AUSTRIAN JOURNAL OF STATISTICS
Volume 41 (2012), Number 3, 197–210

Estimation of $P(Y < X)$ in a Four-Parameter Generalized Gamma Distribution

M. Masoom Ali[1], Manisha Pal[2] and Jungsoo Woo[3]

[1] Department of Mathematical Sciences, Ball State University, USA
[2] Department of Statistics, Calcutta University, India
[3] Department of Statistics, Yeungnam University, South Korea

Abstract: In this paper we consider estimation of $R = P(Y < X)$, when X and Y are distributed as two independent four-parameter generalized gamma random variables with same location and scale parameters. A modified maximum likelihood method and a Bayesian technique have been used to estimate R on the basis of independent samples. As the Bayes estimator cannot be obtained in a closed form, it has been implemented using importance sampling procedure. A simulation study has also been carried out to compare the two methods.

Zusammenfassung: In diesem Beitrag betrachten wir die Schätzung von $R = P(Y < X)$, wenn die beiden unabhängigen Zufallsvariablen X und Y aus einer Vier-Parameter generalisierten Gammaverteilung mit gleichen Lokations- und Skalen-Parametern stammen. Eine modifizierte Maximum-Likelihood Methode und eine Bayes-Technik wurden verwendet, um R auf der Grundlage von unabhängigen Stichproben zu schätzen. Da der Bayes-Schätzer nicht in geschlossener Form dargestellt werden kann, wurde dieser mittels einer Importance Sampling Prozedur implementiert. Eine Simulationsstudie wurde ebenfalls durchgeführt, um beide Methoden zu vergleichen.

Keywords: Generalized Gamma Distribution; Modified Maximum Likelihood Estimation; Profile Likelihood; Bayesian Estimation; Importance Sampling; HDP Intervals; Parametric Bootstrap Confidence Intervals.

1 Introduction

Stress-strength reliability is one of the main tools of reliability analysis of structures. A stress-strength system fails as soon as the applied stress Y is at least as large as its strength X. This model is also known as the load-capacity model in the context of solid mechanics or structural engineering. Inference regarding $P(Y < X)$, defining the reliability of the system, has been widely discussed in literature, when $X$and Y are assumed to be independent random variables. See, for example, Basu (1964), Downtown (1973), Tong (1974, 1977), Kelley, Kelley, and Suchany (1976), Beg (1980), Iwase (1987), McCool (1991), Ivshin (1996), Ali, Woo, and Pal (2004); Ali, Pal, and Woo (2005, 2010), Ali and Woo (2005a, 2005b), Pal, Ali, and Woo (2005), Raqab and Kundu (2005), and Raqab, Madi, and Kundu (2008). Besides, system reliability, $P(Y < X)$ finds importance in other fields too. For example, in biometry, suppose X represents a patient's remaining years of life when treated with drug A and Y represents the same when treated with drug

B. Then, if choice of drug is left to the patient, his deliberation will center on whether $P(Y < X)$ is less than or greater than $1/2$. In the context of statistical tolerance, if X denotes the diameter of a shaft and Y the diameter of a bearing that is to be mounted on the shaft, then the probability that the bearing fits without any interference is given by $P(Y < X)$. Hence, it is very important to consider inference on $P(Y < X)$.

In this paper, we consider the problem of estimating $R = P(Y < X)$, where X and Y are distributed independently as generalized gamma distributions. A four-parameter generalized gamma distribution may be defined as having the cumulative distribution function (cdf)

$$F(x;\alpha,\beta,\gamma,\theta) = \left[\int_\theta^x \frac{\beta^\alpha}{\Gamma(\alpha)}(u-\theta)^{\alpha-1}e^{-\beta(u-\theta)}du\right]^\gamma ,$$

and the probability density function (pdf)

$$f(x;\alpha,\beta,\gamma,\theta) = \frac{\beta^\alpha\gamma}{\Gamma(\alpha)}(x-\theta)^{\alpha-1}e^{-\beta(x-\theta)}\left[\int_\theta^x \frac{\beta^\alpha}{\Gamma(\alpha)}(u-\theta)^{\alpha-1}e^{-\beta(u-\theta)}du\right]^{\gamma-1} ,$$

for $x > \theta$ and $\alpha,\beta,\gamma > 0$. Here θ and β are the location and scale parameters, respectively, and (α,γ) are the shape parameters. We shall denote the distribution by GG$(\alpha,\beta,\gamma,\theta)$. For $\gamma = 1$ and $\theta = 0$ this distribution reduces to the standard two-parameter gamma distribution, whereas for $\alpha = 1$, it reduces to the three-parameter generalized exponential distribution studied by Gupta and Kundu (1999).

In this paper, we assume that $X \sim$ GG$(\alpha,\beta,\gamma_1,\theta)$ and $Y \sim$ GG$(\alpha,\beta,\gamma_2,\theta)$. It has been observed that the usual maximum likelihood estimator of the parameters may not exist. In Section 2, we study modified maximum likelihood estimators of the unknown parameters and hence of R. In Section 3, importance sampling is used to obtain Bayes estimates of the model parameters and of R. In Section 4, the procedures are illustrated by analyzing a simulated and a real data set. Finally, in Section 5 some simulation studies are provided and in Section 6 a discussion on our findings is given.

2 Modified Maximum Likelihood Estimation

Let $X = (X_1, X_2, \ldots, X_m)$ and $Y = (Y_1, Y_2, \ldots, Y_n)$ be independent random samples drawn from GG$(\alpha,\beta,\gamma_1,\theta)$ and GG$(\alpha,\beta,\gamma_2,\theta)$, respectively, and let the ordered observations in the two samples be $(X_{(1)} < X_{(2)} < \cdots < X_{(m)})$ and $(Y_{(1)} < Y_{(2)} < \cdots < Y_{(n)})$. Then, the likelihood function of $\rho = (\alpha,\beta,\gamma_1,\gamma_2,\theta)$ is

$$L(\rho|x,y) \propto \frac{\beta^{(m+n)\alpha}\gamma_1^m\gamma_2^n}{\Gamma^{m+n}(\alpha)}\prod_{i=1}^m(x_{(i)}-\theta)^{\alpha-1}\prod_{j=1}^n(y_{(j)}-\theta)^{\alpha-1}e^{D(\beta,\theta)+S(\alpha,\beta,\theta)}I_{\theta<w} ,$$

where

$$D(\beta,\theta) = -\beta\left[\sum_{i=1}^{m}(x_{(i)}-\theta)+\sum_{j=1}^{n}(y_{(j)}-\theta)\right]$$

$$S(\alpha,\beta,\theta) = (\gamma_1-1)\sum_{i=1}^{m}\log S_1(x_{(i)};\alpha,\beta,\theta)+(\gamma_2-1)\sum_{j=1}^{n}\log S_1(y_{(j)};\alpha,\beta,\theta) \quad (1)$$

$$S_1(x;\alpha,\beta,\theta) = \int_{\theta}^{x}\frac{\beta^{\alpha}}{\Gamma(\alpha)}(u-\theta)^{\alpha-1}e^{-\beta(u-\theta)}du$$

with $w=\min(x_{(1)},y_{(1)})$ and

$$I_{\theta<w} = \begin{cases} 1, & \text{if } \theta < w \\ 0, & \text{otherwise.} \end{cases} \quad (2)$$

Now, for $\beta > 0$ and $\alpha,\gamma_1,\gamma_2 < 1$, as θ approaches w, the likelihood function tends to ∞. This means that the MLEs of $\alpha,\beta,\gamma_1,\gamma_2$ do not exist. We therefore obtain modified MLEs of the unknown parameters using the procedure proposed by Raqab (2007).

Since the likelihood function is maximized at $\theta = w$, the modified MLE of θ is $\hat{\theta} = w$. The modified likelihood function of $\rho^* = (\alpha,\beta,\gamma_1,\gamma_2)$ is then defined as follows.

Case 1: $y_{(1)} < x_{(1)}$: Here the modified likelihood function is given by

$$L_{\mathrm{mod}}(\rho^*|x,y) \propto \frac{\beta^{(m+n-1)\alpha}\gamma_1^{m}\gamma_2^{n-1}}{\Gamma^{m+n-1}(\alpha)}\prod_{i=1}^{m}(x_{(i)}-y_{(1)})^{\alpha-1}\prod_{j=2}^{n}(y_{(j)}-y_{(1)})^{\alpha-1}e^{D_{\mathrm{mod}}(\beta)+S_{\mathrm{mod}}(\alpha,\beta)},$$

where

$$D_{\mathrm{mod}}(\beta) = -\beta\left[\sum_{i=1}^{m}(x_{(i)}-y_{(1)})+\sum_{j=2}^{n}(y_{(j)}-y_{(1)})\right]$$

$$S_{\mathrm{mod}}(\alpha,\beta) = (\gamma_1-1)\sum_{i=1}^{m}\log S_{1\mathrm{mod}}(x_{(i)};\alpha,\beta)+(\gamma_2-1)\sum_{j=2}^{n}\log S_{1\mathrm{mod}}(y_{(j)};\alpha,\beta)$$

$$S_{1\mathrm{mod}}(x;\alpha,\beta) = \int_{y_{(1)}}^{x}\frac{\beta^{\alpha}}{\Gamma(\alpha)}(u-y_{(1)})^{\alpha-1}e^{-\beta(u-y_{(1)})}du\,.$$

Then,

$$\begin{aligned}\log L_{\mathrm{mod}}(\rho^*|x,y) \propto & -(m+n-1)\log\Gamma(\alpha)+(m+n-1)\alpha\log\beta+m\log\gamma_1 \\ & +(n-1)\log\gamma_2+(\alpha-1)\left\{\sum_{i=1}^{m}\log(x_{(i)}-y_{(1)})+\sum_{j=2}^{n}\log(y_{(j)}-y_{(1)})\right\} \\ & +D_{\mathrm{mod}}(\beta)+S_{\mathrm{mod}}(\alpha,\beta)\,.\end{aligned}$$

The modified MLEs of γ_1 and γ_2 are obtained by solving the equations

$$\frac{\partial}{\partial\gamma_1}\log L_{\mathrm{mod}}(\rho^*|x,y)=0 \qquad \text{and} \qquad \frac{\partial}{\partial\gamma_2}\log L_{\mathrm{mod}}(\rho^*|x,y)=0\,,$$

which gives

$$\hat{\gamma}_1 = -\frac{1}{\frac{1}{m}\sum_{i=1}^{m}\log S_{1\text{mod}}(x_{(i)};\hat{\alpha},\hat{\beta})}\,, \qquad \hat{\gamma}_2 = -\frac{1}{\frac{1}{n-1}\sum_{j=2}^{n-1}\log S_{1\text{mod}}(y_{(j)};\hat{\alpha},\hat{\beta})}\,.$$

Here $\hat{\alpha}$ and $\hat{\beta}$ are the modified MLEs of α and β, respectively, satisfying the following non-linear equations, which are obtained by maximizing the modified profile likelihood function of (α, β), i.e.

$$\beta = g_1(\alpha,\beta) \tag{3}$$

$$\alpha = g_2(\alpha,\beta)\,, \tag{4}$$

where

$$\begin{aligned}
g_1(\alpha,\beta) &= \frac{(m+n-1)\alpha}{A_1(\alpha,\beta)}\,, \qquad g_2(\alpha,\beta) = 1+\exp(A_2(\alpha,\beta))\,,\\
A_1(\alpha,\beta) &= \sum_{i=1}^{m}(x_{(i)}-y_{(1)}) + \sum_{j=2}^{n}(y_{(j)}-y_{(1)})\\
&\quad -\frac{\beta^\alpha}{\Gamma(\alpha)}\left[(\hat{\gamma}_1-1)\sum_{i=1}^{m}\frac{(x_{(i)}-y_{(1)})^\alpha e^{-\beta(x_{(i)}-y_{(1)})}}{S_{1\text{mod}}(x_{(i)};\alpha,\beta)}\right.\\
&\qquad\left. + (\hat{\gamma}_2-1)\sum_{j=2}^{n}\frac{(y_{(j)}-y_{(1)})^\alpha e^{-\beta(y_{(j)}-y_{(1)})}}{S_{1\text{mod}}(y_{(j)};\alpha,\beta)}\right]\,,\\
A_2(\alpha,\beta) &= \frac{1}{m(\hat{\gamma}_1-1)+(n-1)(\hat{\gamma}_2-1)}\\
&\quad \left[\frac{\psi(\alpha)}{\Gamma(\alpha)}(m\hat{\gamma}_1+(n-1)\hat{\gamma}_2) - (m+n-1)\log\beta\right.\\
&\qquad \left. -\left\{\sum_{i=1}^{m}\log(x_{(i)}-y_{(1)}) + \sum_{j=2}^{n}\log(y_{(j)}-y_{(1)})\right\}\right]\\
\psi(\alpha) &= \frac{d}{d\alpha}\Gamma(\alpha)\,.
\end{aligned}$$

Case 2: $x_{(1)} < y_{(1)}$: Here the modified likelihood function is given by

$$L_{\text{mod}}(\rho^*|x,y) \propto \frac{\beta^{(m+n-1)\alpha}\gamma_1^{m-1}\gamma_2^{n}}{\Gamma^{m+n-1}(\alpha)}\prod_{i=2}^{m}(x_{(i)}-x_{(1)})^{\alpha-1}\prod_{j=1}^{n}(y_{(j)}-x_{(1)})^{\alpha-1}e^{D_{\text{mod}}(\beta)+S_{\text{mod}}(\alpha,\beta)}\,,$$

where

$$\begin{aligned}
D_{\text{mod}}(\beta) &= -\beta\left[\sum_{i=2}^{m}(x_{(i)}-x_{(1)}) + \sum_{j=1}^{n}(y_{(j)}-x_{(1)})\right]\\
S_{\text{mod}}(\alpha,\beta) &= (\gamma_1-1)\sum_{i=2}^{m}\log S_{1\text{mod}}(x_{(i)};\alpha,\beta) + (\gamma_2-1)\sum_{j=1}^{n}\log S_{1\text{mod}}(y_{(j)};\alpha,\beta)\\
S_{1\text{mod}}(x;\alpha,\beta) &= \int_{x_{(1)}}^{x}\frac{\beta^\alpha}{\Gamma(\alpha)}(u-x_{(1)})^{\alpha-1}e^{-\beta(u-x_{(1)})}du\,.
\end{aligned}$$

The modified MLE of ρ^* can be obtained in the same way as in Case 1. Here we get,

$$\hat{\gamma}_1 = -\frac{1}{\frac{1}{m-1}\sum_{i=2}^{m} \log S_{1\text{mod}}(x_{(i)}; \hat{\alpha}, \hat{\beta})}, \qquad \hat{\gamma}_2 = -\frac{1}{\frac{1}{n}\sum_{j=1}^{n-1} \log S_{1\text{mod}}(y_{(j)}; \hat{\alpha}, \hat{\beta})},$$

where $\hat{\alpha}$ and $\hat{\beta}$ are the modified MLEs of α and β, respectively, obtained by maximizing the modified profile likelihood function of (α, β), which give the following non-linear equations

$$\beta = g_1^*(\alpha, \beta), \qquad \alpha = g_2^*(\alpha, \beta), \tag{5}$$

with

$$g_1^*(\alpha, \beta) = \frac{(m+n-1)\alpha}{A_1^*(\alpha, \beta)}, \qquad g_2^*(\alpha, \beta) = 1 + \exp(A_2^*(\alpha, \beta)),$$

$$A_1^*(\alpha, \beta) = \sum_{i=2}^{m}(x_{(i)} - y_{(1)}) + \sum_{j=1}^{n}(y_{(j)} - y_{(1)}) - \frac{\beta^\alpha}{\Gamma(\alpha)}\left[(\hat{\gamma}_1 - 1)\sum_{i=2}^{m} \frac{(x_{(i)} - y_{(1)})^\alpha e^{-\beta(x_{(i)} - y_{(1)})}}{S_{1\text{mod}}(x_{(i)}; \alpha, \beta)} + (\hat{\gamma}_2 - 1)\sum_{j=1}^{n} \frac{(y_{(j)} - y_{(1)})^\alpha e^{-\beta(y_{(j)} - y_{(1)})}}{S_{1\text{mod}}(y_{(j)}; \alpha, \beta)}\right],$$

$$A_2^*(\alpha, \beta) = \frac{1}{(m-1)(\hat{\gamma}_1 - 1) + n(\hat{\gamma}_2 - 1)} \left[\frac{\psi(\alpha)}{\Gamma(\alpha)}\{(m-1)\hat{\gamma}_1 + n\hat{\gamma}_2\} - (m+n-1)\log\beta - \left\{\sum_{i=2}^{m}\log(x_{(i)} - y_{(1)}) + \sum_{j=1}^{n}\log(y_{(j)} - y_{(1)})\right\}\right].$$

The expression of R can be easily shown to be $R = \gamma_1/(\gamma_1 + \gamma_2)$. Hence, the modified MLE of R is given by

$$\hat{R} = \frac{\hat{\gamma}_1}{\hat{\gamma}_1 + \hat{\gamma}_2}.$$

It is difficult to obtain both the exact and the asymptotic distributions of $\hat{R}$, and thereby find a confidence interval for R. One may, however, use the parametric bootstrap technique proposed by Efron (1982) to get a confidence interval.

3 Bayesian Estimation

A natural and simple choice for the priors of $\alpha, \beta, \gamma_1, \gamma_2$ and θ is to assume that these are independently distributed as follows:

$$\alpha \sim \text{Exp}(a), \quad \beta \sim \text{Exp}(b), \quad \gamma_1 \sim \text{Exp}(c_1), \quad \gamma_2 \sim \text{Exp}(c_2),$$

and that θ has a truncated exponential distribution with pdf

$$h(\theta) = \xi \frac{e^{\xi(\theta-\theta_0)}}{1-e^{-\xi\theta_0}}\,, \qquad 0 < \theta < \theta_0\,, \quad \xi, \theta_0 > 0\,.$$

The prior parameters should be chosen so as to reflect the prior knowledge about $\alpha, \beta, \gamma_1, \gamma_2$ and θ.

3.1 Posterior Distribution

The joint distribution of X, Y, ρ has pdf

$$\begin{aligned} g(x,y,\rho) \propto abc_1c_2\xi \frac{\beta^{(m+n)\alpha}\gamma_1^m\gamma_2^n}{\Gamma^{m+n}(\alpha)(1-e^{-\xi\theta_0})} \prod_{i=1}^{m}(x_{(i)}-\theta)^{\alpha-1}\prod_{j=1}^{n}(y_{(j)}-\theta)^{\alpha-1} \\ \exp(D(\beta,\theta)+S(\alpha,\beta,\theta)-P(\alpha,\beta,\gamma_1,\gamma_2,\theta))I_{0<\theta<\theta_0}I_{\theta<w}\,, \end{aligned}$$

where $D(\beta,\theta)$, $S(\alpha,\beta,\theta)$ are given in (1) and and $I_{\theta<w}$ is defined in (2). Moreover,

$$\begin{aligned} P(\alpha,\beta,\gamma_1,\gamma_2,\theta) &= a\alpha + b\beta + c_1\gamma_1 + c_2\gamma_2 + \xi(\theta_0-\theta) \\ I_{0<\theta<\theta_0} &= \begin{cases} 1, & \text{if } 0<\theta<\theta_0 \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

The posterior distribution of ρ, given $X = x$ and $Y = y$ then comes out to be

$$\begin{aligned} \pi(\rho|x,y) = {} & g_\alpha\left(1, a-(m+n)\log\beta - \sum_{i=1}^{m}\log(x_i-\theta) - \sum_{j=1}^{n}\log(y_j-\theta)\right) \\ & g_\beta\left(1, b + \sum_{i=1}^{m}(x_i-w_0) + \sum_{j=1}^{n}(y_j-w_0)\right) \\ & g_{\gamma_1}\left(m+1, c_1 - \sum_{i=1}^{m}\log S_1(x_i;\alpha,\beta,\theta)\right) \\ & g_{\gamma_2}\left(n+1, c_2 - \sum_{j=1}^{n}\log S_1(y_j;\alpha,\beta,\theta)\right) \\ & h_\theta\left(\xi+(m+n)\beta, w_0\right)U(\alpha,\beta,\theta)\,, \end{aligned} \tag{6}$$

where

$$\begin{aligned} U(\alpha,\beta,\theta) = {} & \exp\left(-\sum_{i=1}^{m}\log(x_i-\theta) - \sum_{j=1}^{n}\log(y_j-\theta) - \sum_{i=1}^{m}\log S_1(x_i;\cdot) - \sum_{j=1}^{n}\log S_1(y_j;\cdot)\right) \\ & \left\{(\xi+(m+n)\beta)\Gamma^{m+n}(\alpha)\right\}^{-1}\left(a-(m+n)\log\beta - \sum_{i=1}^{m}\log(x_i-\theta) - \sum_{j=1}^{n}\log(y_j-\theta)\right)^{-1} \\ & \left\{c_1 - \sum_{i=1}^{m}\log S_1(x_i;\cdot)\right\}^{-m-1}\left\{c_2 - \sum_{j=1}^{n}\log S_1(y_j;\cdot)\right\}^{-n-1}\,, \end{aligned}$$

and $w_0 = \min(\theta_0, w)$, $g_Z(t,s)$ is the pdf of a gamma variable Z with shape and scale parameters t and s, respectively, and $h_Z(t,s)$ is the pdf of a truncated exponential variable Z with parameters t and s.

The posterior joint density of (α, β, θ) is obtained by integrating (6) over (γ_1, γ_2) and is given by

$$\begin{aligned}\pi_0(\alpha,\beta,\theta|x,y) = {} & g_\alpha\left(1, a-(m+n)\log\beta-\sum_{i=1}^{m}\log(x_i-\theta)-\sum_{j=1}^{n}\log(y_j-\theta)\right)\\ & g_\beta\left(1, b+\sum_{i=1}^{m}(x_i-w_0)+\sum_{j=1}^{n}(y_j-w_0)\right)\\ & h_\theta\left(\xi+(m+n)\beta, w_0\right)U(\alpha,\beta,\theta)\,.\end{aligned}$$

And the marginal posterior distributions of γ_1 and γ_2, given α, β, θ, are respectively Gamma$\left(m+1, c_1-\sum_{i=1}^{m}\log S_1(x_i;\alpha,\beta,\theta)\right)$ and Gamma$\left(n+1, c_2-\sum_{j=1}^{n}\log S_1(y_j;\alpha,\beta,\theta)\right)$.

3.2 Posterior Expectation

For any continuous function $k(\cdot)$ of ρ, the posterior expectation is given by

$$\begin{aligned}\mathrm{E}(k(\rho)|x,y) = H^{-1} & \int\!\!\int\!\!\int\!\!\int\!\!\int k(\alpha,\beta,\gamma_1,\gamma_2,\theta)U(\alpha,\beta,\theta)h_\theta(\xi+(m+n)\beta,w_0)\\ & g_\alpha\left(1, a-(m+n)\log\beta-\sum_{i=1}^{m}\log(x_i-\theta)-\sum_{j=1}^{n}\log(y_j-\theta)\right)\\ & g_\beta\left(1, b+\sum_{i=1}^{m}(x_i-w_0)+\sum_{j=1}^{n}(y_j-w_0)\right)\\ & g_{\gamma_1}\left(m+1, c_1-\sum_{i=1}^{m}\log S_1(x_i;\alpha,\beta,\theta)\right)\\ & g_{\gamma_2}\left(n+1, c_2-\sum_{j=1}^{n}\log S_1(y_j;\alpha,\beta,\theta)\right)d\gamma_1 d\gamma_2 d\alpha d\theta d\beta\,,\end{aligned}$$

where

$$\begin{aligned}H = \int\!\!\int\!\!\int & U(\alpha,\beta,\theta)h_\theta(\xi+(m+n)\beta,w_0)\\ & g_\alpha\left(1, a-(m+n)\log\beta-\sum_{i=1}^{m}\log(x_i-\theta)-\sum_{j=1}^{n}\log(y_j-\theta)\right)\\ & g_\beta\left(1, b+\sum_{i=1}^{m}(x_i-w_0)+\sum_{j=1}^{n}(y_j-w_0)\right)d\alpha d\theta d\beta\,.\end{aligned}$$

Therefore, we get

$$\mathrm{E}(k(\rho)|x,y) = \frac{\mathrm{E}_1(k(\alpha,\beta,\gamma_1,\gamma_2,\theta)U(\alpha,\beta,\theta))}{\mathrm{E}_1(U(\alpha,\beta,\theta))}\,, \tag{7}$$

where $\mathrm{E}_1(\cdot)$ denotes the expectation under

$$\begin{aligned}
\gamma_1 &\sim \text{Gamma}\left(m+1, c_1 - \sum_{i=1}^{m} \log S_1(x_i;\alpha,\beta,\theta)\right) \\
\gamma_2 &\sim \text{Gamma}\left(n+1, c_2 - \sum_{j=1}^{n} \log S_1(y_j;\alpha,\beta,\theta)\right) \\
\alpha &\sim \text{Gamma}\left(1, a - (m+n)\log\beta - \sum_{i=1}^{m}\log(x_i-\theta) - \sum_{j=1}^{n}\log(y_j-\theta)\right) \\
\theta &\sim \text{truncated exponential}\left(\xi + (m+n)\beta, w_0\right) \\
\beta &\sim \text{Gamma}\left(1, b + \sum_{i=1}^{m}(x_i - w_0) + \sum_{j=1}^{n}(y_j - w_0)\right)\,.
\end{aligned} \tag{8}$$

Hence, to find the posterior expectation of a function, we can use the following general importance sampling procedure:

Step 1: Generate β from the Gamma$(1, b + \sum_{i=1}^{m}(x_i - w_0) + \sum_{j=1}^{n}(y_j - w_0))$.

Step 2: For β obtained in step 1, generate θ from the truncated exponential$(\xi + (m+n)\beta, w_0)$.

Step 3: For β and θ obtained in steps 1 and 2, generate α from Gamma$(1, a - (m+n)\log\beta - \sum_{i=1}^{m}\log(x_i-\theta) - \sum_{j=1}^{n}\log(y_j-\theta))$.

Step 4: For the values of β, θ and α obtained, generate γ_1 from Gamma$(m+1, c_1 - \sum_{i=1}^{m}\log S_1(x_i;\alpha,\beta,\theta))$ and γ_2 from Gamma$(n+1, c_2 - \sum_{j=1}^{n}\log S_1(y_j;\alpha,\beta,\theta))$.

Step 5: From steps 1 to 4 compute (7) by averaging the numerator and denominator with respect to the simulations.

3.3 Highest Probability Density Intervals

A Monte Carlo method has been developed by Chen and Shao (1999) for using importance sampling to compute highest probability density (HPD) intervals for parameters and any function of them. The method can be used to find HPD intervals for the model parameters $\alpha, \beta, \gamma_1, \gamma_2, \theta$ and also for R.

Let $\lambda = p(\rho)$ be a continuous function of ρ. To find a HPD interval for λ, let ρ_i, $i = 1, 2, \ldots, q$ denote a sample of size q from the importance sampling distribution (8), where $\rho_i = (\alpha_i, \beta_i, \gamma_{1i}, \gamma_{2i}, \theta_i)$. Let the corresponding sample for λ be λ_i, $i = 1, 2, \ldots, q$ and let the ordered sample observations be denoted by $\lambda_{(1)} < \lambda_{(2)} < \cdots < \lambda_{(q)}$. Suppose

$\rho_{(i)} = (\alpha_{(i)}, \beta_{(i)}, \gamma_{1(i)}, \gamma_{2(i)}, \theta_{(i)})$ denotes the observation on ρ corresponding to $\lambda_{(i)}$, $i = 1, 2, \ldots, q$.

We compute

$$v_i = \frac{U(\alpha_{(i)}, \beta_{(i)}, \theta_{(i)})}{\sum_{j=1}^{q} U(\alpha_{(j)}, \beta_{(j)}, \theta_{(j)})}, \qquad i = 1, 2, \ldots, q .$$

Then, for q sufficiently large, the $100(1-\gamma)$ % HPD interval for λ is given by the shortest interval among $I_j(q)$, $j = 1, 2, \ldots, \gamma q$, where

$$I_j(q) = \left(\hat{\lambda}^{(j/q)}, \hat{\lambda}^{(j/q+(1-\gamma))}\right),$$

$\hat{\lambda}^{(\delta)}$ is an estimate of the δ-th quantile of λ and is given by

$$\hat{\lambda}^{(\delta)} = \begin{cases} \lambda_{(1)} & \text{if } \delta = 0 \\ \lambda_{(i)} & \text{if } \sum_{j=1}^{i-1} v_j \le \delta \le \sum_{j=1}^{i} v_j . \end{cases}$$

We can find HPD intervals for $\alpha, \beta, \gamma_1, \gamma_2, \theta$ and R in this way.

4 Data Analysis

In this section we illustrate the procedures discussed by analyzing simulated data sets and real life data sets.

Example 1: We generate data sets of 20 observations each from the distributions GG($\alpha, \beta, \gamma_1, \theta$) and GG($\alpha, \beta, \gamma_2, \theta$) with $\alpha = 1.15$, $\beta = 0.5$, $\theta = 1$, $\gamma_1 = 2.5$, $\gamma_2 = 1.5$.

Here the true value of R is 0.6250. The modified MLEs of the unknown parameters are obtained as $\hat{\theta} = 1.09$, $\hat{\alpha} = 1.0228$, $\hat{\beta} = 0.4563$, $\hat{\gamma}_1 = 1.8379$, $\hat{\gamma}_2 = 1.33588$. Hence, the modified MLE of R is $\hat{R} = 0.5791$. The 95 % parametric bootstrap confidence interval of R has been computed using 1000 bootstrap samples and it came out to be $(0.4027, 0.7401)$.

To find the Bayes estimates, we take $a = b = c_1 = c_2 = 1$, $\xi = 1$, $\theta_0 = 2$. Forty thousand simulated values of $\theta, \alpha, \beta, \gamma_1$ and γ_2 are used to implement the importance sampling procedure. The Bayes estimates of the parameters have been obtained as $\tilde{\theta} = 0.9250$, $\tilde{\alpha} = 1.0982$, $\tilde{\beta} = 0.4705$, $\tilde{\gamma}_1 = 2.462$, $\tilde{\gamma}_2 = 1.4108$, and $\tilde{R} = 0.6357$. The 95 % HPD interval came out to be $(0.4320, 0.7664)$.

In order to check which estimation procedure gives better fit to the given data sets, we have computed the Kolmogorov-Smirnov (K-S) distances between the empirical and the fitted distribution function, based on the modified MLEs and on the Bayes estimators and tested at a 5 % level of significance. For data set 1, The K-S distance based on modified MLEs (Bayes estimates) is 0.2217 (0.1523) and the corresponding p-value is 0.391 (0.629). Similarly, for data set 2, the K-S distance based on modified MLEs (Bayes estimates) is 0.1636 (0.2011) and the corresponding p-value is 0.716 (0.322). Thus, for data set 1, Bayes estimates provide better fit than modified MLEs while for data set 2, modified MLEs give better fit than Bayes estimates.

Example 2: Here we analyze the strength data, reported by Badar and Priest (1982), using the generalized gamma distribution. Estimates of the unknown parameters, and hence of R, are obtained by both the methods discussed. It may be noted that Raqab et al. (2008) fitted the 3-parameter generalized exponential distribution to the same data set.

Badar and Priest (1982) reported strength data measured in GPA for single carbon fibre and impregnated 1000 carbon fibre tows. Single fibres were tested at gauge lengths of 1, 10, 20 and 50 mm. Impregnated tows of 1000 fibres were tested at gauge lengths of 20, 50, 150 and 300 mm. The transformed data sets that were considered by Raqab and Kundu (2005) are used here. Data Set 1 (of size 69) and Data Set 2 (of size 63) correspond to single fibre with 20 mm and 10 mm of gauge length, respectively.

Data Set 1 (x:) 0.0312, 0.314, 0.479, 0.552, 0.700, 0.803, 0.861, 0.865, 0.944, 0.958, 0.966, 0.977, 1.006, 1.021, 1.027, 1.055, 1.063, 1.098, 1.140, 1.179, 1.224, 1.240, 1.253, 1.270, 1.272, 1.274, 1.301, 1.301, 1.359, 1.382, 1.382, 1.426, 1.434, 1.435, 1.478, 1.490, 1.511, 1.514, 1.535, 1.554, 1.566, 1.570, 1.586, 1.629, 1.633, 1.642, 1.648, 1.684, 1.697, 1.726, 1.770, 1.773, 1.800, 1.809, 1.818, 1.821, 1.848, 1.880, 1.954, 2.012, 2.067, 2.084, 2.090, 2.096, 2.128, 2.233, 2.433, 2.585, 2.585.

Data Set 2 (y:) 0.101, 0.332, 0.403, 0.428, 0.457, 0.550, 0.561, 0.596, 0.597, 0.645, 0.954, 0.674, 0.718, 0.722, 0.725, 0.732, 0.775, 0.814, 0.816, 0.818, 0.824, 0.859, 0.875, 0.938, 0.940, 1.056, 1.117, 1.128, 1.137, 1.137, 1.177, 1.196, 1.230, 1.325, 1.339, 1.345, 1.420, 1.423, 1.435, 1.443, 1.464, 1.472, 1.494, 1.532, 1.546, 1.577, 1.608, 1.635, 1.693, 1.701, 1.737, 1.754, 1.762, 1.828, 2.052, 2.071, 2.086, 2.171, 2.224, 2.227, 2.425, 2.595, 3.220.

Here, the modified MLE of θ is $\hat{\theta} = y_{(1)} = 0.101$. To find the MLEs of α and β, we carry out an iterative procedure as follows: Taking a starting value of α as 1, we solve (3) to get β. Then, using that value of β in (4), we solve for α. The procedure is continued till the values of α and β converge. We obtain the MLEs as $\hat{\alpha} = 1.7250$, $\hat{\beta} = 2.871$, and therefore, $\hat{\gamma}_1 = 3.5439$, $\hat{\gamma}_2 = 7.5672$. Hence, the MLE of R is $\hat{R} = 0.3189$.

To find the Bayes estimates, we take the prior parameters as $a = b = c_1 = c_2 = 1$, $\xi = 1$, $\theta_0 = 2$, $\theta_0 = 2$, $\xi = 1$. Based on these priors and implementing the importance sampling procedure using forty thousand simulated values of $\theta, \alpha, \beta, \gamma_1$ and γ_2, the Bayes estimates are obtained as $\tilde{\alpha} = 1.2573$, $\tilde{\beta} = 2.145$, $\tilde{\gamma}_1 = 1.2062$, $\tilde{\gamma}_2 = 3.0242$, $\tilde{\theta} = 0.0867$ and $\tilde{R} = 0.2852$. Further, the 95 % HPD credible interval of R is $(0.2423, 0.3865)$.

To examine which set of parameter estimates gives better fit to the data sets, we compute the K-S distance between the empirical and the fitted distributions based on the modified MLEs and the Bayes estimators and test at a 5 % level of significance. For data set 1, the p-value comes out to be 0.3003 (0.0225) for the modified MLEs (Bayes estimators), and for data set 2, the p-value is 0.7123 (0.4476) for the modified MLEs (Bayes estimators). Hence, the modified MLEs give a better fit than the Bayes estimates.

The following figures show the plots of the empirical survival functions and the fitted survival functions. The plots also indicate that the modified maximum likelihood method of estimation provides better fit than the Bayes method of estimation.

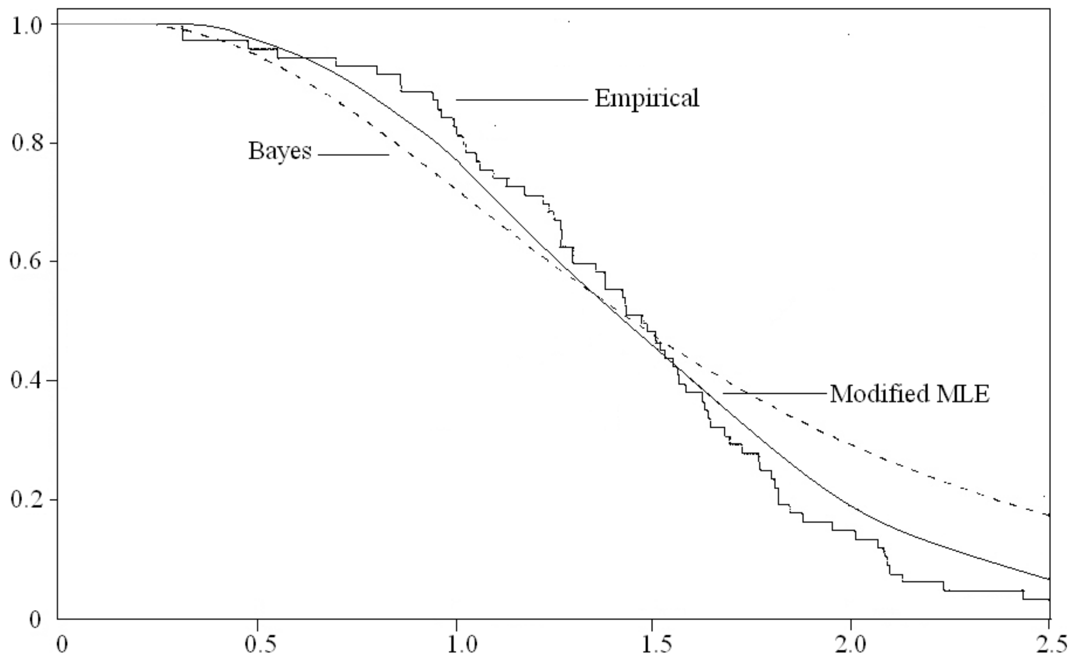

Figure 1: Empirical survival function and the fitted survival functions for Data Set 1.

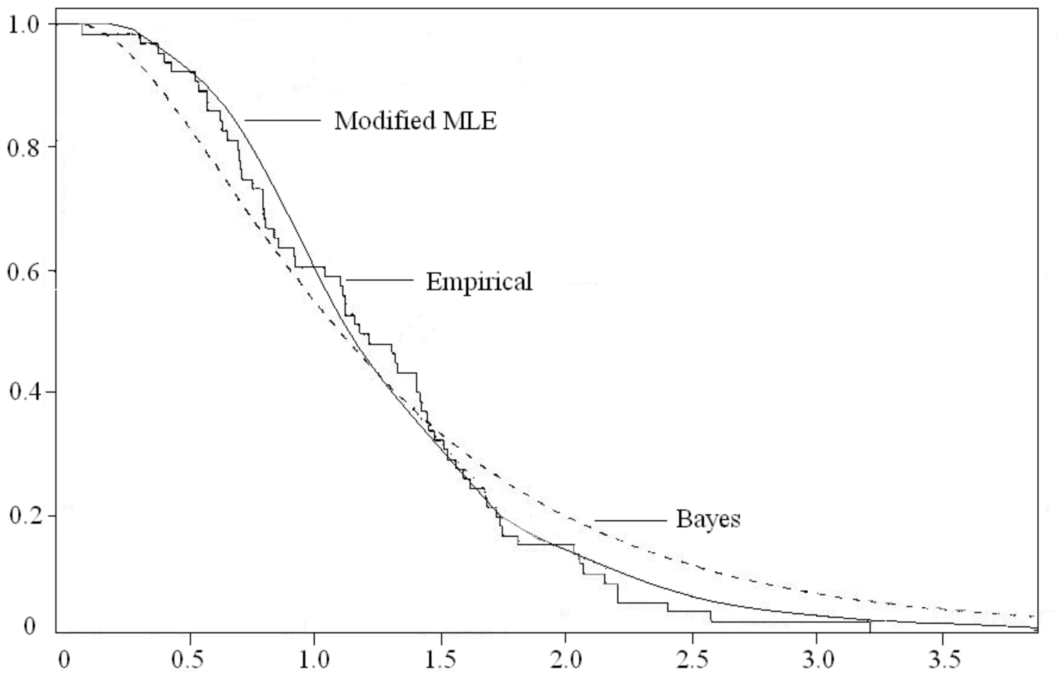

Figure 2: Empirical survival function and the fitted survival functions for Data Set 2.

5 A Monte Carlo Simulation Study

A simulation study has been carried out to compare the two methods of estimation used. We take parameter values to be $\gamma_1, \gamma_2 = 0.5, 1.0, 1.5$ and 2.0. Without loss of generality, we have taken $\theta = 0$, $\alpha = 1.5$ and $\beta = 1.0$. We consider sample sizes to be (m, n) = (10, 10), (20, 20), (40, 40). For a particular set of parameters and from a given generated sample, we compute the modified MLEs and Bayes estimators of R and replicate the process 1000 times. For the Bayes estimator of R, we have taken small values of the exponential hyper parameters to reflect vague prior information, viz. $a = b = c_1 = c_2 = 1$. We also assumed that $\xi = 1$ and $\theta_0 = 2$. Forty thousand simulated values of $\theta, \alpha, \beta, \gamma_1$ and γ_2 are used to implement the importance sampling procedure. Next we compute the mean squared errors in each case. The results are reported in Table 1 to 4.

Table 1: MSEs of $\hat{R}$ and $\tilde{R}$ when $\gamma_1 = 0.5$.

γ_2	0.5		1.0		1.5		2.0	
(m, n)	$\hat{R}$	$\tilde{R}$	$\hat{R}$	$\tilde{R}$	$\hat{R}$	$\tilde{R}$	$\hat{R}$	$\tilde{R}$
(10, 10)	0.0129	0.0142	0.0127	0.0139	0.0111	0.0119	0.0098	0.0121
(20, 20)	0.0078	0.0950	0.0086	0.0100	0.0079	0.0072	0.0068	0.0820
(40, 40)	0.0055	0.0610	0.0053	0.0068	0.0036	0.0057	0.0037	0.0500

Table 2: MSEs of $\hat{R}$ and $\tilde{R}$ when $\gamma_1 = 1.0$.

γ_2	0.5		1.0		1.5		2.0	
(m, n)	$\hat{R}$	$\tilde{R}$	$\hat{R}$	$\tilde{R}$	$\hat{R}$	$\tilde{R}$	$\hat{R}$	$\tilde{R}$
(10, 10)	0.0138	0.0175	0.0135	0.0143	0.0279	0.0314	0.0245	0.0224
(20, 20)	0.0085	0.0113	0.0109	0.0098	0.0126	0.0131	0.0087	0.0104
(40, 40)	0.0067	0.0072	0.0064	0.0074	0.0091	0.0102	0.0059	0.0068

Table 3: MSEs of $\hat{R}$ and $\tilde{R}$ when $\gamma_1 = 1.5$.

γ_2	0.5		1.0		1.5		2.0	
(m, n)	$\hat{R}$	$\tilde{R}$	$\hat{R}$	$\tilde{R}$	$\hat{R}$	$\tilde{R}$	$\hat{R}$	$\tilde{R}$
(10, 10)	0.0113	0.0147	0.0108	0.0210	0.0102	0.0202	0.0107	0.0124
(20, 20)	0.0084	0.0099	0.0093	0.0112	0.0087	0.0124	0.0086	0.0093
(40, 40)	0.0063	0.0075	0.0058	0.0079	0.0069	0.0056	0.0059	0.0051

Table 4: MSEs of $\hat{R}$ and $\tilde{R}$ when $\gamma_1 = 2.0$.

γ_2	0.5		1.0		1.5		2.0	
(m, n)	$\hat{R}$	$\tilde{R}$	$\hat{R}$	$\tilde{R}$	$\hat{R}$	$\tilde{R}$	$\hat{R}$	$\tilde{R}$
(10, 10)	0.0150	0.0175	0.0113	0.0109	0.0243	0.0314	0.0162	0.0234
(20, 20)	0.0086	0.0096	0.0091	0.0098	0.0125	0.0213	0.0097	0.0120
(40, 40)	0.0056	0.0076	0.0067	0.0072	0.0079	0.0089	0.0064	0.0077

It is observed that for both the methods of estimation, as the sample sizes increase the MSEs decrease for all sets of parameters considered. However, though in most of the cases the MSE is lower in modified maximum likelihood method than in Bayes method, it is not consistently so. Thus, it is not possible to conclude that the modified maximum likelihood method of estimation always gives better fit than the Bayes method of estimation.

6 Discussion

The paper studies the estimation of $R = P(Y < X)$ when X and Y have independent four-parameter generalized gamma distributions. It is seen that the usual maximum likelihood estimators of the distribution parameters may not exist. Thus, a modified maximum likelihood procedure has been used for parameter estimation. Further, Bayesian estimation with importance sampling procedure has been employed to estimate the model parameters and hence R. Simulated data sets and real-life data sets have been analyzed using the two methods of estimation. Also, a simulation study has been conducted to compare the two methods of estimation. It may be noted that the maximum likelihood method is a classical approach to estimation of parameters, while Bayes method is advised when one has informative priors. The present paper uses both the methods of estimation with the intention of studying how the estimators can be obtained in a complex situation as discussed in the paper.

Acknowledgement

The authors thank the anonymous referee for his fruitful suggestions, which immensely helped to improve the presentation of the paper.

References

Ali, M., Pal, M., and Woo, J. (2005). Inference on $P(Y < X)$ in generalized uniform distributions. *Calcutta Statistical Association Bulletin*, *57*, 35-48.

Ali, M., Pal, M., and Woo, J. (2010). Estimation of $Pr(Y < X)$ when X and Y belong to different distribution families. *Journal of Probability and Statistical Science*, *8*, 35-48.

Ali, M., and Woo, J. (2005a). Inference on reliability $P(Y < X)$ in a p-dimensional Raleigh distribution. *Mathematical and Computer Modelling*, *42*, 367-373.

Ali, M., and Woo, J. (2005b). Inference on $P(Y < X)$ in a Pareto distribution. *Journal of Modern Applied Statistical Methods*, *4*, 583-586.

Ali, M., Woo, J., and Pal, M. (2004). Inference on reliability $P(Y < X)$ in two-parameter exponential distributions. *International Journal of Statistical Sciences*, *3*, 119-125.

Badar, M. G., and Priest, A. M. (1982). Statistical aspects of fiber and bundle strength in hybrid composites. In T. Hayashi, K. Kawata, and S. Umekawa (Eds.), *Progress in Science and Engineering Composites, ICCM-IV, Tokyo (1982)* (p. 1129-1136).

Basu, A. P. (1964). Estimates of reliability for some distributions useful in reliability. *Technometrics*, *6*, 215-219.

Beg, M. A. (1980). On the estimation of $P(Y < X)$ for two-parameter exponential distribution. *Metrika*, *27*, 29-34.

Chen, M. H., and Shao, Q. M. (1999). Estimation of Bayesian credible intervals and HDP intervals. *Journal of Computational and Graphical Statistics*, *8*, 69-92.

Downtown, F. (1973). The estimation of $\Pr\{Y < X\}$ in the normal case. *Technometrics*, *15*, 551-558.

Efron, B. (1982). *The Jackknife, the Bootstrap and Other Resampling Plans*. Philadelphia, Pennsylvania, USA: Society for Industrial and Applied Mathematics.

Ivshin, V. V. (1996). Unbiased estimators of $P(X < Y)$ and their variances in the case of uniform and two-parameter exponential distributions. *Journal of Mathematical Sciences*, *81*, 2790-2793.

Iwase, K. (1987). On UMVU estimators of $\Pr(Y < X)$ in the 2-parameter exponential case. *Memoirs of the Faculty of Engineering, Hiroshima University*, *9*, 21-24.

Kelley, G. D., Kelley, J. A., and Suchany, W. R. (1976). Efficient estimation of $P(Y < X)$ in the exponential case. *Technometrics*, *18*, 359-360.

McCool, J. I. (1991). Inference on $P(X < Y)$ in the Weibull case. *Communications in Statistcs – Simulation and Computation*, *20*, 129-148.

Pal, M., Ali, M., and Woo, J. (2005). Estimation and testing of $P(Y < X)$ in two-parameter exponential distributions. *Statistics*, *39*, 415-428.

Raqab, M. Z., and Kundu, D. (2005). Comparison of different estimators of $P(Y < X)$ for a scaled Burr Type X distribution. *Communications in Statistcs – Simulation and Computation*, *34*, 465-483.

Raqab, M. Z., Madi, M. T., and Kundu, D. (2008). Estimation of $P(Y < X)$ for a 3-parameter generalized exponential distribution. *Communications in Statistcs – Theory and Methods*, *37*, 2854-2864.

Tong, H. (1974). A note on the estimation of $\Pr(Y < X)$ in the exponential case. *Technometrics*, *16*, 625.

Tong, H. (1977). On the estimation of $\Pr(Y < X)$ for exponential families. *IEEE Transactions on Reliability*, *R-26*, 54-56.

Authors' addresses:

M. Masoom Ali

Department of Mathematical Sciences
Ball State University
USA

Manisha Pal

Department of Statistics
Calcutta University
India

Jungsoo Woo

Department of Statistics
Yeungnam University
South Korea