^*\mathbf{T}_i|^{k_i-k_{i+1}} = \prod_{i=1}^{m} t_{ii}^{2k_i},
\end{aligned}$$

and by Theorem 2.2 noting that $dt_{ii}^2 = 2t_{ii}dt_{ii}$, then

$$\begin{aligned}
(d\mathbf{X}) &= 2^m \prod_{i=1}^{m} t_{ii}^{\beta(m-i)+1}\left(\bigwedge_{i\leq j} dt_{ij}\right),\\
&= \prod_{i=1}^{m} \left(t_{ii}^2\right)^{\beta(m-i)/2}\left(\bigwedge_{i=1} dt_{ii}^2\right)\wedge\left(\bigwedge_{i<j} dt_{ij}\right).
\end{aligned}$$

Substituting this expression in (11) we find that the joint density of the t_{ij} $(1 \leq i \leq j \leq m)$ can be written as

$$\prod_{i=1}^{m} \frac{\beta^{a+k_i-(i-1)\beta/2}}{\Gamma[a+k_i-(i-1)\beta/2]}\exp\{-\beta t_{ii}^2\}\left(t_{ii}^2\right)^{a+k_i-(i-1)\beta/2-1}\left(dt_{ii}^2\right)$$

$$\times \prod_{i<j}^{m} \frac{1}{(\pi/\beta)^{\beta/2}}\exp\{-\beta t_{ij}^2\}(dt_{ij}),$$

only observe that

$$\begin{aligned}
\frac{\beta^{am+\sum_{i=1}^{m} k_i}}{\Gamma_m^\beta[a,\kappa]} &= \frac{\beta^{am+\sum_{i=1}^{m} k_i - m(m-1)\beta/4}}{\beta^{-m(m-1)\beta/4}\pi^{m(m-1)\beta/4}\prod_{i=1}^{m}\Gamma[a+k_i-(i-1)\beta/2]}\\
&= \prod_{i=1}^{m} \frac{\beta^{a+k_i-(i-1)\beta/2}}{\Gamma[a+k_i-(i-1)\beta/2]}\prod_{i<j}^{m}\frac{1}{(\pi/\beta)^{\beta/2}}.
\end{aligned}$$

□

In analogy to generalised variance for Wishart case, the following result gives the distribution of $|\mathbf{X}|$ when $\mathbf{X}$ has a Riesz distribution type I or type II.

Theorem 3.3 *Let $v = |\mathbf{X}|/|\mathbf{\Sigma}|$. Then*

1. *if $\mathbf{X}$ has a Riesz distribution of type I, (7), the density of v is*

$$\prod_{i=1}^{m} \mathcal{G}^\beta\left(t_{ii}^2 : a+k_i-(i-1)\beta/2, 1\right).$$

2. *if $\mathbf{X}$ has a Riesz distribution of type II, (8), the density of v is*

$$\prod_{i=1}^{m} \mathcal{G}^\beta\left(t_{ii}^2 : a-k_i-(i-1)\beta/2, 1\right).$$

where t_{ii}^2, $i = 1, \dots, m$, are independent random variables.

Proof. This is immediately from Theorem 3.2, noting that if

$$\mathbf{B} = \mathbf{\Sigma}^{-1/2}\mathbf{X}\mathbf{\Sigma}^{-1/2} = \mathbf{T}^*\mathbf{T},$$

with $\mathbf{T} \in \mathfrak{T}_U^\beta(m)$ and $t_{ii} > 0$, $i = 1, 2, \dots, m$, then

$$|\mathbf{B}| = \prod_{i=1}^m t_{ii}^2 = |\mathbf{X}|/|\mathbf{\Sigma}| = v.$$

□

Theorem 3.4 *Let $\mathbf{\Sigma} = \mathbf{I}_m$ and $\kappa = (k_1, k_2, \dots, k_m)$, $k_1 \geq k_2 \geq \cdots \geq k_m \geq 0$, $k_1, k_2, \dots, k_m$ are nonnegative integers.*

1. *Let $\lambda_1, \dots, \lambda_m$, $\lambda_1 > \cdots > \lambda_m > 0$ be the eigenvalues of $\mathbf{X}$. Then if $\mathbf{X}$ has a Riesz distribution of type I, the joint density of $\lambda_1, \dots, \lambda_m$ is*

$$\frac{\beta^{am+\sum_{i=1}^m k_i}\ \pi^{m^2\beta/2+\varrho}}{\Gamma_m^\beta[m\beta/2]\Gamma_m^\beta[a,\kappa]} \prod_{i<j}^m (\lambda_i - \lambda_j)^\beta \exp\left\{-\beta \sum_{i=1}^m \lambda_i\right\}$$
$$\times \prod_{i=1}^m \lambda_i^{a-(m-1)\beta/2-1} \frac{C_\kappa^\beta(\mathbf{L})}{C_\kappa^\beta(\mathbf{I}_m)}. \qquad (12)$$

where $\mathbf{L} = \operatorname{diag}(\lambda_1, \dots, \lambda_m)$ and $\operatorname{Re}(a) \geq (m-1)\beta/2 - k_m$.

2. *Let $\delta_1, \dots, \delta_m$, $\delta_1 > \cdots > \delta_m > 0$ be the eigenvalues of $\mathbf{X}$. Then if $\mathbf{X}$ has a Riesz distribution of type II, the joint density of their eigenvalues is*

$$\frac{\beta^{am-\sum_{i=1}^m k_i}\ \pi^{m^2\beta/2+\varrho}}{\Gamma_m^\beta[m\beta/2]\Gamma_m^\beta[a,\kappa]} \prod_{i<j}^m (\delta_i - \delta_j)^\beta \exp\left\{-\beta \sum_{i=1}^m \delta_i\right\}$$
$$\times \prod_{i=1}^m \delta_i^{a-(m-1)\beta/2-1} \frac{C_\kappa^\beta(\mathbf{D}^{-1})}{C_\kappa^\beta(\mathbf{I}_m)}). \qquad (13)$$

where $\mathbf{D} = \operatorname{diag}(\delta_1, \dots, \delta_m)$, $\operatorname{Re}(a) > (m-1)\beta/2 + k_1$.

Where ϱ is defined in Lemma 3 and $C_\kappa^\beta(\cdot)$ denotes the zonal spherical functions or spherical polynomials, see Gross and Richards (1987) and Faraut and Korányi (1994, Chapter XI, Section 3).

Proof. 1. From Lemma 3

$$\frac{\beta^{am+\sum_{i=1}^m k_i}\ \pi^{m^2\beta/2+\varrho}}{\Gamma_m^\beta[m\beta/2]\Gamma_m^\beta[a,\kappa]|\mathbf{\Sigma}|^a q_\kappa(\mathbf{\Sigma})} \prod_{i<j}^m (\lambda_i - \lambda_j)^\beta$$

$$\int_{\mathbf{H}\in\mathfrak{U}^\beta(m)} \operatorname{etr}\{-\beta\mathbf{HLH}^*\}|\mathbf{HLH}^*|^{a-(m-1)\beta/2-1} q_\kappa(\mathbf{HLH}^*)(d\mathbf{H}).$$

Therefore,

$$\frac{\beta^{am+\sum_{i=1}^m k_i}\ \pi^{m^2\beta/2+\varrho}}{\Gamma_m^\beta[m\beta/2]\Gamma_m^\beta[a,\kappa]|\mathbf{\Sigma}|^a q_\kappa(\mathbf{\Sigma})} \prod_{i<j}^m (\lambda_i - \lambda_j)^\beta \prod_{i=1}^m \lambda_i^{a-(m-1)\beta/2-1} \exp\left\{-\beta \sum_{i=1}^m \lambda_i\right\}$$

$$\int_{\mathbf{H}\in\mathfrak{U}^\beta(m)} q_\kappa(\mathbf{HLH}^*)(d\mathbf{H}),$$

the result is follow from (Gross and Richards 1987, Equation 4.8(2) and Definition 5.3) and Faraut and Korányi (1994, Chapter XI, Section 3).

2. Is proved similarly. □

4. Generalised beta distributions: Beta-Riesz distributions

This section defines several versions for the beta functions and their relation with the gamma functions type I and II. In these terms, the beta-Riesz distributions type I and II are defined. Finally, diverse properties are studied.

4.1. Generalised c-beta function

A generalised of *multivariate beta function* for the cone $\mathfrak{P}_m^\beta$, denoted as $\mathcal{B}_m^\beta[a,\kappa;b,\tau]$, can be defined as

$$\int_{\mathbf{0}<\mathbf{S}<\mathbf{I}_m} |\mathbf{S}|^{a-(m-1)\beta/2-1} q_\kappa(\mathbf{S}) |\mathbf{I}_m-\mathbf{S}|^{b-(m-1)\beta/2-1} q_\tau(\mathbf{I}_m-\mathbf{S})(d\mathbf{S}) \tag{14}$$

where $\kappa=(k_1,k_2,\dots,k_m)\in\Re^m$, $\tau=(t_1,t_2,\dots,t_m)\in\Re^m$, $\mathrm{Re}(a)>(m-1)\beta/2-k_m$ and $\mathrm{Re}(b)>(m-1)\beta/2-t_m$. This is defined by Faraut and Korányi (1994, p. 130) for Euclidean simple Jordan algebras. In the context of multivariate analysis, this generalised beta function can be termed *generalised c-beta function type I*, as analogy to the correspondence case of matrix multivariate beta distribution, and using the term c-beta as abbreviation of classical-beta. In the next theorem we introduce the *generalised c-beta function type II* and its relation with the generalised gamma function.

Theorem 4.1 *The generalised c-beta function type I can be expressed as*

$$\int_{\mathbf{R}\in\mathfrak{P}_m^\beta} |\mathbf{R}|^{a-(m-1)\beta/2-1} q_\kappa(\mathbf{R}) |\mathbf{I}_m+\mathbf{R}|^{-(a+b)} q_{-(\kappa+\tau)}(\mathbf{I}_m+\mathbf{R})(d\mathbf{R})$$
$$= \frac{\Gamma_m^\beta[a,\kappa]\Gamma_m^\beta[b,\tau]}{\Gamma_m^\beta[a+b,\kappa+\tau]},$$

where $\kappa=(k_1,k_2,\dots,k_m)\in\Re^m$, $\tau=(t_1,t_2,\dots,t_m)\in\Re^m$, *Re*$(a)>(m-1)\beta/2-k_m$ *and Re*$(b)>(m-1)\beta/2-t_m$. *The integral expression is termed* generalised c-beta function type II.

Proof. Let $\mathcal{U}(\mathbf{I}_m-\mathbf{S})^*\mathcal{U}(\mathbf{I}_m-\mathbf{S})=(\mathbf{I}_m-\mathbf{S})$ the Cholesky decomposition of $(\mathbf{I}_m-\mathbf{S})$ where $\mathcal{U}(\mathbf{I}_m-\mathbf{S})\in\mathfrak{T}_U^\beta(m)$ and define $\mathbf{R}=\mathcal{U}(\mathbf{I}_m-\mathbf{S})^{*-1}\mathbf{S}\,\mathcal{U}(\mathbf{I}_m-\mathbf{S})^{-1}$ then

$$\begin{aligned}\mathbf{R} &= \mathcal{U}(\mathbf{I}_m-\mathbf{S})^{*-1}(\mathbf{I}_m-(\mathbf{I}_m-\mathbf{S}))\,\mathcal{U}(\mathbf{I}_m-\mathbf{S})^{-1}\\ &= \mathcal{U}(\mathbf{I}_m-\mathbf{S})^{*-1}\mathcal{U}(\mathbf{I}_m-\mathbf{S})^{-1}-\mathbf{I}_m\end{aligned}$$

Thus $(\mathbf{I}_m+\mathbf{R})=\mathcal{U}(\mathbf{I}_m-\mathbf{S})^{*-1}\mathcal{U}(\mathbf{I}_m-\mathbf{S})^{-1}$. By Lemma 2.3

$$\begin{aligned}(d\mathbf{R}) &= |\mathcal{U}(\mathbf{I}_m-\mathbf{S})^{*-1}\mathcal{U}(\mathbf{I}_m-\mathbf{S})^{-1}|^{(m-1)\beta+2}(d\mathbf{S})\\ &= |\mathbf{I}_m+\mathbf{R}|^{(m-1)\beta+2}(d\mathbf{S}),\end{aligned}$$

therefore $(d\mathbf{S})=(\mathbf{I}+\mathbf{R})^{-(m-1)\beta-2}(d\mathbf{R})$. Now remember that $q_\kappa(\mathbf{T}^{*-1}\mathbf{A}\mathbf{T}^{-1})=q_\kappa(\mathbf{A})q_{-\kappa}(\mathbf{B})=q_\kappa(\mathbf{A})q_\kappa^{-1}(\mathbf{B})$ for $\mathbf{B}=\mathbf{T}^*\mathbf{T}$, we have

$$|\mathbf{R}|^{a-(m-1)\beta/2-1}q_\kappa(\mathbf{R})=|\mathbf{I}_m-\mathbf{S}|^{-a+(m-1)\beta/2+1}|\mathbf{S}|^{a-(m-1)\beta/2-1}q_\kappa(\mathbf{S})q_{-\kappa}(\mathbf{I}_m-\mathbf{S})$$

and
$$|\mathbf{I}_m + \mathbf{R}|^{b-(m-1)\beta/2-1} q_\tau(\mathbf{I} + \mathbf{R}) = |\mathbf{I}_m - \mathbf{S}|^{-b+(m-1)\beta/2+1} q_{-\kappa}(\mathbf{I}_m - \mathbf{S}),$$
then
$$|\mathbf{I}_m + \mathbf{R}|^{-b+(m-1)\beta/2+1} q_{-\tau}(\mathbf{I} + \mathbf{R}) = |\mathbf{I}_m - \mathbf{S}|^{b-(m-1)\beta/2-1} q_\kappa(\mathbf{I}_m - \mathbf{S}).$$
from where the desired result is obtained.

For the expression in terms of generalised gamma function, let $\mathbf{B} = \mathcal{U}(\mathbf{\Xi})^* \mathbf{S} \mathcal{U}(\mathbf{\Xi})$ in (14), such that $\mathbf{\Xi} = \mathcal{U}(\mathbf{\Xi})^* \mathcal{U}(\mathbf{\Xi})$. Then $(d\mathbf{S}) = |\mathbf{\Xi}|^{-(m-1)\beta/2-1}(d\mathbf{B})$, and
$$\mathcal{B}_m^\beta[a, \kappa; b, \tau] |\mathbf{\Xi}|^{a+b-(m-1)\beta/2-1} q_{\kappa+\tau}(\mathbf{\Xi})$$
$$= \int_{\mathbf{0}}^{\mathbf{\Xi}} |\mathbf{B}|^{a-(m-1)\beta/2-1} q_\kappa(\mathbf{B}) |\mathbf{\Xi} - \mathbf{B}|^{b-(m-1)\beta/2-1} q_\tau(\mathbf{\Xi} - \mathbf{B})(d\mathbf{B}).$$
Taking Laplace transform of both size, by (7), the left size is
$$\int_{\mathbf{\Xi} \in \mathfrak{P}_m^\beta} \mathcal{B}_m^\beta[a, \kappa; b, \tau] \operatorname{etr}\{-\mathbf{\Xi}\mathbf{Z}\} |\mathbf{\Xi}|^{a+b-(m-1)\beta/2-1} q_{\kappa+\tau}(\mathbf{\Xi})(d\mathbf{\Xi})$$
$$= \mathcal{B}_m^\beta[a, \kappa; b, \tau] \Gamma_m^\beta[a + b; \kappa + \tau] |\mathbf{Z}|^{-(a+b)} q_{\kappa+\tau}(\mathbf{Z}^{-1}),$$
and applying Lemma 2.5, $g_1(\mathbf{Z})$ is
$$\int_{\mathbf{\Xi} \in \mathfrak{P}_m^\beta} \operatorname{etr}\{-\mathbf{\Xi}\mathbf{Z}\} |\mathbf{\Xi}|^{a-(m-1)\beta/2-1} q_\kappa(\mathbf{\Xi})(d\mathbf{\Xi}) = \Gamma_m^\beta[a; \kappa] |\mathbf{Z}|^{-a} q_\kappa(\mathbf{Z}^{-1}),$$
and $g_2(\mathbf{Z})$ is given by
$$\int_{\mathbf{B} \in \mathfrak{P}_m^\beta} \operatorname{etr}\{-\mathbf{B}\mathbf{Z}\} |\mathbf{B}|^{b-(m-1)\beta/2-1} q_\kappa(\mathbf{B})(d\mathbf{B}) = \Gamma_m^\beta[b; \tau] |\mathbf{Z}|^{-b} q_\tau(\mathbf{Z}^{-1}).$$
Thus, equally
$$\mathcal{B}_m^\beta[a, \kappa; b, \tau] = \frac{\Gamma_m^\beta[a, \kappa] \Gamma_m^\beta[b, \tau]}{\Gamma_m^\beta[a + b, \kappa + \tau]}.$$
□

4.2. Generalised k-beta function

Alternatively, a generalised of *multivariate beta function* for the cone $\mathfrak{P}_m^\beta$, can be defined and denoted as
$$\mathcal{B}_m^\beta[a, -\kappa; b, -\tau] = \int_{\mathbf{0}<\mathbf{S}<\mathbf{I}_m} |\mathbf{S}|^{a-(m-1)\beta/2-1} q_\kappa\left(\mathbf{S}^{-1}\right) |\mathbf{I}_m - \mathbf{S}|^{b-(m-1)\beta/2-1} q_\tau\left((\mathbf{I}_m - \mathbf{S})^{-1}\right)(d\mathbf{S}) \tag{15}$$
where $\kappa = (k_1, k_2, \dots, k_m) \in \Re^m$, $\tau = (t_1, t_2, \dots, t_m) \in \Re^m$, $\operatorname{Re}(a) > (m-1)\beta/2 + k_1$ and $\operatorname{Re}(b) > (m-1)\beta/2 + t_1$. Again, in the context of multivariate analysis, this generalised k-beta function can be termed *generalised k-beta function type I*, as an analogy to the corresponding case of matrix multivariate beta distribution and using the term k-beta as abbreviation of Khatri-beta. Next theorem introduces the *generalised k-beta function type II* and its relation with the generalised gamma function proposed by Khatri (1966).

Theorem 4.2 *The generalised k-beta function type II can be expressed as*
$$\int_{\mathbf{R} \in \mathfrak{P}_m^\beta} |\mathbf{R}|^{a-(m-1)\beta/2-1} q_\kappa(\mathbf{R}^{-1}) |\mathbf{I}_m + \mathbf{R}|^{-(a+b)} q_{-(\kappa+\tau)}\left((\mathbf{I}_m + \mathbf{R})^{-1}\right)(d\mathbf{R})$$

$$= \frac{\Gamma_m^\beta[a, -\kappa]\Gamma_m^\beta[b, -\tau]}{\Gamma_m^\beta[a+b, -\kappa-\tau]},$$

where $\kappa = (k_1, k_2, \dots, k_m) \in \Re^m$, $\tau = (t_1, t_2, \dots, t_m) \in \Re^m$, $Re(a) > (m-1)\beta/2 + k_1$ *and* $Re(b) > (m-1)\beta/2 + t_1$. *The integral expression is termed* generalised k-beta function type II.

Proof. The proof is analogous to the given for Theorem 4.1. □

Observe that if $\kappa = (0, \dots, 0) \in \Re^m$ and $\tau = (0, \dots, 0) \in \Re^m$ in (14), Theorem 4.1, (15) and Theorem 4.2 the classical beta function is obtained, see Herz (1955).

4.3. c-beta-Riesz and k-beta-Riesz distributions

As an immediate consequence of the results of the previous section, next the c-beta-Riesz and k-beta-Riesz distributions types I and II are defined.

Definition 4.1 *Let* $\kappa = (k_1, k_2, \dots, k_m) \in \Re^m$ *and* $\tau = (t_1, t_2, \dots, t_m) \in \Re^m$.

1. *Then it said that* $\mathbf{S}$ *has a* c-beta-Riesz distribution of type I *if its density function is*

$$\frac{1}{\mathcal{B}_m^\beta[a, \kappa; b, \tau]} |\mathbf{S}|^{a-(m-1)\beta/2-1} q_\kappa(\mathbf{S}) |\mathbf{I}_m - \mathbf{S}|^{b-(m-1)\beta/2-1} q_\tau(\mathbf{I}_m - \mathbf{S})(d\mathbf{S}), \qquad (16)$$

where $\mathbf{0} < \mathbf{S} < \mathbf{I}_m$ *and* $Re(a) > (m-1)\beta/2 - k_m$ *and* $Re(b) > (m-1)\beta/2 - t_m$.

2. *Then it said that* $\mathbf{R}$ *has a* c-beta-Riesz distribution of type II *if its density function is*

$$\frac{1}{\mathcal{B}_m^\beta[a, \kappa; b, \tau]} |\mathbf{R}|^{a-(m-1)\beta/2-1} q_\kappa(\mathbf{R}) |\mathbf{I}_m + \mathbf{R}|^{-(a+b)} q_{-(\kappa+\tau)}(\mathbf{I}_m + \mathbf{R})(d\mathbf{R}), \qquad (17)$$

where $\mathbf{R} \in \mathfrak{P}_m^\beta$ *and* $Re(a) > (m-1)\beta/2 - k_m$ *and* $Re(b) > (m-1)\beta/2 - t_m$.

Similarly we have

Definition 4.2 *Let* $\kappa = (k_1, k_2, \dots, k_m) \in \Re^m$ *and* $\tau = (t_1, t_2, \dots, t_m) \in \Re^m$.

1. *Then it said that* $\mathbf{S}$ *has a* k-beta-Riesz distribution of type I *if its density function is*

$$\frac{1}{\mathcal{B}_m^\beta[a, -\kappa; b, -\tau]} |\mathbf{S}|^{a-(m-1)\beta/2-1} q_\kappa(\mathbf{S}^{-1}) |\mathbf{I}_m - \mathbf{S}|^{b-(m-1)\beta/2-1} q_\tau\left((\mathbf{I}_m - \mathbf{S})^{-1}\right)(d\mathbf{S}), \qquad (18)$$

where $\mathbf{0} < \mathbf{S} < \mathbf{I}_m$ *and* $Re(a) > (m-1)\beta/2 + k_1$ *and* $Re(b) > (m-1)\beta/2 + t_1$.

2. *Then it said that* $\mathbf{R}$ *has a* k-beta-Riesz distribution of type II *if its density function is*

$$\frac{1}{\mathcal{B}_m^\beta[a, -\kappa; b, -\tau]} |\mathbf{R}|^{a-(m-1)\beta/2-1} q_\kappa(\mathbf{R}^{-1}) |\mathbf{I}_m + \mathbf{R}|^{-(a+b)} q_{-(\kappa+\tau)}\left((\mathbf{I}_m + \mathbf{R})^{-1}\right)(d\mathbf{R}), \qquad (19)$$

where $\mathbf{R} \in \mathfrak{P}_m^\beta$ *and* $Re(a) > (m-1)\beta/2 + k_1$ *and* $Re(b) > (m-1)\beta/2 + t_1$.

Observe that the relationship between the densities (16) and (17), and between the densities (18) and (19) are easily obtained from the theorems 4.1 and 4.2, respectively.

The following result state the relation between the Riesz and beta-Riesz distributions.

Theorem 4.3 *Let* $\mathbf{X}_1$ *and* $\mathbf{X}_2$ *be independently distributed as Riesz distribution type I, such that* $\mathbf{X}_1 \sim \mathfrak{R}_m^{\beta,I}(a,\kappa,\mathbf{\Sigma})$ *and* $\mathbf{X}_2 \sim \mathfrak{R}_m^{\beta,I}(b,\tau,\mathbf{\Sigma})$*,* $Re(a) > (m-1)\beta/2 + k_1$ *and* $Re(b) > (m-1)\beta/2 + t_1$*. Let*

$$\mathbf{S} = \mathcal{U}(\mathbf{X}_1+\mathbf{X}_2)^{*-1}\mathbf{X}_1\mathcal{U}(\mathbf{X}_1+\mathbf{X}_2)^{-1},$$

where $\mathcal{U}(\mathbf{X}_1+\mathbf{X}_2) \in \mathfrak{T}_U^\beta(m)$ *is such that* $(\mathbf{X}_1+\mathbf{X}_2) = \mathcal{U}(\mathbf{X}_1+\mathbf{X}_2)^*\mathcal{U}(\mathbf{X}_1+\mathbf{X}_2)$*. Then* $\mathbf{S}$ *and* $(\mathbf{X}_1+\mathbf{X}_2)$ *are independent,* $\mathbf{S}$ *has a c-beta-Riesz distribution type I and* $(\mathbf{X}_1+\mathbf{X}_2) \sim \mathfrak{R}_m^{\beta,I}(a+b,\kappa+\tau,\mathbf{I}_m)$*.*

Proof. The joint density of $\mathbf{X}_1$ and $\mathbf{X}_2$ is given by

$$\frac{\beta^{(a+b)m+\sum_{i=1}^m(k_i+t_i)}}{\Gamma_m^\beta[a,\kappa]\Gamma_m^\beta[b,\tau]|\mathbf{\Sigma}|^{a+b}q_{\kappa+\tau}(\mathbf{\Sigma})}\operatorname{etr}\{-\beta\mathbf{\Sigma}^{-1}(\mathbf{X}_1+\mathbf{X}_2)\}|\mathbf{X}_1|^{a-(m-1)\beta/2-1}q_\kappa(\mathbf{X}_1)$$

$$\times|\mathbf{X}_2|^{b-(m-1)\beta/2-1}q_\tau(\mathbf{X}_2)(d\mathbf{X}_1)\wedge(d\mathbf{X}_2).$$

Let $\mathbf{Y} = \mathbf{X}_1+\mathbf{X}_2$ and $\mathbf{Z} = \mathbf{X}_1$, then, $(d\mathbf{X}_1)\wedge(d\mathbf{X}_2) = (d\mathbf{Y})\wedge(d\mathbf{Z})$. Then the joint density of $\mathbf{Y}$ and $\mathbf{Z}$ is given by

$$\frac{\beta^{(a+b)m+\sum_{i=1}^m(k_i+t_i)}}{\Gamma_m^\beta[a,\kappa]\Gamma_m^\beta[b,\tau]|\mathbf{\Sigma}|^{a+b}q_{\kappa+\tau}(\mathbf{\Sigma})}\operatorname{etr}\{-\beta\mathbf{\Sigma}^{-1}\mathbf{Y}\}|\mathbf{Z}|^{a-(m-1)\beta/2-1}q_\kappa(\mathbf{Z})$$

$$\times|\mathbf{Y}-\mathbf{Z}|^{b-(m-1)\beta/2-1}q_\tau(\mathbf{Y}-\mathbf{Z})(d\mathbf{Y})\wedge(d\mathbf{Z}).$$

Let $\mathbf{W} = \mathcal{U}(\mathbf{Y})^*\mathcal{U}(\mathbf{Y})$, with $\mathcal{U}(\mathbf{Y}) \in \mathfrak{T}_U^\beta(m)$ and $\mathbf{Z} = \mathcal{U}(\mathbf{Y})^*\mathbf{S}\,\mathcal{U}(\mathbf{Y})$. Observing that $\mathcal{U}(\mathbf{Y})$ is a function of $\mathbf{W}$

$$(d\mathbf{Y})\wedge(d\mathbf{Z}) = |\mathcal{U}(\mathbf{Y})^*\mathcal{U}(\mathbf{Y})|^{\beta(m-1)/2+1}(d\mathcal{U}(\mathbf{Y})^*\mathcal{U}(\mathbf{Y}))\wedge(d\mathbf{S})$$

Hence the joint density of $\mathbf{S}$ and $\mathbf{W} = \mathcal{U}(\mathbf{Y})^*\mathcal{U}(\mathbf{Y})$ is

$$\frac{\beta^{(a+b)m+\sum_{i=1}^m(k_i+t_i)}}{\Gamma_m^\beta[a+b,\kappa+\tau]|\mathbf{\Sigma}|^{a+b}q_{\kappa+\tau}(\mathbf{\Sigma})}\operatorname{etr}\{-\beta\mathbf{\Sigma}^{-1}\mathcal{U}(\mathbf{Y})^*\mathcal{U}(\mathbf{Y})\}|\mathcal{U}(\mathbf{Y})^*\mathcal{U}(\mathbf{Y})|^{a+b-\beta(m-1)/2-1}$$

$$q_{\kappa+\tau}(\mathcal{U}(\mathbf{Y})^*\mathcal{U}(\mathbf{Y}))(d\mathcal{U}(\mathbf{Y})^*\mathcal{U}(\mathbf{Y}))$$

$$\times\frac{\Gamma_m^\beta[a+b,\kappa+\tau]}{\Gamma_m^\beta[a,\kappa]\Gamma_m^\beta[b,\tau]}|\mathbf{S}|^{a-(m-1)\beta/2-1}q_\kappa(\mathbf{S})|\mathbf{I}-\mathbf{S}|^{b-(m-1)\beta/2-1}q_\tau(\mathbf{I}-\mathbf{S})(d\mathbf{S}).$$

which shows that $\mathbf{W} = \mathcal{U}(\mathbf{Y})^*\mathcal{U}(\mathbf{Y}) = \mathbf{X}_1+\mathbf{X}_2 \sim \mathfrak{R}_m^{\beta,I}(a+b,\kappa+\tau,\mathbf{\Sigma})$ independently of $\mathbf{S}$ with a c-beta-Riesz distribution type I. □

Theorem 4.4 *Let* $\mathbf{X}_1$ *and* $\mathbf{X}_2$ *be independently distributed as Riesz distribution type I, such that* $\mathbf{X}_1 \sim \mathfrak{R}_m^{\beta,I}(a,\kappa,\mathbf{\Sigma})$ *and* $\mathbf{X}_2 \sim \mathfrak{R}_m^{\beta,I}(b,\tau,\mathbf{\Sigma})$*,* $Re(a) > (m-1)\beta/2 + k_1$ *and* $Re(b) > (m-1)\beta/2 + t_1$*. Let*

$$\mathbf{R} = \mathcal{U}(\mathbf{X}_2)^{*-1}\mathbf{X}_1\mathcal{U}(\mathbf{X}_2)^{-1},$$

where $\mathcal{U}(\mathbf{X}_2) \in \mathfrak{T}_U^\beta(m)$ *is such that* $\mathbf{X}_1 = \mathcal{U}(\mathbf{X}_2)^*\mathcal{U}(\mathbf{X}_2)$*. Then* $\mathbf{S}$ *has a c-beta-Riesz distribution type II.*

Proof. From Theorem 4.1 we know that if $\mathbf{S}$ has a c-beta-Riesz distribution type I then $\mathbf{R} = \mathcal{U}(\mathbf{I}_m-\mathbf{S})^{*-1}\mathbf{S}\,\mathcal{U}(\mathbf{I}_m-\mathbf{S})^{-1}$ has a c-beta-Riesz distribution type II. In addition the theorem establish that if $\mathbf{X}_1$ and $\mathbf{X}_2$ be independently distributed as Riesz distribution type I, such that $\mathbf{X}_1 \sim \mathfrak{R}_m^{\beta,I}(a,\kappa,\mathbf{\Sigma})$ and $\mathbf{X}_2 \sim \mathfrak{R}_m^{\beta,I}(b,\tau,\mathbf{\Sigma})$ then $\mathbf{R} = \mathcal{U}(\mathbf{X}_1)^{*-1}\mathbf{X}_1\mathcal{U}(\mathbf{X}_2)^{-1}$. Thus, the desired result is follow if we proof that

$$\mathbf{R} = \mathcal{U}(\mathbf{I}_m-\mathbf{S})^{*-1}\mathbf{S}\,\mathcal{U}(\mathbf{I}_m-\mathbf{S})^{-1} = \mathcal{U}(\mathbf{X}_2)^{*-1}\mathbf{X}_1\mathcal{U}(\mathbf{X}_2)^{-1}.$$

With this aim in mind, let $\mathcal{U}(\mathbf{Y}) \in \mathfrak{T}_U^\beta(m)$, such that $\mathbf{X}_1 + \mathbf{X}_2 = \mathcal{U}(\mathbf{Y})^*\mathcal{U}(\mathbf{Y})$, then $\mathbf{S} = \mathcal{U}(\mathbf{Y})^{*-1}\mathbf{X}_1\mathcal{U}(\mathbf{Y})^{-1}$. Now, if $\mathbf{X}_2 = \mathcal{U}(\mathbf{X}_2)^*\mathcal{U}(\mathbf{X}_2)$, with $\mathcal{U}(\mathbf{X}_2) \in \mathfrak{T}_U^\beta(m)$. We have

$$\begin{aligned}
\mathbf{I}_m - \mathbf{S} &= \mathbf{I}_m - \mathcal{U}(\mathbf{Y})^{*-1}\mathbf{X}_1\mathcal{U}(\mathbf{Y})^{-1} \\
&= \mathcal{U}(\mathbf{Y})^{*-1}(\mathcal{U}(\mathbf{Y})^*\mathcal{U}(\mathbf{Y}) - \mathbf{X}_1)\mathcal{U}(\mathbf{Y})^{-1} \\
&= \mathcal{U}(\mathbf{Y})^{*-1}\mathbf{X}_2\mathcal{U}(\mathbf{Y})^{-1} \\
&= \mathcal{U}(\mathbf{Y})^{*-1}\mathcal{U}(\mathbf{X}_2)^*\mathcal{U}(\mathbf{X}_2)\mathcal{U}(\mathbf{Y})^{-1} \\
&= \left(\mathcal{U}(\mathbf{X}_2)\mathcal{U}(\mathbf{Y})^{-1}\right)^*\left(\mathcal{U}(\mathbf{X}_2)\mathcal{U}(\mathbf{Y})^{-1}\right) \\
&= \mathcal{U}(\mathbf{I}_m - \mathbf{S})^*\mathcal{U}(\mathbf{I}_m - \mathbf{S}).
\end{aligned}$$

This least equally is obtained observing that $\left(\mathcal{U}(\mathbf{X}_2)\mathcal{U}(\mathbf{Y})^{-1}\right) \in \mathfrak{T}_U^\beta(m)$, then $\left(\mathcal{U}(\mathbf{X}_2)\mathcal{U}(\mathbf{Y})^{-1}\right) = \mathcal{U}(\mathbf{I}_m - \mathbf{S})$. Therefore

$$\begin{aligned}
\mathcal{U}(\mathbf{I}_m - \mathbf{S})^{*-1}\mathbf{S}\,\mathcal{U}(\mathbf{I}_m - \mathbf{S})^{-1} &= \left(\mathcal{U}(\mathbf{X}_2)\mathcal{U}(\mathbf{Y})^{-1}\right)^{*-1}\mathbf{S}\left(\mathcal{U}(\mathbf{X}_2)\mathcal{U}(\mathbf{Y})^{-1}\right)^{-1} \\
&= \mathcal{U}(\mathbf{X}_2)^{*-1}\mathcal{U}(\mathbf{Y})^*\mathbf{S}\,\mathcal{U}(\mathbf{Y})\mathcal{U}(\mathbf{X}_2)^{-1} \\
&= \mathcal{U}(\mathbf{X}_2)^{*-1}\mathbf{X}_1\mathcal{U}(\mathbf{X}_2)^{-1}.
\end{aligned}$$

From where the desired result is obtained. □

The following theorems 4.5 and 4.6 contain versions for k-beta-Riesz distributions of theorems 4.3 and 4.4, whose proofs are similar.

Theorem 4.5 *Let* $\mathbf{X}_1$ *and* $\mathbf{X}_2$ *be independently distributed as Riesz distribution type II, such that* $\mathbf{X}_1 \sim \mathfrak{R}_m^{\beta,II}(a, \kappa, \mathbf{\Sigma})$ *and* $\mathbf{X}_2 \sim \mathfrak{R}_m^{\beta,II}(b, \tau, \mathbf{\Sigma})$, $Re(a) > (m-1)\beta/2 + k_1$ *and* $Re(b) > (m-1)\beta/2 + t_1$. *Let*

$$\mathbf{S} = \mathcal{U}(\mathbf{X}_1 + \mathbf{X}_2)^{*-1}\mathbf{X}_1\mathcal{U}(\mathbf{X}_1 + \mathbf{X}_2)^{-1},$$

where $\mathcal{U}(\mathbf{X}_1 + \mathbf{X}_2) \in \mathfrak{T}_U^\beta(m)$ *is such that* $(\mathbf{X}_1 + \mathbf{X}_2) = \mathcal{U}(\mathbf{X}_1 + \mathbf{X}_2)^*\mathcal{U}(\mathbf{X}_1 + \mathbf{X}_2)$. *Then* $\mathbf{S}$ *has a* k*-beta-Riesz distribution type I.*

Theorem 4.6 *Let* $\mathbf{X}_1$ *and* $\mathbf{X}_2$ *be independently distributed as Riesz distribution type I, such that* $\mathbf{X}_1 \sim \mathfrak{R}_m^{\beta,II}(a, \kappa, \mathbf{\Sigma})$ *and* $\mathbf{X}_2 \sim \mathfrak{R}_m^{\beta,II}(b, \tau, \mathbf{\Sigma})$, $Re(a) > (m-1)\beta/2 + k_1$ *and* $Re(b) > (m-1)\beta/2 + t_1$. *Let*

$$\mathbf{R} = \mathcal{U}(\mathbf{X}_2)^{*-1}\mathbf{X}_1\mathcal{U}(\mathbf{X}_2)^{-1},$$

where $\mathcal{U}(\mathbf{X}_2) \in \mathfrak{T}_U^\beta(m)$ *is such that* $\mathbf{X}_2 = \mathcal{U}(\mathbf{X}_1)^*\mathcal{U}(\mathbf{X}_1)$. *Then* $\mathbf{S}$ *has a* k*-beta-Riesz distribution type II.*

4.4. Some properties of the c-beta-Riesz and k-beta-Riesz distributions

This section derives the distributions of eigenvalues for c-beta-Riesz and k-beta-Riesz distributions type I and II. First consider the following integrals:

$$Q(\kappa, \tau, \mathbf{A}, \mathbf{B}) = \int_{\mathfrak{U}^\beta(m)} q_\kappa(\mathbf{HAH}^*)q_\tau\left(\mathbf{HBH}^*\right)(d\mathbf{H})$$

and

$$Q_1(\kappa, \tau, \mathbf{A}, \mathbf{B}) = \int_{\mathfrak{U}^\beta(m)} q_\kappa(\mathbf{HAH}^*)q_{-(\kappa+\tau)}\left(\mathbf{HBH}^*\right)(d\mathbf{H})$$

Theorem 4.7 *Let* $\mathbf{\Sigma} \in \mathbf{\Phi}_m^\beta$, $\kappa = (k_1, k_2, \dots, k_m)$, $k_1 \geq k_2 \geq \cdots \geq k_m \geq 0$, $k_1, k_2, \dots, k_m$ *are nonnegative integers and* $\tau = (t_1, t_2, \dots, t_m)$, $t_1 \geq t_2 \geq \cdots \geq t_m \geq 0$, $t_1, t_2, \dots, t_m$ *are nonnegative integers.*

1. *Let* $\mathbf{L} = \text{diag}(\lambda_1, \dots, \lambda_m)$*,* $\lambda_1 > \dots > \lambda_m > 0$ *be the eigenvalues of* $\mathbf{S}$*. Then if* $\mathbf{S}$ *has a c-beta-Riesz distribution of type I, the joint density of* $\lambda_1, \dots, \lambda_m$ *is*

$$\frac{\pi^{m^2\beta/2+\varrho}}{\Gamma_m^\beta[m\beta/2]\mathcal{B}_m^\beta[a,\kappa;b,\tau]}\prod_{i<j}^m(\lambda_i-\lambda_j)^\beta\prod_{i=1}^m\lambda_i^{a-(m-1)\beta/2-1}\prod_{i=1}^m(1-\lambda_i)^{b-(m-1)\beta/2-1}$$

$$Q(\kappa,\tau,\mathbf{L},\mathbf{I}_m-\mathbf{L})\left(\bigwedge_{i=1}^m d\lambda_i\right),$$

where $0 < \lambda_i < 1$*,* $i = 1, \dots, m$ *and* $Re(a) > (m-1)\beta/2 - k_m$ *and* $Re(b) > (m-1)\beta/2 - t_m$*.*

2. *Let* $\Delta = \text{diag}(\delta_1, \dots, \delta_m)$*,* $\delta_1 > \dots > \delta_m > 0$ *be the eigenvalues of* $\mathbf{R}$*. Then if* $\mathbf{R}$ *has a c-beta-Riesz distribution of type II, the joint density of their eigenvalues is*

$$\frac{\pi^{m^2\beta/2+\varrho}}{\Gamma_m^\beta[m\beta/2]\mathcal{B}_m^\beta[a,\kappa;b,\tau]}\prod_{i<j}^m(\delta_i-\delta_j)^\beta\prod_{i=1}^m\delta_i^{a-(m-1)\beta/2-1}\prod_{i=1}^m(1-\delta_i)^{-(a+b)}$$

$$Q_1(\kappa,\tau,\mathbf{\Delta},(\mathbf{I}_m+\mathbf{\Delta}))\left(\bigwedge_{i=1}^m d\delta_i\right),$$

where $\delta_i > 0$*,* $i = 1, \dots, m$ *and* $Re(a) > (m-1)\beta/2 - k_m$ *and* $Re(b) > (m-1)\beta/2 - t_m$*.*

ϱ *is defined in Lemma 2.4.*

Proof. This is due to applying the Lemma 2.4 in (16) and (17). □

This section conclude establishing the Theorem 4.7 for the case of the k-beta-Riesz distributions.

Theorem 4.8 *Let* $\mathbf{\Sigma} \in \mathbf{\Phi}_m^\beta$*,* $\kappa = (k_1, k_2, \dots, k_m)$*,* $k_1 \geq k_2 \geq \dots \geq k_m \geq 0$*,* $k_1, k_2, \dots, k_m$ *are nonnegative integers and* $\tau = (t_1, t_2, \dots, t_m)$*,* $t_1 \geq t_2 \geq \dots \geq t_m \geq 0$*,* $t_1, t_2, \dots, t_m$ *are nonnegative integers.*

1. *Let* $\mathbf{L} = \text{diag}(\lambda_1, \dots, \lambda_m)$*,* $\lambda_1 > \dots > \lambda_m > 0$ *be the eigenvalues of* $\mathbf{S}$*. Then if* $\mathbf{S}$ *has a k-beta-Riesz distribution of type I, the joint density of* $\lambda_1, \dots, \lambda_m$ *is*

$$\frac{\pi^{m^2\beta/2+\varrho}}{\Gamma_m^\beta[m\beta/2]\mathcal{B}_m^\beta[a,-\kappa;b,-\tau]}\prod_{i<j}^m(\lambda_i-\lambda_j)^\beta\prod_{i=1}^m\lambda_i^{a-(m-1)\beta/2-1}\prod_{i=1}^m(1-\lambda_i)^{b-(m-1)\beta/2-1}$$

$$Q\left(\kappa,\tau,\mathbf{L}^{-1},(\mathbf{I}_m-\mathbf{L})^{-1}\right)\left(\bigwedge_{i=1}^m d\lambda_i\right),$$

where $0 < \lambda_i < 1$*,* $i = 1, \dots, m$ *and* $Re(a) > (m-1)\beta/2 + k_1$ *and* $Re(b) > (m-1)\beta/2 + t_1$*.*

2. *Let* $\Delta = \text{diag}(\delta_1, \dots, \delta_m)$*,* $\delta_1 > \dots > \delta_m > 0$ *be the eigenvalues of* $\mathbf{R}$*. Then if* $\mathbf{R}$ *has a k-beta-Riesz distribution of type II, the joint density of their eigenvalues is*

$$\frac{\pi^{m^2\beta/2+\varrho}}{\Gamma_m^\beta[m\beta/2]\mathcal{B}_m^\beta[a,-\kappa;b,-\tau]}\prod_{i<j}^m(\delta_i-\delta_j)^\beta\prod_{i=1}^m\delta_i^{a-(m-1)\beta/2-1}\prod_{i=1}^m(1-\delta_i)^{-(a+b)}$$

$$Q_1\left(\kappa,\tau,\mathbf{\Delta}^{-1},(\mathbf{I}_m+\mathbf{\Delta})^{-1}\right)\left(\bigwedge_{i=1}^m d\delta_i\right),$$

where $\delta_i > 0$*,* $i = 1, \dots, m$ *and* $Re(a) > (m-1)\beta/2 + k_1$ *and* $Re(b) > (m-1)\beta/2 + t_1$*.*

ϱ is defined in Lemma 2.4.

Finally observe that if in all result of this section are taking $\kappa = (0, 0, \dots, 0) \in \Re^m$ and $\tau = (0, 0, \dots, 0) \in \Re^m$ the obtained results are the corresponding to matrix multivariate beta distributions of type I and II.

Conclusions

Finally, note that the real dimension of real normed division algebras can be expressed as powers of 2, $\beta = 2^n$ for $n = 0, 1, 2, 3$. On the other hand, as observed from Kabe (1984), the results obtained in this work can be extended to hypercomplex cases; that is, for complex, bicomplex, biquaternion and bioctonion (or sedenionic) algebras, which of course are not division algebras (except the complex algebra). Also note, that hypercomplex algebras are obtained by replacing the real numbers with complex numbers in the construction of real normed division algebras. Thus, the results for hypercomplex algebras are obtained by simply replacing β with 2β in our results. Alternatively, following Kabe (1984), it can be concluded that, results are true for '2^n-ions', $n = 0, 1, 2, 3, 4, 5$, emphasising that only for $n = 0, 1, 2, 3$ are the result algebras, in fact, real normed division algebras.

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