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Efficient Generalized Ratio-Product Type Estimators for Finite Population Mean with Ranked Set Sampling

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Abstract: In this paper we suggest two modified estimators of the population mean using the power transformation based on ranked set sampling (RSS). The first order approximation of the bias and of the mean squared error of the proposed estimators are obtained. A generalized version of the suggested estimators by applying the power transformation is also presented. Theoretically, it is shown that these suggested estimators are more efficient than the estimators in simple random sampling (SRS). A numerical illustration is also carried out to demonstrate the merits of the proposed estimators using RSS over the usual estimators in SRS.

Zusammenfassung: In dieser Arbeit schlagen wir zwei modifizierte Schätzer für das Populationsmittel vor, indem wir die Power-Transformation basierend auf "Ranked Set Sampling" (RSS) verwenden. Approximationen erster Ordnung für den Bias und den mittleren quadratischen Fehlers der vorgeschlagenen Schätzer werden erhalten. Eine verallgemeinerte Version der vorgeschlagenen Schätzer durch die Anwendung der Power-Transformation wird auch vorgestellt. Theoretisch wird gezeigt, dass diese vorgeschlagenen Schätzer effizienter sind als die Schätzer unter "Simple Random Sampling" (SRS). Eine numerische Darstellung wird auch durchgeführt, um die Vorzüge dieser Schätzer unter RSS über die üblichen Schätzer unter SRS aufzuzeigen.

Keywords: Ranked Set Sampling, Ratio Estimator, Power Transformation Estimator, Auxiliary Variable, Coefficient of Variation, Coefficient of Kurtosis.

1 Introduction

The literature on ranked set sampling describes a great variety of techniques for using auxiliary information to obtain more efficient estimators. Ranked set sampling was first suggested by McIntyre (1952) to increase the efficiency of estimator of population mean. Kadilar, Unyazici, and Cingi (2009) used this technique to improve ratio estimator given by Prasad (1989). Here we shall propose two modified estimators of population mean using power transformation using RSS based on auxiliary variable.

The classical ratio estimators given by Cochran (1940) for estimating the population mean $\overline{Y}$ is defined as

$$\overline{y}_R = \overline{y}\left(\frac{\overline{X}}{\overline{x}}\right) ,$$

where $\overline{y}$ is the sample mean for study variable y and $\overline{x}$, $\overline{X}$ are the sample mean and population mean, respectively, for the auxiliary variable x.

When the population coefficient of variation C_x of the auxiliary variable x is known, Sisodia and Dwivedi (1981) give a modified ratio estimator for $\overline{Y}$ as

$$\overline{y}_{SD} = \overline{y}\left(\frac{\overline{X} + C_x}{\overline{x} + C_x}\right) . \tag{1}$$

Motivated by Sisodia and Dwivedi (1981), H. P. Singh and Kakran (1993) developed a ratio-type estimator for $\overline{Y}$ as

$$\overline{y}_{SK} = \overline{y}\left(\frac{\overline{X} + \beta_2(x)}{\overline{x} + \beta_2(x)}\right) , \tag{2}$$

where $\beta_2(x)$ is the known value of the coefficient of kurtosis of an auxiliary variable x.

Utilizing the information on the coefficient of variation C_x and the coefficient of kurtosis $\beta_2(x)$ of the auxiliary variable x, Upadhyaya and Singh (1999) suggested the following ratio type estimators

$$\overline{y}_{UP1} = \overline{y}\left(\frac{\overline{X}\beta_2(x) + C_x}{\overline{x}\beta_2(x) + C_x}\right) , \tag{3}$$

$$\overline{y}_{UP2} = \overline{y}\left(\frac{\overline{X}C_x + \beta_2(x)}{\overline{x}C_x + \beta_2(x)}\right) . \tag{4}$$

By applying the power transformation on the Upadhyaya and Singh (1999) estimators, H. P. Singh, Tailor, Singh, and Kim (2008) suggested modified estimators as

$$\overline{y}_{UP1(\alpha)} = \overline{y}\left(\frac{\overline{X}\beta_2(x) + C_x}{\overline{x}\beta_2(x) + C_x}\right)^{\alpha} \tag{5}$$

$$\overline{y}_{UP2(\delta)} = \overline{y}\left(\frac{\overline{X}C_x + \beta_2(x)}{\overline{x}C_x + \beta_2(x)}\right)^{\delta} , \tag{6}$$

where α and δ are suitably chosen scalars such that the mean squared errors of $\overline{y}_{UP1(\alpha)}$ and $\overline{y}_{UP2(\delta)}$ are minimum.

To the first degree of approximation, the mean squared error (MSE) of these estimators are

$$\text{MSE}(\overline{y}_R) = \frac{1}{n}\overline{Y}^2\left(C_y^2 + C_x^2 - 2\rho_{yx}C_yC_x\right) ,$$

$$\text{MSE}(\overline{y}_{SD}) = \frac{1}{n}\overline{Y}^2\left(C_y^2 + \lambda_1^2C_x^2 - 2\lambda_1\rho_{yx}C_yC_x\right) , \tag{7}$$

$$\text{MSE}(\overline{y}_{SK}) = \frac{1}{n}\overline{Y}^2\left(C_y^2 + \lambda_2^2C_x^2 - 2\lambda_2\rho_{yx}C_yC_x\right) , \tag{8}$$

$$\text{MSE}(\overline{y}_{UP1}) = \frac{1}{n}\overline{Y}^2\left(C_y^2 + \gamma_1^2C_x^2 - 2\gamma_1\rho_{yx}C_yC_x\right) , \tag{9}$$

$$\text{MSE}(\overline{y}_{UP2}) = \frac{1}{n}\overline{Y}^2\left(C_y^2 + \gamma_2^2C_x^2 - 2\gamma_2\rho_{yx}C_yC_x\right) , \tag{10}$$

$$\text{MSE}(\overline{y}_{UP1(\alpha)}) = \frac{1}{n}\overline{Y}^2\left(C_y^2 + \phi_1^2\alpha^2C_x^2 - 2\phi_1\alpha\rho_{yx}C_yC_x\right) , \tag{11}$$

$$\text{MSE}(\overline{y}_{UP2(\delta)}) = \frac{1}{n}\overline{Y}^2\left(C_y^2 + \phi_2^2\delta^2C_x^2 - 2\phi_2\delta\rho_{yx}C_yC_x\right) , \tag{12}$$

with

$$C_y = \frac{S_y}{\overline{Y}}, \quad C_x = \frac{S_x}{\overline{X}}, \quad \lambda_1 = \frac{\overline{X}}{\overline{X} + C_x}, \quad \lambda_2 = \frac{\overline{X}}{\overline{X} + \beta_2(x)},$$
$$\gamma_1 = \phi_1 = \frac{\overline{X}\beta_2(x)}{\overline{X}\beta_2(x) + C_x}, \quad \gamma_2 = \phi_2 = \frac{\overline{X}C_x}{\overline{X}C_x + \beta_2(x)}, \quad \rho_{yx} = \frac{S_{yx}}{S_y S_x},$$

where ρ_{yx} denotes the correlation coefficient between y and x.

2 Ratio Estimator in Ranked Set Sampling

In ranked set sampling (RSS) m independent random sets are chosen (each of size m) and the units in each set are selected with equal probability and without replacement from a finite population of size N. The members of each random set are ranked with respect to the characteristic of the study variable or auxiliary variable. Then, the smallest unit is selected from the first ordered set and the second smallest unit is selected from the second ordered set. By this way, this procedure is continued until the unit with the largest rank is chosen from the m-th set. This cycle may be repeated r times, so $mr = n$ units have been measured during this process. Thus, RSS and SRS have equivalent sample sizes n for comparison of their biases and efficiencies.

When we rank on the auxiliary variable, let $(y_{[i]}, x_{(i)})$ denote the i-th judgment ordering of the study variable and the i-th perfect ordering for the auxiliary variable in the i-th set, where $i = 1, \ldots, m$.

Samawi and Muttlak (1996) define the ratio estimator for the population mean as

$$\overline{y}_{R,RSS} = \overline{y}_{[n]} \left(\frac{\overline{X}}{\overline{x}_{(n)}} \right), \tag{13}$$

where

$$\overline{y}_{[n]} = \frac{1}{n} \sum_{i=1}^{n} y_{[i]}, \quad \overline{x}_{(n)} = \frac{1}{n} \sum_{i=1}^{n} x_{(i)}$$

are the ranked set sample means for the variables y and x, respectively.

To the first degree of approximation, the MSE of the estimator $\overline{y}_{R,RSS}$ is given by (after ignoring the finite population correction factor)

$$\text{MSE}(\overline{y}_{R,RSS}) = \overline{Y}^2 \left(\frac{1}{mr}(C_y^2 + C_x^2 - 2\rho_{yx} C_y C_x) - (W_{y[i]} - W_{x(i)})^2 \right),$$

where

$$W_{x(i)}^2 = \frac{1}{m^2 r} \frac{1}{\overline{X}^2} \sum_{i=1}^{m} \tau_{x(i)}^2, \qquad W_{y[i]}^2 = \frac{1}{m^2 r} \frac{1}{\overline{Y}^2} \sum_{i=1}^{m} \tau_{y[i]}^2$$

with

$$\tau_{x(i)} = \mu_{x(i)} - \overline{X}, \qquad \tau_{y[i]} = \mu_{y[i]} - \overline{Y}.$$

3 Suggested Estimators Based on Ranked Set Sampling

Motivated by Sisodia and Dwivedi (1981) we suggest a ratio-type estimator for $\overline{Y}$ using RSS, when the population coefficient of variation of the auxiliary variable C_x is known as

$$\overline{y}_{RSS,MM1} = \overline{y}_{[n]} \left(\frac{\overline{X} + C_x}{\overline{x}_{(n)} + C_x} \right) . \tag{14}$$

To obtain the bias and the MSE of this estimator we put $\overline{y}_{[n]} = \overline{Y}(1+\varepsilon_0)$ and $\overline{x}_{(n)} = \overline{X}(1+\varepsilon_1)$ so that $\mathrm{E}(\varepsilon_0) = \mathrm{E}(\varepsilon_1) = 0$ as $\mathrm{E}\left(\overline{y}_{[n]}\right) = \overline{Y}$ and $\mathrm{E}\left(\overline{x}_{(n)}\right) = \overline{X}$ under ranked set sampling. Moreover,

$$\begin{aligned} \mathrm{var}(\varepsilon_0) &= \mathrm{E}(\varepsilon_0^2) = \frac{\mathrm{var}(\overline{y}_{[n]})}{\overline{Y}^2} = \frac{1}{mr}\frac{1}{\overline{Y}^2}\left(S_y^2 - \frac{1}{m}\sum_{i=1}^{m} \tau_{y[i]}^2 \right) \\ &= \theta C_y^2 - W_{y[i]}^2 \qquad \text{with} \quad \theta = \frac{1}{mr} \end{aligned}$$

and similarly

$$\mathrm{var}(\varepsilon_1) = \mathrm{E}(\varepsilon_1^2) = \theta C_x^2 - W_{x(i)}^2$$

and

$$\begin{aligned} \mathrm{cov}(\varepsilon_0, \varepsilon_1) &= \mathrm{E}(\varepsilon_0 \varepsilon_1) = \frac{\mathrm{cov}\left(\overline{y}_{[n]}, \overline{x}_{(n)}\right)}{\overline{XY}} = \frac{1}{\overline{XY}}\frac{1}{mr}\left(S_{yx} - \frac{1}{m}\sum_{i=1}^{m} \tau_{yx(i)} \right) \\ &= \theta \rho_{yx} C_y C_x - W_{yx(i)} , \end{aligned}$$

where

$$W_{yx(i)} = \frac{1}{m^2 r}\frac{1}{\overline{XY}}\sum_{i=1}^{m} \tau_{yx(i)} \qquad \text{with} \quad \tau_{yx(i)} = (\mu_{y[i]} - \overline{Y})(\mu_{x(i)} - \overline{X}) .$$

Further to validate the first degree of approximation, we assume that the sample size is large enough to get $|\varepsilon_0|$ and $|\varepsilon_1|$ as small as possible such that the terms involving ε_0 or ε_1 in a degree greater than two are negligible.

The bias and the MSE of $\overline{y}_{RSS,MM1}$ are found next. Since

$$\mathrm{bias}(\overline{y}_{RSS,MM1}) = \mathrm{E}(\overline{y}_{RSS,MM1}) - \overline{Y}$$

and

$$\overline{y}_{RSS,MM1} = \overline{Y}(1+\varepsilon_0)(1+\lambda_1\varepsilon_1)^{-1} , \qquad \lambda_1 = \frac{\overline{X}}{\overline{X} + C_x}$$

we suppose $|\lambda_1 \varepsilon_1| < 1$ so that $(1+\lambda_1\varepsilon_1)^{-1}$ is expandable and derive

$$\overline{y}_{RSS,MM1} = \overline{Y}\,(1+\varepsilon_0)\left\{1 - \lambda_1\varepsilon_1 + \lambda_1^2\varepsilon_1^2 + O(\lambda_1\varepsilon_1)\right\}$$

Because $\mathrm{E}(\varepsilon_0) = \mathrm{E}(\varepsilon_1) = 0$ we have

$$\begin{aligned} \mathrm{bias}(\overline{y}_{RSS,MM1}) &= \overline{Y}\left(\lambda_1^2 \mathrm{E}(\varepsilon_1^2) - \lambda_1 \mathrm{E}(\varepsilon_0\varepsilon_1)\right) \\ &= \overline{Y}\left(\lambda_1^2\left\{\theta C_x^2 - W_{x(i)}^2\right\} - \lambda_1\left\{\theta\rho_{yx}C_yC_x - W_{yx(i)}\right\}\right) \\ &= \overline{Y}\left(\theta(\lambda_1^2 C_x^2 - \lambda_1\rho_{yx}C_yC_x) - \left\{\lambda_1^2 W_{x(i)}^2 - \lambda_1 W_{yx(i)}\right\}\right) . \end{aligned}$$

Now

$$\begin{aligned}
\text{MSE}(\overline{y}_{RSS,MM1}) &= \text{E}(\overline{y}_{RSS,MM1}-\overline{Y})^2 \\
&= \overline{Y}^2\text{E}\left(\varepsilon_0-\lambda_1\varepsilon_1+\lambda_1^2\varepsilon_1^2-2\lambda_1\varepsilon_0\varepsilon_1\right)^2 \\
&= \overline{Y}^2\text{E}\left(\varepsilon_0^2+\lambda_1^2\varepsilon_1^2-2\lambda_1\varepsilon_0\varepsilon_1\right) \\
&= \overline{Y}^2\left(\theta C_y^2-W_{y[i]}^2+\lambda_1^2(\theta C_x^2-W_{x(i)}^2)-2\lambda_1\left(\theta\rho_{yx}C_yC_x-W_{yx(i)}\right)\right) \\
&= \overline{Y}^2\left(\theta\{C_y^2+\lambda_1^2C_x^2-2\lambda_1\rho_{yx}C_yC_x\}-\{W_{y[i]}^2+\lambda_1^2W_{x(i)}^2-2\lambda_1W_{yx(i)}\}\right) \\
&= \overline{Y}^2\left(\theta\{C_y^2+\lambda_1^2C_x^2-2\lambda_1\rho_{yx}C_yC_x\}-\{W_{y[i]}-\lambda_1W_{x(i)}\}^2\right) . \qquad (15)
\end{aligned}$$

Motivated by H. P. Singh and Kakran (1993) we suggest another new ratio estimator in ranked set sampling as

$$\overline{y}_{RSS,MM2} = \overline{y}_{[n]}\left(\frac{\overline{X}+\beta_2(x)}{\overline{x}_{(n)}+\beta_2(x)}\right) . \qquad (16)$$

Similarly, the bias and the mean squared error are obtained, respectively, as

$$\text{bias}(\overline{y}_{RSS,MM2}) = \overline{Y}\left(\theta(\lambda_2^2C_x^2-\lambda_2\rho_{yx}C_yC_x)-\{\lambda_2^2W_{x(i)}^2-\lambda_2W_{yx(i)}\}\right)$$

and

$$\text{MSE}(\overline{y}_{RSS,MM2}) = \overline{Y}^2\left(\theta\{C_y^2+\lambda_2^2C_x^2-2\lambda_2\rho_{yx}C_yC_x\}-\{W_{y[i]}-\lambda_2W_{x(i)}\}^2\right) . \qquad (17)$$

Motivated by Upadhyaya and Singh (1999) we also proposed two more ratio type estimators considering both coefficients of variation and kurtosis using ranked set sampling as

$$\overline{y}_{RSS,MM3} = \overline{y}_{[n]}\left(\frac{\overline{X}\beta_2(x)+C_x}{\overline{x}_{(n)}\beta_2(x)+C_x}\right) \qquad (18)$$

$$\overline{y}_{RSS,MM4} = \overline{y}_{[n]}\left(\frac{\overline{X}C_x+\beta_2(x)}{\overline{x}_{(n)}C_x+\beta_2(x)}\right) . \qquad (19)$$

The bias and the MSE of $\overline{y}_{RSS,MM3}$ can be found as follows. Since

$$\text{bias}(\overline{y}_{RSS,MM3}) = \text{E}(\overline{y}_{RSS,MM3})-\overline{Y}$$

and

$$\overline{y}_{RSS,MM3} = \overline{Y}(1+\varepsilon_0)(1+\gamma_1\varepsilon_1)^{-1}, \qquad \gamma_1 = \frac{\overline{X}\beta_2(x)}{\overline{X}\beta_2(x)+C_x},$$

we suppose that $|\gamma_1\varepsilon_1| < 1$ such that $(1+\gamma_1\varepsilon_1)^{-1}$ is expandable and get

$$\overline{y}_{RSS,MM3} = \overline{Y}\left(1+\varepsilon_0\right)\left\{1-\gamma_1\varepsilon_1+\gamma_1^2\varepsilon_1^2+O(\gamma_1\varepsilon_1)\right\} .$$

Therefore,

$$\begin{aligned}
\text{bias}(\overline{y}_{RSS,MM3}) &= \overline{Y}\left(\lambda^2\text{E}(\varepsilon_1^2)-\lambda\text{E}(\varepsilon_0\varepsilon_1)\right) \\
&= \overline{Y}\left(\gamma_1^2\left\{\theta C_x^2-W_{x(i)}^2\right\}-\gamma_1\left\{\theta\rho_{yx}C_yC_x-W_{yx(i)}\right\}\right) \\
&= \overline{Y}\left(\theta(\gamma_1^2C_x^2-\gamma_1\rho_{yx}C_yC_x)-\{\gamma_1^2W_{x(i)}^2-\gamma_1W_{yx(i)}\}\right)
\end{aligned}$$

and

$$\begin{aligned}
\text{MSE}(\overline{y}_{RSS,MM3}) &= \text{E}(\overline{y}_{RSS,MM3}-\overline{Y})^2 \\
&= \overline{Y}^2\text{E}\left(\varepsilon_0-\gamma_1\varepsilon_1+\gamma_1^2\varepsilon_1^2-2\gamma_1\varepsilon_0\varepsilon_1\right)^2 \\
&= \overline{Y}^2\text{E}\left(\varepsilon_0^2+\gamma_1^2\varepsilon_1^2-2\gamma_1\varepsilon_0\varepsilon_1\right) \\
&= \overline{Y}^2\left(\theta C_y^2-W_{y[i]}^2+\lambda^2(\theta C_x^2-W_{x(i)}^2)-2\lambda\left(\theta\rho_{yx}C_yC_x-W_{yx(i)}\right)\right) \\
&= \overline{Y}^2\left(\theta\{C_y^2+\gamma_1^2C_x^2-2\gamma_1\rho_{yx}C_yC_x\}-\{W_{y[i]}^2+\gamma_1^2W_{x(i)}^2-2\gamma_1W_{yx(i)}\}\right) \\
&= \overline{Y}^2\left(\theta\{C_y^2+\gamma_1^2C_x^2-2\gamma_1\rho_{yx}C_yC_x\}-\{W_{y[i]}-\gamma_1W_{x(i)}\}^2\right) . \qquad (20)
\end{aligned}$$

Similarly, the bias and mean squared error of the estimator $\overline{y}_{RSS,MM4}$ can be obtained, respectively, by changing the place of coefficient of kurtosis and coefficient of variation, as

$$\text{bias}(\overline{y}_{RSS,MM4})=\overline{Y}\left(\theta(\gamma_2^2C_x^2-\gamma_2\rho_{yx}C_yC_x)-\{\gamma_2^2W_{x(i)}^2-\gamma_2W_{yx(i)}\}\right)$$

and

$$\text{MSE}(\overline{y}_{RSS,MM4})=\overline{Y}^2\left(\theta\{C_y^2+\gamma_2^2C_x^2-2\gamma_2\rho_{yx}C_yC_x\}-\{W_{y[i]}-\gamma_2W_{x(i)}\}^2\right) . \qquad (21)$$

By applying the power transformation on $\overline{y}_{RSS,MM3}$ and $\overline{y}_{RSS,MM4}$ given in (18) and (19), we now propose generalized estimators as

$$\overline{y}_{RSS,MM3(\alpha)} = \overline{y}_{[n]}\left(\frac{\overline{X}\beta_2(x)+C_x}{\overline{x}_{(n)}\beta_2(x)+C_x}\right)^{\alpha} \qquad (22)$$

$$\overline{y}_{RSS,MM4(\delta)} = \overline{y}_{[n]}\left(\frac{\overline{X}C_x+\beta_2(x)}{\overline{x}_{(n)}C_x+\beta_2(x)}\right)^{\delta} . \qquad (23)$$

The bias and MSE of the estimator $\overline{y}_{RSS,MM3(\alpha)}$ to the first degree of approximation, are obtained next. We have

$$\text{bias}(\overline{y}_{RSS,MM3(\alpha)})=\text{E}(\overline{y}_{RSS,MM3(\alpha)})-\overline{Y}$$

where

$$\begin{aligned}
\overline{y}_{RSS,MM3(\alpha)} &= \overline{Y}(1+\varepsilon_0)(1+\phi_1\varepsilon_1)^{-\alpha} \\
&= \overline{Y}\left((1+\varepsilon_0)\left\{1-\phi_1\alpha\varepsilon_1+\frac{\alpha(\alpha+1)}{2}\phi_1^2\varepsilon_1^2+O(\varepsilon_1)\right\}\right)
\end{aligned}$$

Thus

$$\begin{aligned}
\text{bias}(\overline{y}_{RSS,MM3(\alpha)}) &= \overline{Y}\left(\phi_1^2\frac{\alpha(\alpha+1)}{2}\left\{\theta C_x^2-W_{x(i)}^2\right\}-\phi_1\alpha\left\{\theta\rho_{yx}C_yC_x-W_{yx(i)}\right\}\right) \\
&= \frac{\overline{Y}}{2}\left(\phi_1\alpha\theta\left\{(\alpha+1)C_x^2-2\rho_{yx}C_yC_x\right\}-\phi_1\alpha\left\{(\alpha+1)W_{x(i)}^2-2W_{yx(i)}\right\}\right)
\end{aligned}$$

and

$$\begin{aligned}
\text{MSE}(\overline{y}_{RSS,MM3(\alpha)}) &= \text{E}(\overline{y}_{RSS,MM3(\alpha)} - \overline{Y})^2 \\
&= \overline{Y}^2 \text{E}\left(\varepsilon_0 - \phi_1\alpha\varepsilon_1 + \phi_1^2\frac{\alpha(\alpha+1)}{2}\varepsilon_1^2 - \phi_1\alpha\varepsilon_0\varepsilon_1\right)^2 \\
&= \overline{Y}^2 \text{E}\left(\varepsilon_0^2 + \phi_1^2\alpha^2\varepsilon_1^2 - 2\phi_1\alpha\varepsilon_0\varepsilon_1\right) \\
&= \overline{Y}^2\big(\theta C_y^2 - W_{y[i]}^2 + \phi_1^2\alpha^2(\theta C_x^2 - W_{x(i)}^2) - 2\phi_1\alpha\left(\theta\rho_{yx}C_yC_x - W_{yx(i)}\right)\big) \\
&= \overline{Y}^2\big(\theta\{C_y^2 + \alpha^2\phi_1^2C_x^2 - 2\alpha\phi_1\rho_{yx}C_yC_x\} - \{W_{y[i]} - \phi_1\alpha W_{x(i)}\}^2\big)\,. \quad (24)
\end{aligned}$$

Similarly, the bias and mean squared error of the estimator $\overline{y}_{RSS,MM4(\delta)}$ can be obtained, respectively, by changing the place of coefficient of kurtosis and coefficient of variation, as

$$\text{bias}(\overline{y}_{RSS,MM4(\delta)}) = \frac{\overline{Y}}{2}\left(\phi_2\delta\theta\left\{(\delta+1)C_x^2 - 2\rho_{yx}C_yC_x\right\} - \phi_2\delta\left\{(\delta+1)W_{x(i)}^2 - 2W_{yx(i)}\right\}\right)$$

and

$$\text{MSE}(\overline{y}_{RSS,MM4(\delta)}) = \overline{Y}^2\left(\theta\{C_y^2 + \delta^2\phi_2^2C_x^2 - 2\delta\phi_2\rho_{yx}C_yC_x\} - \{W_{y[i]} - \phi_2\delta W_{x(i)}\}^2\right)\,. \quad (25)$$

4 Optimality of α and δ

The optimum value of α to minimize the MSE of $\overline{y}_{RSS,MM3(\alpha)}$ can easily be found as zero of its derivative, i.e.

$$\frac{\partial}{\partial\alpha}\text{MSE}(\overline{y}_{RSS,MM3(\alpha)}) = 0\,,$$

which results in

$$\alpha = \frac{\theta\rho_{xy}C_xC_y - W_{yx(i)}}{\phi_1\left(\theta C_x^2 - W_{x(i)}^2\right)}$$

and because $\text{cov}(\overline{x}_{(n)}, \overline{y}_{[n]}) = \beta\text{var}\left(\overline{x}_{(n)}\right)$ this is equivalent to

$$\alpha = \rho_{xy}\frac{C_y}{\phi_1C_x}\,. \quad (26)$$

Similarly,

$$\delta = \rho_{xy}\frac{C_y}{\phi_2C_x}\,. \quad (27)$$

After substituting (26) and (27), respectively, in (24) and (25), we obtain the minimum mean squared error of the proposed estimators as

$$\begin{aligned}
\min\text{MSE}(\overline{y}_{RSS,MM3(\alpha)}) &= \min\text{MSE}(\overline{y}_{RSS,MM4(\delta)}) \\
&= \overline{Y}^2\left(\theta C_y^2(1-\rho_{yx}^2) - \{W_{y[i]} - KW_{x(i)}\}^2\right)\,,
\end{aligned}$$

where $K = \rho\frac{C_y}{C_x}$.

5 Efficiency Comparison

On comparing (7) to (12) with (15), (17), (20), (21), (24) and (25), respectively, we obtain

1. $\text{MSE}(\overline{y}_{SD}) - \text{MSE}(\overline{y}_{RSS,MM1}) = A_1 \geq 0$, where $A_1 = \left[W_{y[i]} - \lambda_1 W_{x(i)}\right]^2$
   $\Rightarrow \text{MSE}(\overline{y}_{RSS,MM1}) \leq \text{MSE}(\overline{y}_{SD})$

2. $\text{MSE}(\overline{y}_{SK}) - \text{MSE}(\overline{y}_{RSS,MM2}) = A_2 \geq 0$, where $A_2 = \left[W_{y[i]} - \lambda_2 W_{x(i)}\right]^2$
   $\Rightarrow \text{MSE}(\overline{y}_{RSS,MM2}) \leq \text{MSE}(\overline{y}_{SK})$

3. $\text{MSE}(\overline{y}_{UP1}) - \text{MSE}(\overline{y}_{RSS,MM3}) = A_3 \geq 0$, where $A_3 = \left[W_{y[i]} - \gamma_1 W_{x(i)}\right]^2$
   $\Rightarrow \text{MSE}(\overline{y}_{RSS,MM3}) \leq \text{MSE}(\overline{y}_{UP1})$

4. $\text{MSE}(\overline{y}_{UP2}) - \text{MSE}(\overline{y}_{RSS,MM4}) = A_4 \geq 0$, where $A_4 = \left[W_{y[i]} - \gamma_2 W_{x(i)}\right]^2$
   $\Rightarrow \text{MSE}(\overline{y}_{RSS,MM4}) \leq \text{MSE}(\overline{y}_{UP2})$

5. $\text{MSE}(\overline{y}_{UP1(\alpha)}) - \text{MSE}(\overline{y}_{RSS,MM3(\alpha)}) = A_5 \geq 0$, where $A_5 = \left[W_{y[i]} - \phi_1 \alpha W_{x(i)}\right]^2$
   $\Rightarrow \text{MSE}(\overline{y}_{RSS,MM3(\alpha)}) \leq \text{MSE}(\overline{y}_{UP1(\alpha)})$

6. $\text{MSE}(\overline{y}_{UP2(\delta)}) - \text{MSE}(\overline{y}_{RSS,MM4(\delta)}) = A_6 \geq 0$, where $A_6 = \left[W_{y[i]} - \phi_1 \delta W_{x(i)}\right]^2$
   $\Rightarrow \text{MSE}(\overline{y}_{RSS,MM4(\delta)}) \leq \text{MSE}(\overline{y}_{UP2(\delta)})$

It is easily seen that the MSE of the suggested estimators given in (14), (16), (18), (19), (22), and (23) are always smaller than those of the estimators given in (1) to (6), respectively, because A_1, A_2, A_3, A_4, A_5, and A_6 are all non-negative values. As a result, the proposed generalized estimators $\overline{y}_{RSS,MM3(\alpha)}$ and $\overline{y}_{RSS,MM4(\delta)}$ for the population mean using power transformation in ranked set sampling are more efficient than the usual estimators $\overline{y}_{UP1(\alpha)}$ and $\overline{y}_{UP2(\delta)}$.

For $\alpha = \delta = -1$, the generalized estimators given in (22) and (23) turn to product type estimators.

6 Numerical Illustration

To compare the efficiencies of the various estimators of our study, we take a population of size $N = 50$ (see page 1111 in the appendix of S. Singh, 2003). The example considers the data of agricultural loans outstanding of all operating banks in different states of the USA in 1997, where y is the real estate farm loans (study variable) in 1000 \$ and x is the non-real estate loans (auxiliary variable) in 1000 \$.

For the above population, the parameters are summarized as: $Y = 27771.73, X = 43908.12, \overline{Y} = 555.43, \overline{X} = 878.16, S_x^2 = 1176526, S_y^2 = 342021.5, C_x^2 = 1.5256, C_y^2 = 1.1086, \rho = 0.8038, \beta_2(x) = 1.9215, \lambda_1 = 0.9986, \lambda_2 = 0.9978, \gamma_1 = \phi_1 = 0.9993, \gamma_2 = \phi_2 = 0.9982$, and $K = 0.6852$.

From this population we have taken 100 ranked set samples with size $m = 4$ and number of cycles $r = 3$, so that $n = mr = 12$. For these 100 ranked set samples chosen, we have computed estimated MSE's of the proposed estimators $\overline{y}_{MM1,RSS}$, $\overline{y}_{MM2,RSS}$, $\overline{y}_{MM3,RSS}$, $\overline{y}_{MM4,RSS}$, $\overline{y}_{RSS,MM3(\alpha)}$, and $\overline{y}_{RSS,MM4(\delta)}$ which are given in Table 2. Table 1

Table 1: Estimated MSE's of various estimators using SRS (Note that $\overline{y}_{UP1(\alpha)} = \overline{y}_{UP2(\delta)}$)

Estimator	$\overline{y}_{SD}$	$\overline{y}_{SK}$	$\overline{y}_{UP1}$	$\overline{y}_{UP2}$	$\overline{y}_{UP1(\alpha)}$
MSE	19936.6	19313.5	14006.0	13925.8	10088.0

shows the MSE's of the estimators $\overline{y}_{SD}$, $\overline{y}_{SK}$, $\overline{y}_{UP1}$, $\overline{y}_{UP2}$, $\overline{y}_{UP1(\alpha)}$, and $\overline{y}_{UP2(\delta)}$, given in H. P. Singh et al. (2008).

7 Conclusion

On comparing Table 1 with Table 2 and Table 3 for the 100 ranked set samples, we see that the MSE's of the proposed estimators are smaller than those of the estimators given by previous authors. As a result, all the proposed new ratio type estimators $\overline{y}_{MM1,RSS}$, $\overline{y}_{MM2,RSS}$, $\overline{y}_{MM3,RSS}$, $\overline{y}_{MM4,RSS}$, $\overline{y}_{RSS,MM3(\alpha)}$ and $\overline{y}_{RSS,MM4(\delta)}$ for the population mean using RSS are more efficient than the respective estimators $\overline{y}_{SD}$, $\overline{y}_{SK}$, $\overline{y}_{UP1}$, $\overline{y}_{UP2}$, $\overline{y}_{UP1(\alpha)}$ and $\overline{y}_{UP2(\delta)}$ under SRS. Thus, if the coefficient of variation and the coefficient of kurtosis are known for the auxiliary variable, then these proposed estimators are recommended for use in practice.

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Table 2: Estimated MSE's of different new estimators using RSS. Note that $\overline{y}_{RSS,MM3(\alpha)} = \overline{y}_{RSS,MM4(\delta)}$

Estimator	$\overline{y}_{RSS,MM1}$	$\overline{y}_{RSS,MM2}$	$\overline{y}_{RSS,MM3}$	$\overline{y}_{RSS,MM4}$	$\overline{y}_{RSS,MM3(\alpha)}$
1	12414.7	12400.0	12476.7	12408.1	10075.5
2	12988.8	12971.6	13053.0	12981.0	10040.1
3	13167.9	13149.8	13232.9	13159.7	10026.6
4	13890.6	13868.7	13958.8	13880.4	9102.2
5	13576.7	13550.1	13649.2	13564.2	7036.4
6	11703.5	11671.2	11780.9	11688.1	2807.5
7	12834.1	12817.5	12897.8	12826.6	10058.4
8	10293.4	10286.5	10348.6	10290.7	10029.0
9	13714.0	13687.2	13786.5	13701.3	6141.9
10	13921.1	13897.2	13991.2	13909.9	8136.1
11	11171.5	11164.6	11226.8	11168.8	10028.7
12	13888.4	13864.4	13958.6	13877.2	9056.6
13	10645.2	10635.0	10703.3	10640.9	9910.5
14	13926.9	13904.4	13995.8	13916.4	9023.7
15	13397.1	13370.7	13469.4	13384.7	7522.3
16	13815.7	13795.5	13882.5	13806.3	8379.8
17	13640.3	13615.1	13711.4	13628.4	8460.5
18	13598.9	13579.8	13664.8	13590.1	9470.2
19	13868.4	13846.8	13936.6	13858.4	9368.3
20	13916.5	13894.1	13985.3	13906.1	9265.6
21	11283.3	11250.6	11361.0	11267.7	2399.1
22	13782.3	13761.0	13850.1	13772.4	9801.3
23	13881.2	13859.5	13949.2	13871.1	9009.6
24	13182.5	13165.6	13246.5	13174.8	9802.9
25	10544.8	10537.9	10600.0	10542.1	9292.4
26	13811.1	13790.2	13878.5	13801.4	9292.4
27	13551.3	13533.2	13616.3	13543.0	9127.4
28	13003.3	12987.2	13066.6	12996.1	9876.6
29	8886.9	8880.4	8941.6	8884.4	9466.6
30	13911.7	13887.4	13982.1	13900.3	7443.7
31	10588.9	10577.6	10647.9	10584.0	9781.1
32	12641.1	12625.7	12703.6	12634.2	10043.6
33	11778.5	11765.6	11838.9	11772.8	10078.9
34	12300.9	12285.2	12363.7	12293.8	10069.8
35	13838.6	13814.0	13909.3	13827.1	8646.4
36	11356.7	11347.0	11414.4	11352.7	10086.8
37	13758.1	13737.9	13825.0	13748.8	9256.5
38	13351.6	13332.7	13417.3	13342.8	9999.4
39	13434.6	13416.8	13499.4	13426.5	9500.7
40	13771.4	13745.3	13843.3	13759.1	6824.4
41	12928.9	12912.7	12992.3	12921.6	9975.0
42	12870.4	12851.9	12935.9	12861.9	10060.0
43	12576.9	12563.8	12637.6	12571.1	9825.2
44	13860.9	13839.0	13929.2	13850.7	9686.9
45	12544.7	12531.5	12605.4	12538.9	9866.8
46	13818.6	13793.1	13890.0	13806.6	7272.4
47	13776.8	13756.7	13843.6	13767.5	8917.3
48	9061.8	9058.5	9113.9	9060.9	9887.5
49	12958.0	12929.7	13032.0	12944.6	5995.0
50	13869.5	13848.5	13937.0	13859.7	8231.8

Table 3: Estimated MSE's of different new estimators using RSS. Note that $\overline{y}_{RSS,MM3(\alpha)} = \overline{y}_{RSS,MM4(\delta)}$

Estimator	$\overline{y}_{RSS,MM1}$	$\overline{y}_{RSS,MM2}$	$\overline{y}_{RSS,MM3}$	$\overline{y}_{RSS,MM4}$	$\overline{y}_{RSS,MM3(\alpha)}$
51	13357.4	13340.5	13421.3	13349.7	9365.7
52	7661.9	7654.3	7717.7	7658.9	8419.4
53	8235.4	8237.4	8282.7	8237.2	9983.6
54	10449.1	10440.2	10506.0	10445.4	9947.6
55	13073.1	13044.3	13147.5	13059.5	5575.9
56	1806.8	1815.9	1848.0	1812.1	7294.8
57	13933.8	13910.4	14003.4	13922.9	9173.3
58	13677.3	13658.4	13743.1	13668.6	8846.3
59	12227.8	12213.4	12289.6	12221.4	10088.0
60	13366.9	13339.9	13439.7	13354.2	7012.6
61	13674.2	13647.6	13746.6	13661.7	6768.0
62	13926.8	13902.9	13996.8	13915.6	7530.2
63	8494.8	8494.9	8543.9	8495.6	9941.9
64	13017.1	12988.7	13091.1	13003.7	6003.9
65	13875.4	13850.6	13946.3	13863.8	7538.5
66	13377.5	13349.7	13450.9	13364.4	6305.2
67	13931.9	13909.0	14001.1	13921.2	9662.0
68	13688.4	13669.7	13754.0	13679.8	8571.0
69	10301.1	10290.5	10359.6	10296.6	9706.8
70	9995.6	9992.5	10047.5	9994.9	10088.0
71	12745.8	12729.7	12809.0	12738.6	10055.7
72	13884.8	13863.1	13952.8	13874.7	8934.0
73	13850.5	13829.2	13918.3	13840.6	9174.8
74	13925.9	13902.1	13995.9	13914.8	8154.6
75	13586.8	13566.8	13653.5	13577.6	9883.3
76	13834.9	13813.7	13902.6	13825.1	9282.6
77	13141.6	13125.9	13204.4	13134.5	9522.3
78	12733.2	12705.0	12807.1	12719.9	6059.3
79	11764.0	11750.2	11825.2	11757.9	10035.3
80	13869.0	13847.5	13937.0	13859.1	9120.3
81	10669.4	10660.3	10726.5	10665.6	10005.9
82	13093.9	13076.3	13158.4	13085.9	10010.7
83	13618.9	13600.6	13684.1	13610.6	8857.9
84	10486.9	10482.1	10540.3	10485.3	10082.6
85	10540.6	10508.5	10617.9	10525.4	2269.6
86	1755.0	1771.5	1789.7	1764.0	8492.2
87	11823.1	11791.1	11900.2	11807.9	3092.4
88	12753.5	12723.8	12828.6	12739.5	4973.8
89	2060.9	2077.0	2096.0	2069.7	8607.7
90	13925.9	13903.2	13994.8	13915.3	9375.5
91	12665.4	12650.1	12727.9	12658.5	10032.8
92	13634.5	13607.9	13706.9	13621.9	6949.3
93	13935.8	13912.8	14005.0	13925.1	8337.9
94	9761.7	9758.5	9813.7	9760.9	10070.1
95	13854.0	13832.9	13921.6	13844.2	8828.4
96	13926.6	13902.9	13996.5	13915.5	8781.9
97	12045.1	12034.5	12103.5	12040.5	9916.0
98	9633.5	9628.9	9686.7	9632.0	9984.5
99	4198.5	4206.4	4240.8	4203.2	8710.8
100	8512.7	8509.2	8565.0	8511.7	9623.5

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