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Transmuted Modified Inverse Rayleigh Distribution

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Abstract

We introduce the transmuted modified Inverse Rayleigh distribution by using quadratic rank transmutation map (QRTM), which extends the modified Inverse Rayleigh distribution. A comprehensive account of the mathematical properties of the transmuted modified Inverse Rayleigh distribution are discussed. We derive the quantile, moments, moment generating function, entropy, mean deviation, Bonferroni and Lorenz curves, order statistics and maximum likelihood estimation The usefulness of the new model is illustrated using real lifetime data.

Keywords: modified inverse Rayleigh distribution, moments, order statistics, maximum likelihood estimation.

1. Introduction

The inverse Rayleigh (IR) distribution is the special case of the inverse Weibull (IW) distribution for modeling lifetime data. Trayer (1964) introduced the (IR)distribution. Gharraph (1993), Mukarjee and Maitim (1996) discussed some properties of the (IR) distribution. Voda (1972) also discussed some properties of the maximum likelihood estimator for the IR distribution. Mohsin and Shahbaz (2005) studied the comparison of the negative moment estimator with maximum likelihood estimator of the IR distribution. Recently Khan (2014), studied the modified inverse Rayleigh (MIR) distribution and discussed its theoretical properties. The cumulative distribution function (cdf) of the MIR distribution is given by

$$G(x;\alpha,\beta) = \exp\left\{-\frac{\alpha}{x} - \beta\left(\frac{1}{x}\right)^2\right\}, \quad x>0, \tag{1.1}$$

where $\alpha > 0$ and $\beta > 0$ are the scale parameters. The density function corresponding to (1.1) is

$$g(x;\alpha,\beta) = \left(\alpha + \frac{2\beta}{x}\right)\left(\frac{1}{x}\right)^2 \exp\left\{-\frac{\alpha}{x} - \beta\left(\frac{1}{x}\right)^2\right\}, \quad x>0. \tag{1.2}$$

The behavior of instantaneous failure rate of the modified inverse Rayleigh distribution has increasing and decreasing reliability patterns for engineering system or component failure rate for lifetime data. The two parameter modified inverse Rayleigh distribution is the extended model of the inverse Rayleigh distribution and has nice physical interpretation. The inverse Rayleigh (IR) distribution is the special case of the modified inverse Rayleigh (MIR) distribution when $\alpha = 0$ and the MIR distribution

coincides with the inverse exponential distribution for $\beta = 0$.
Khan and King (2012), proposed the modified inverse Weibull distribution and presented comprehensive description of the mathematical properties along with its reliability behavior. Khan et al. (2008), studied the flexibility of the inverse Weibull distribution. Aryal et al. (2009) studied the transmuted extreme value distribution with application to climate data. Aryal et al. (2011), proposed the transmuted Weibull distribution and studied various structural properties of this model for analyzing reliability data. More recently Khan and King (2013), proposed the transmuted modified Weibull distribution and studied its mathematical properties. Khan and King (2013) also proposed the transmuted generalized inverse Weibull distribution with application to reliability data. Khan, King and Hudson (2013), studied the transmuted generalized exponential distribution and studied its various structural properties with an application to survival data. More recently Merovci (2013), studied the transmuted Rayleigh distribution. In this research article, we propose the three parameter transmuted modified inverse Rayleigh distribution denoted as the TMIR which is a new generalization of the modified inverse Rayleigh distribution and discuss its statistical properties and applications. The new extended distribution contains five submodels such as the TIR (transmuted inverse Rayleigh), TIE (transmuted inverse exponential), modified inverse Rayleigh, inverse Rayleigh and inverse exponential distributions.
A random variable *X* is said to have transmuted distribution if its distribution function is given by

$$F(x) = (1+\lambda)\, G(x) - \lambda\, G(x)^2, \tag{1.3}$$

where $G(x)$ is the CDF of the base distribution. It is important to note that at $\lambda = 0$ we have the distribution of the base random variable, Shaw et al.(2009).

The article is organized as follows, In Section 2, we present the analytical shapes of the probability density, distribution function, reliability function and hazard function of the subject model. A range of mathematical properties are considered in Section 3, we demonstrate the quantile functions, moment estimation, moment generating function. In Section 4, we derived the entropies, mean deviation, Bonferroni and Lorenz curves. In Section 5, we derive density functions of the pdf of rth order statistics and the rth moment of order statistics $X_{(r)}$. In Section 6, Maximum likelihood estimates (MLE_s) of the unknown parameters and the asymptotic confidence intervals of the TMIR distribution are discussed. In Section 7. we fit the TMIR distribution to illustrate its usefulness. Concluding remarks are addressed in Section 8.

2. Transmuted modified inverse Rayleigh distribution

A positive random variable *x* has the three parameters TMIR distribution with scale parameters $\alpha, \beta > 0$ and the transmuted parameter $|\lambda| \le 1$ is given by

$$f(x;\alpha,\beta,\lambda) = \left(\alpha + \frac{2\beta}{x}\right)\left(\frac{1}{x}\right)^2 \exp\left\{-\frac{\alpha}{x} - \beta\left(\frac{1}{x}\right)^2\right\} u_2(x), \tag{2.1}$$

$$u_g(x) = \left\{1 + \lambda - g\lambda \exp\left\{-\frac{\alpha}{x} - \beta\left(\frac{1}{x}\right)^2\right\}\right\}, \quad g = 1,2, \tag{2.2}$$

The cumulative distribution function CDF corresponding to (2.1) is given by

$$F(x;\alpha,\beta,\lambda) = \exp\left\{-\frac{\alpha}{x} - \beta\left(\frac{1}{x}\right)^2\right\} u_1(x). \tag{2.3}$$

Here α and β are the scale parameters and λ is the transmuting parameter representing the different patterns of the TMIR distribution. The probability density function given in (2.1) with their associated reliability function, hazard function and cumulative hazard function are given in (2.4-2.6) respectively

$$R(x;\alpha,\beta,\lambda) = 1 - \exp\left\{-\frac{\alpha}{x} - \beta\left(\frac{1}{x}\right)^2\right\} u_1(x), \tag{2.4}$$

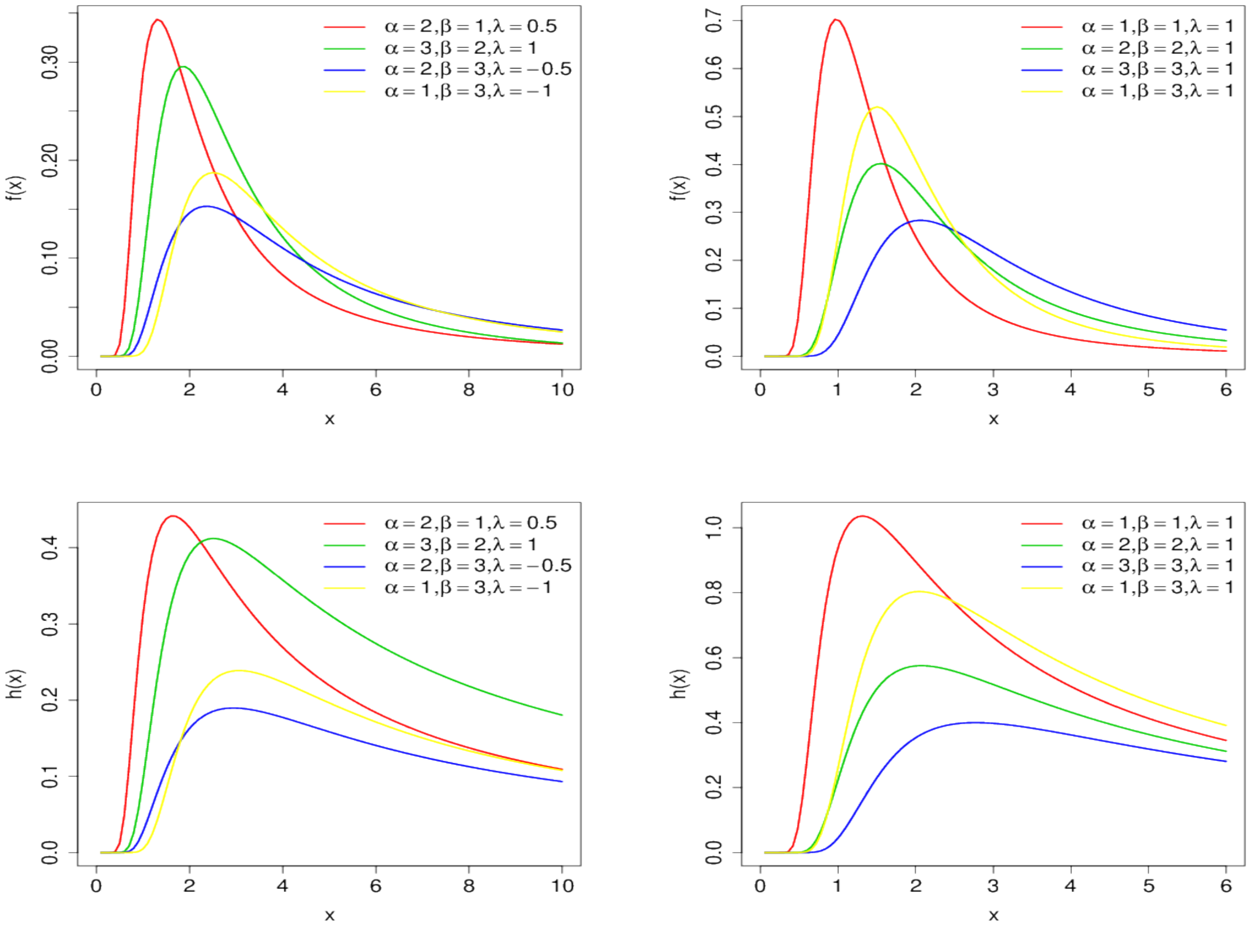

Figure 1: Plots of the TMIR pdf and hf for some parameter values.

$$h(x;\alpha,\beta,\lambda) \quad = \quad \frac{\left(\alpha+\frac{2\beta}{x}\right)\left(\frac{1}{x}\right)^2\exp\left\{-\frac{\alpha}{x}-\beta\left(\frac{1}{x}\right)^2\right\}u_2(x)}{1-\exp\left\{-\frac{\alpha}{x}-\beta\left(\frac{1}{x}\right)^2\right\}u_1(x)}, \tag{2.5}$$

and

$$H(x;\alpha,\theta,\lambda) \quad = \quad -\ln\left[1-\exp\left\{-\frac{\alpha}{x}-\beta\left(\frac{1}{x}\right)^2\right\}u_1(x)\right]. \tag{2.6}$$

Fig. 1 shows the different patterns of the density function (pdf) and hazard function (hf) of the TMIR distribution. It illustrate that the behavior of instantaneous failure rate of the TMIR distribution has upside-down bathtub shape curves.

3. Moments and quantiles

In this section we obtain some statistical properties of the TMIR distribution.

3.1. Quantile and median

The quantile $F^{-1}(u)$ of the TMIR distribution is the real solution of the following equation

$$F^{-1}(u) = \frac{2\beta}{-\alpha+\sqrt{\alpha^2-4\beta\ln\left(\frac{(1+\lambda)-\sqrt{(1+\lambda)^2-4\lambda u}}{2\lambda}\right)}}, \tag{3.1}$$

where u has the uniform $U(0,1)$ distribution.

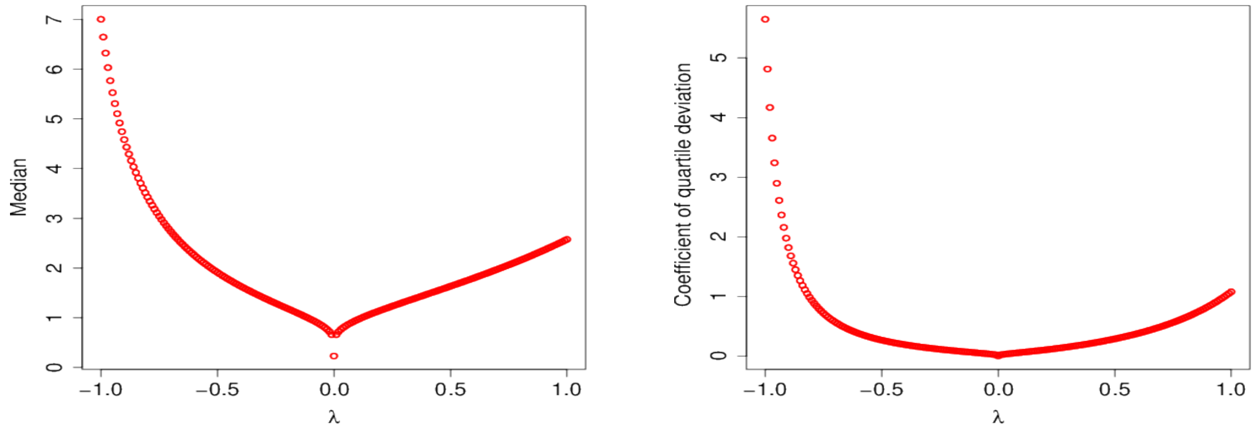

Figure 2: Median and coefficient of quantile deviation of the TMIRD.

The random number for the TMIR distribution is performed by generating uniform numbers and then applying the quantile function using equation (3.1). By substituting $u = 0.5$ in (3.1) we obtain the median of the TMIR distribution. Fig. 2 shows the median and quartile deviation life of the TMIR distribution when $\alpha = 2$ and $\beta = 3$. To illustrate the skewness and kurtosis we consider the measure based on quantiles. The skewness and kurtosis measures can now be calculated from quantiles using Bowley and Percentile coefficient of kurtosis. The Bowley Skewness and Percentile coeffficient of kurtosis when $\alpha = 2$, $\lambda = 0.5$ as a function of β are illustrated in Fig. 3 respectively. It is important to note that as the parameter β increases the behavior of the Bowley Skewness and Percentile coeffficient of kurtosis are decreases asymptotically.

3.2. Moments

Theorem 1. *If X has the* TMIR$(x;\alpha,\beta,\lambda)$ *with* $|\lambda| \le 1$, *then the kth moment of X is given by*

$$\mu_k = (1+\lambda)\sum_{p=0}^{\infty}\frac{(-1)^p\beta^p\alpha^{k-2p}}{p!}z_2(k,p) - \lambda\sum_{p=0}^{\infty}\frac{(-1)^p(2\beta)^p(2\alpha)^{k-2p}}{p!}z_1(k,p),$$

$$z_g(k,p) = \left[\Gamma\left(2p-k+1\right) + \frac{g\beta}{\alpha^2}\Gamma\left(2p-k+2\right)\right], \quad g=1,2.$$

Proof. By definition

$$\mu_k = \int_0^\infty x^{k-2}\left(\alpha + \frac{2\beta}{x}\right)\exp\left\{-\frac{\alpha}{x} - \beta\left(\frac{1}{x}\right)^2\right\}u_2(x)dx.$$

so that

$$\mu_k = (1+\lambda)\int_0^\infty x^{k-2}\left(\alpha+\frac{2\beta}{x}\right)\exp\left\{-\frac{\alpha}{x}-\beta\left(\frac{1}{x}\right)^2\right\}dx$$
$$-2\lambda\int_0^\infty x^{k-2}\left(\alpha+\frac{2\beta}{x}\right)\exp\left\{-\frac{2\alpha}{x}-2\beta\left(\frac{1}{x}\right)^2\right\}dx.$$

The above expression can be obtained by using

$$\exp\left\{-g\beta\left(\frac{1}{x}\right)^2\right\} = \sum_{p=0}^{\infty}\frac{(-1)^p(g\beta)^p x^{-2p}}{p!}, \quad g=1,2.$$

the above integral yields the following kth moment,

$$\mu_k = (1+\lambda)\sum_{p=0}^{\infty}\frac{(-1)^p\beta^p\alpha^{k-2p}}{p!}\left[\Gamma(2p-k+1)+\frac{2\beta}{\alpha^2}\Gamma(2p-k+2)\right] -\lambda\sum_{p=0}^{\infty}\frac{(-1)^p(2\beta)^p(2\alpha)^{k-2p}}{p!}\left[\Gamma(2p-k+1)+\frac{\beta}{\alpha^2}\Gamma(2p-k+2)\right]. \quad (3.2)$$

□

Theorem 2. *If X is a random variable that has the* TMIR$(x;\alpha,\beta,\lambda)$ *with* $|\lambda|\le 1$*, then the moment generating function* (mgf) *of X is given by*

$$M_x(t) = (1+\lambda)\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{(-1)^p\beta^p t^q}{\alpha^{2p-q}p!q!}J_2(p,q)-\lambda\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{(-1)^p(2\beta)^p t^q}{(2\alpha)^{2p-q}p!q!}J_1(p,q),$$

$$J_h(p,q)=\left[\Gamma(2p-q+1)+\frac{h\beta}{\alpha^2}\Gamma(2p-q+2)\right], \quad h=1,2.$$

Proof. By definition

$$M_x(t) = \int_0^{\infty}\left(\alpha+\frac{2\beta}{x}\right)\left(\frac{1}{x}\right)^2\exp\left\{tx-\frac{\alpha}{x}-\beta\left(\frac{1}{x}\right)^2\right\}u_2(x)dx.$$

so that

$$M_x(t) = (1+\lambda)\int_0^{\infty}\left(\alpha+\frac{2\beta}{x}\right)\left(\frac{1}{x}\right)^2\exp\left\{tx-\frac{\alpha}{x}-\beta\left(\frac{1}{x}\right)^2\right\}dx -2\lambda\int_0^{\infty}\left(\alpha+\frac{2\beta}{x}\right)\left(\frac{1}{x}\right)^2\exp\left\{tx-\frac{2\alpha}{x}-2\beta\left(\frac{1}{x}\right)^2\right\}dx.$$

Using the Taylor series expansions the above integral reduces to

$$M_x(t) = (1+\lambda)\sum_{q=0}^{\infty}\frac{t^q}{q!}\int_0^{\infty}x^{q-2}\left(\alpha+\frac{2\beta}{x}\right)\exp\left\{-\frac{\alpha}{x}-\beta\left(\frac{1}{x}\right)^2\right\}dx -2\lambda\sum_{q=0}^{\infty}\frac{t^q}{q!}\int_0^{\infty}x^{q-2}\left(\alpha+\frac{2\beta}{x}\right)\exp\left\{-\frac{2\alpha}{x}-2\beta\left(\frac{1}{x}\right)^2\right\}dx,$$

the above integral yields the following moment generating function

$$M_x(t) = (1+\lambda)\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{(-1)^p\beta^p t^q}{\alpha^{2p-q}p!q!}\left[\Gamma(2p-q+1)+\frac{2\beta}{\alpha^2}\Gamma(2p-q+2)\right] -\lambda\sum_{p=0}^{\infty}\sum_{q=0}^{\infty}\frac{(-1)^p(2\beta)^p t^q}{(2\alpha)^{2p-q}p!q!}\left[\Gamma(2p-q+1)+\frac{\beta}{\alpha^2}\Gamma(2p-q+2)\right]. \quad (3.3)$$

□

Based on Theorem 1, the coefficient of skewness and coefficient of kurtosis of the TMIR$(x;\alpha,\beta,\lambda)$ are obtained from the well known relations $\beta_1=\mu_3/\mu_2^{\frac{3}{2}}$ and $\beta_2=\mu_4/\mu_2^2$, respectively.

4. Entropy and mean deviation

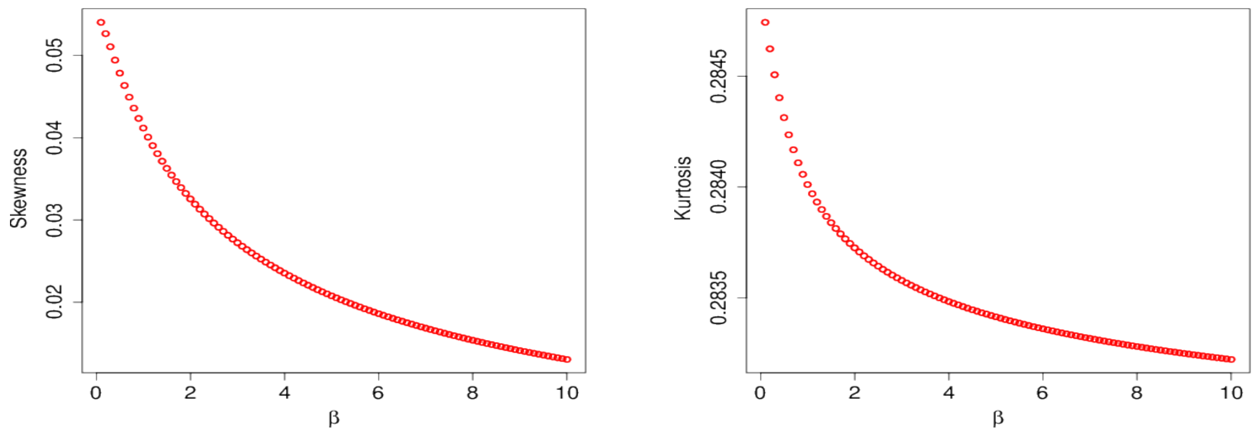

Figure 3: Bowley skewness and percentile kurtosis of the TMIRD.

The entropy of a random variable X with probability density $\text{TMIR}(x;\alpha,\beta,\lambda)$ is a measure of variation of the uncertainty. A large value of entropy indicates the greater uncertainty in the data. The Rényi entropy (1960), $I_R(\rho)$, for X is a measure of variation of uncertainty and is defined as

$$I_R(\rho) = \frac{1}{1-\rho}\log\left\{\int_0^\infty f(x)^\rho dx\right\}, \tag{4.1}$$

where $\rho > 0$ and $\rho \neq 1$. Suppose X has the $\text{TMIR}(x;\alpha,\beta,\lambda)$ then by substituting (2.1) and (2.2) in (4.1), we obtain

$$I_R(\rho) = \frac{1}{1-\rho}\log\left\{\int_0^\infty \left(\alpha+\frac{2\beta}{x}\right)^\rho \left(\frac{1}{x}\right)^{2\rho} \exp\left\{-\frac{\alpha\rho}{x}-\beta\rho\left(\frac{1}{x}\right)^2\right\} u_2(x)^\rho dx\right\},$$

the TMIR Rényi entropy reduces to

$$I_R(\rho) = \frac{1}{1-\rho}\log\left\{\sum_{i=0}^\infty\sum_{j=0}^\infty(-1)^j(1+\lambda)^\rho\alpha^\rho\binom{\rho}{i}\binom{\rho}{j}\left(\frac{2\beta}{\alpha}\right)^i\left(\frac{2\lambda}{1+\lambda}\right)^i \xi_{i,j}dx\right\},$$

where

$$\xi_{i,j} = \int_0^\infty\left(\frac{1}{x}\right)^{2\rho+j}\exp\left\{-(\rho+i)\left\{\frac{\alpha}{x}+\beta\left(\frac{1}{x}\right)^2\right\}\right\}dx.$$

The above integral can be calculated as

$$\xi_{i,j} = \sum_{k=0}^\infty\frac{(-1)^k\beta^k(\rho+i)^k}{k!}\left(\frac{\Gamma\left(j+2(k+\rho)-1\right)}{[\alpha(\rho+i)]^{j+2(k+\rho)+1}}\right).$$

and thus we obtain the TMIR Rényi entropy as

$$I_R(\rho) = \frac{\rho}{1-\rho}\log\alpha+\frac{\rho}{1-\rho}\log(1+\lambda)+\frac{1}{1-\rho}$$
$$\log\left\{\sum_{i=0}^\infty\sum_{j,k=0}^\infty\frac{(-1)^{j+k}\beta^k(\rho+i)^kV_{i,j}}{k!}\left(\frac{\Gamma\left(j+2(k+\rho)-1\right)}{[\alpha(\rho+i)]^{j+2(k+\rho)+1}}\right)\right\}.$$

where

$$V_{i,j} = \binom{\rho}{i}\binom{\rho}{j}\left(\frac{2\beta}{\alpha}\right)^i\left(\frac{2\lambda}{1+\lambda}\right)^i$$

The β-or(q-entropy) was introduced by Havrda and Charvat (1967), and is defined as

$$I_H(q) = \frac{1}{q-1}\left\{1-\int_0^{\infty} f(x)^q dx\right\}, \tag{4.2}$$

where $q > 0$ and $q \neq 1$. Suppose X has the TMIR$(x;\alpha,\beta,\lambda)$ then by substituting (2.1) and (2.2) in (4.2), we obtain

$$I_H(q) = \frac{1}{q-1}\left\{1-\int_0^{\infty}\left(\alpha+\frac{2\beta}{x}\right)^q\left(\frac{1}{x}\right)^{2q}\exp\left\{-\frac{\alpha q}{x}-\beta q\left(\frac{1}{x}\right)^2\right\}u_2(x)^q dx\right\},$$

the above integral yields the TMIR q-entropy is

$$I_H(q) = \frac{1}{q-1}\left\{1-\sum_{i=0}^{\infty}\sum_{j,k=0}^{\infty}\frac{(-1)^{j+k}\beta^k(q+i)^k\xi_{i,j}}{k!}\left(\frac{\Gamma\left(j+2(k+q)-1\right)}{[\alpha(q+i)]^{j+2(k+q)+1}}\right)\right\},$$

where

$$\xi_{i,j} = \alpha^q(1+\lambda)^q\binom{q}{i}\binom{q}{j}\left(\frac{2\beta}{\alpha}\right)^i\left(\frac{2\lambda}{1+\lambda}\right)^i.$$

The degree of scatter in a population is widely measured by the totality of deviations from the mean and median. If X has the TMIR$(x;\alpha,\beta,\lambda)$, then we derive the mean deviation about the mean and about the median from the following equations of Gauss et al. (2013)

$$\delta_1 = \int_0^{\infty}|x-\mu|f(x)dx \qquad \text{and} \qquad \delta_2 = \int_0^{\infty}|x-M|f(x)dx.$$

The mean μ is given in equation (3.2) and the median M is obtained from equation (3.1). These measures are calculated using the relationships:

$$\delta_1 = 2[\mu F(\mu)-\psi(\mu)] \qquad \text{and} \qquad \delta_2 = \mu - 2\psi(M).$$

The quantity $\psi(q)$ used to determine the Bonferroni and Lorenz curves, which have applications in econometrics and finance, reliability and survival analysis, demography, insurance and biomedical sciences is given by

$$\begin{aligned}\psi(q) = {} & (1+\lambda)\sum_{h=0}^{\infty}\frac{(-1)^h\beta^h}{\alpha^{2h}h!}\left[\alpha\gamma\left(2h+1,\frac{\alpha}{q}\right)+\frac{2\beta}{\alpha}\gamma\left(2h+2,\frac{\alpha}{q}\right)\right] \\ & -\lambda\sum_{h=0}^{\infty}\frac{(-1)^h(2\beta)^h}{(2\alpha)^{2h}h!}\left[\alpha\gamma\left(2h+1,\frac{\alpha}{q}\right)+\frac{\beta}{\alpha}\gamma\left(2h+2,\frac{2\alpha}{q}\right)\right].\end{aligned} \tag{4.3}$$

By using (4.3), one obtains the Bonferroni and the Lorenz curve as

$$B(P) = \frac{\psi(q)}{P\mu}, \qquad \text{and} \qquad L(P) = \frac{\psi(q)}{\mu}.$$

5. Order statistics

The density of the rth order statistic $X_{(r)}$ of a random sample drawn from the TMIR$(x;\alpha,\beta,\lambda)$ distribution with $|\lambda| \leq 1$, follows from Arnold et al. (1), with the density function of $X_{(r)}$ is given by

$$f_{r:n}(x) = \frac{(F(x))^{r-1}(1-F(x))^{n-r}f(x)}{B(r,n-r+1)}, \quad x > 0. \tag{5.1}$$

By setting $\delta = \exp\left\{-\frac{\alpha}{x} - \beta\left(\frac{1}{x}\right)^2\right\}$, substituting (2.1) and (2.3) into (5.1), we obtain

$$f_{r:n}(x) = n\begin{pmatrix} n-1 \\ r-1 \end{pmatrix}\sum_{k=0}^{n-r}\begin{pmatrix} n-r \\ k \end{pmatrix}(-1)^k\delta^{r+k}V_{r:k}(x),$$

where

$$V_{r:k}(x) = \left(\alpha + \frac{2\beta}{x}\right)\left(\frac{1}{x}\right)^2 u_1(x)^{r+k-1}u_2(x).$$

This leads to the combining terms of the order statistics of the TMIR distribution, given by

$$f_{r:n}(x) = n\begin{pmatrix} n-1 \\ r-1 \end{pmatrix}\sum_{k=0}^{n-r}\sum_{m=0}^{\infty}\Im_{k,m}\left(\alpha + \frac{2\beta}{x}\right)\left(\frac{1}{x}\right)^2 z_{k,m}u_2(x), \tag{5.2}$$

where

$$\Im_{k,m} = n\begin{pmatrix} n-r \\ k \end{pmatrix}\begin{pmatrix} r+k-1 \\ m \end{pmatrix}(-1)^{k+m}(1+\lambda)^{r+k-1}\left(\frac{\lambda}{1+\lambda}\right)^m,$$

and

$$z_{k,m} = \exp\left\{-(r+k+m)\left(\frac{\alpha}{x} + \beta\left(\frac{1}{x}\right)^2\right)\right\}.$$

Using (5.2), the S th moment of the rth order statistics $X_{(r)}$ is given by

$$\mu_s^{n:r} = n\begin{pmatrix} n-1 \\ r-1 \end{pmatrix}\sum_{k=0}^{n-r}\sum_{m=0}^{\infty}\Im_{k,m}\left\{(1+\lambda)\sum_{i=0}^{\infty}\frac{(-1)^i\beta^i\tau_{i,\mathrm{s},0}}{C^{-i}i!} - 2\lambda\sum_{i=0}^{\infty}\frac{(-1)^i\beta^i\tau_{i,\mathrm{s},1}}{(C+1)^{-i}i!}\right\},$$

where $c = r + k + m$,

$$\tau_{i,\mathrm{s},g} = \alpha\left((c+g)\alpha\right)^{\mathrm{s}-2i-1}\Gamma\left(2i-\mathrm{s}+1\right) + 2\beta\left((c+g)\alpha\right)^{\mathrm{s}-2i-2}\Gamma\left(2i-\mathrm{s}+2\right).$$

6. Maximum likelihood estimation

Consider the random samples $x_1, x_2, ..., x_n$ consisting of n observations from the TMIR distribution and $\Theta = (\alpha, \beta, \lambda)^T$ be the parameter vector. The likelihood function of (2.1)is given by

$$L(\alpha,\beta,\lambda) = \prod_{i=1}^{n}\left(\alpha + \frac{2\beta}{x_i}\right)\left(\frac{1}{x_i}\right)^2\exp\left\{-\frac{\alpha}{x_i} - \beta\left(\frac{1}{x_i}\right)^2\right\} \times\left\{1+\lambda-2\lambda\exp\left\{-\frac{\alpha}{x_i} - \beta\left(\frac{1}{x_i}\right)^2\right\}\right\}. \tag{6.1}$$

By taking the logarithm of (6.1), we have the log-likelihood function

$$\log L = \sum_{i=1}^{n}\log\left(\alpha + \frac{2\beta}{x_i}\right) + \sum_{i=1}^{n}log\left(\frac{1}{x_i}\right)^2 - \sum_{i=1}^{n}\left(\frac{\alpha}{x_i}\right) -\beta\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)^2 + \sum_{i=1}^{n}\log\left\{1+\lambda-2\lambda\exp\left\{-\frac{\alpha}{x_i} - \beta\left(\frac{1}{x_i}\right)^2\right\}\right\}. \tag{6.2}$$

Differentiating (6.2) with respect to α, β and λ, then equating it to zero, we obtain the estimating equations are

$$\frac{\partial \log L}{\partial \alpha} = \sum_{i=1}^{n}\left\{\alpha+\frac{2\beta}{x_i}\right\}^{-1}-\sum_{i=1}^{n}\left(\frac{1}{x_i}\right) +2\lambda\sum_{i=1}^{n}\frac{\exp\left\{-\frac{\alpha}{x_i}-\beta\left(\frac{1}{x_i}\right)^2\right\}\left(\frac{1}{x_i}\right)}{\left\{1+\lambda-2\lambda\exp\left\{-\frac{\alpha}{x_i}-\beta\left(\frac{1}{x_i}\right)^2\right\}\right\}}, \quad (6.3)$$

$$\frac{\partial \log L}{\partial \beta} = \sum_{i=1}^{n}\left\{\alpha+\frac{2\beta}{x_i}\right\}^{-1}\left(\frac{2}{x_i}\right)-\sum_{i=1}^{n}\left(\frac{1}{x_i}\right)^2 +2\lambda\sum_{i=1}^{n}\frac{\exp\left\{-\frac{\alpha}{x_i}-\beta\left(\frac{1}{x_i}\right)^2\right\}\left(\frac{1}{x_i}\right)^2}{\left\{1+\lambda-2\lambda\exp\left\{-\frac{\alpha}{x_i}-\beta\left(\frac{1}{x_i}\right)^2\right\}\right\}}, \quad (6.4)$$

and

$$\frac{\partial \log L}{\partial \lambda} = \sum_{i=1}^{n}\frac{1-2\exp\left\{-\frac{\alpha}{x_i}-\beta\left(\frac{1}{x_i}\right)^2\right\}}{\left\{1+\lambda-2\lambda\exp\left\{-\frac{\alpha}{x_i}-\beta\left(\frac{1}{x_i}\right)^2\right\}\right\}}, \quad (6.5)$$

It is more convenient to use quasi Newton algorithm to numerically maximize the log-likelihood function given in equation (6.2) to yield the ML estimators $\hat{\alpha}$, $\hat{\beta}$ and $\hat{\lambda}$ respectively. For finding the interval estimation and testing the hypothesis of the subject model, we required the observed information matrix is given by

$$\begin{pmatrix}\hat{\alpha}\\ \hat{\beta}\\ \hat{\lambda}\end{pmatrix} \sim N\left[\begin{pmatrix}\alpha\\ \beta\\ \lambda\end{pmatrix}, \begin{pmatrix}\hat{V}_{11} & \hat{V}_{12} & \hat{V}_{13}\\ \hat{V}_{21} & \hat{V}_{22} & \hat{V}_{23}\\ \hat{V}_{31} & \hat{V}_{32} & \hat{V}_{33}\end{pmatrix}\right],$$

the expected information matrix is given by

$$V^{-1} = -E\begin{pmatrix}\frac{\partial^2 \log L}{\partial\alpha^2} & \frac{\partial^2 \log L}{\partial\alpha\partial\beta} & \frac{\partial^2 \log L}{\partial\alpha\partial\lambda}\\ \frac{\partial^2 \log L}{\partial\beta\partial\alpha} & \frac{\partial^2 \log L}{\partial\beta^2} & \frac{\partial^2 \log L}{\partial\beta\partial\lambda}\\ \frac{\partial^2 \log L}{\partial\alpha\partial\lambda} & \frac{\partial^2 \log L}{\partial\lambda\partial\beta} & \frac{\partial^2 \log L}{\partial\lambda^2}\end{pmatrix}. \quad (6.6)$$

Solving the inverse matrix for the observed information matrix (6.6), yields the asymptotic variance and co-variances of the ML estimators $\hat{\alpha}$, $\hat{\beta}$, and $\hat{\lambda}$. By using (6.6) approximate $100(1-\alpha)\%$ asymptotic confidence intervals for α, β and λ are

$$\hat{\alpha} \pm Z_{\frac{\alpha}{2}}\sqrt{\hat{V}_{11}}, \qquad \hat{\beta} \pm Z_{\frac{\alpha}{2}}\sqrt{\hat{V}_{22}}, \qquad \hat{\lambda} \pm Z_{\frac{\alpha}{2}}\sqrt{\hat{V}_{33}},$$

where $Z_{\frac{\alpha}{2}}$ is the upper αth percentile of the standard normal distribution.

7. Application

This section illustrates the usefulness of the TMIR distribution with real data. The data consist of thirty successive values of March precipitation (in inches) given by Hinkley (1977) and given below
0.77, 1.74, 0.81, 1.2, 1.95, 1.2, 0.47, 1.43, 3.37, 2.2, 3, 3.09, 1.51, 2.1, 0.52, 1.62, 1.31, 0.32, 0.59,

0.81, 2.81, 1.87, 1.18, 1.35, 4.75, 2.48, 0.96, 1.89, 0.9, 2.05.
Four distributions are fitted to the precipitation data using maximum likelihood estimation. The estimated parameters for the TMIR distribution with their corresponding 95% C.I are given in Table 2. The summary statistics of the fitted TMIR, TIR, MIR and IR distributions are given in Table 1.

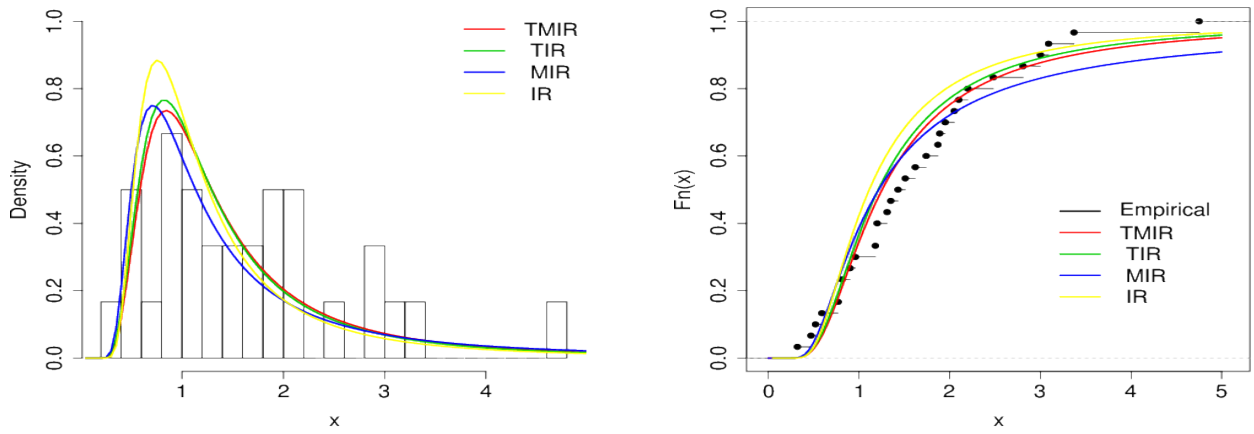

Figure 4: Estimated Reliability and Survival functions for four fitted models.

Table 1: *Summary Statistics for* TMIR, TIR, MIR *and* IR *distributions*

Distribution	Quartile deviation	Bowley Skewness	Percentile kurtosis
TMIR	0.1378	-0.0881	0.2866
TIR	0.1361	-0.0898	0.2864
MIR	0.4672	0.3928	0.1726
IR	0.3741	0.3069	0.2096

Table 2: *Estimated Parameters of the* TMIR *distribution*

			95% Confidence	Interval
Parameter	ML Estimate	Standard Error	Lower	Upper
α	0.0212	0.2388	-0.4469	0.4893
β	0.6472	0.2654	0.1268	1.16758
λ	-0.6703	0.2612	-1.1825^{*}	-0.1581

The MLE$_S$ of the parameters (with their standard errors) and their corresponding values of the Coefficient of determination, mean square error (MSE) and Kolmogorov-Smirnov (K-S) test values are displayed in Table 3. The likelihood ratio (LR) statistic for testing the hypothesis H_0 IR v.s H_A: TMIR is 4.0786 with their corresponding p-value 0.04343. Hence we reject the null hypothesis in favour of the TMIR distribution, because the p-value is small. Fig. 4 illustrates the four fitted models with empirical functions for the precipitation data. These graphs illustrate that the TMIR distribution fits well. The hazard plot of the estimated TMIR distribution has increasing and then decreasing instantaneous failure rate. As we can see from these numerical results in Table 3, the Coefficient of determination of the TMIR distribution is higher than the other three sub-models and the values of mean square error (MSE) and Kolmogorov-Smirnov (K-S) test of the TMIR distribution are the smallest among those of the four fitted distribution. Therefore the TMIR distribution can be chosen as the best model for lifetime data

Table 3: *Estimates of the model parameters for precipitation data and the* K-S *test, Coefficient of determination and associated* MSE *values*

Distribution	TMIR	TIR	MIR	IR
α	0.0212	-	0.3598	-
	(0.2388)		(0.3745)	
β	0.6472	0.6285	0.5881	0.8588
	(0.2654)	(0.1583)	(0.2975)	(0.1568)
λ	-0.6703	-0.6701	-	-
	(0.2613)	(0.2661)		
K-S	0.1395	0.1626	0.1641	0.2206
R^2	0.9442	0.9199	0.9137	0.8382
MSE	0.0726	0.0862	0.0801	0.1212

analysis. Fig. 4 also illustrate that the TMIR distribution gives a better fit than the other three submodels.

8. Conclusion

We proposed a new distribution, named the TMIR distribution, which is an extension of the MIR distribution. The TMIR distribution provides better results than the MIR, TIR and IR distributions. In this model the new parameter λ provides more flexibility in modeling reliability data. We derive the quantile function, moments, moment generating function, entropies, mean deviation, Bonferroni and Lorenz curves. We also derive the Sth moment of rth order statistics and the kth moment of rth median order statistics. We discuss the maximum likelihood estimation and obtain the fisher information matrix. The usefulness of the new model is illustrated in an application to real data using MLE. We hope that the proposed model may attract wider application in the analysis of reliability data.

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