Austrian Journal of Statistics
*July 2021, Volume 50, 13–31.*
*http://www.ajs.or.at/*
doi:10.17713/ajs.v50i3.857

# Integration of Multivariate Loss Function Approach in the Hotelling's Charts under Banerjee and Rahim (1988) Weibull Shock Models

**M. H. Naderi**
Allameh Tabataba'i University

**A. Seif**
Bu-Ali Sina University

**M. Bameni Mogahadm**
Allameh Tabataba'i University

**Abstract**

A proper monitoring of stochastic systems is the control charts of statistical process control and drift in characteristics of output may be due to one or several assignable causes. Although many research works have been done on the economic design of control charts with single assignable cause, the economic statistical design of $T^2$ control chart under Weibull shock model with multiple assignable causes and considering multivariate Taguchi loss function has not been presented yet. Using Taguchi loss function in the concept of quality control charts with economic and economic statistical design leads to better decisions in the industry. Based on the optimization of the average cost per unit of time and taking into account the different combination values of Weibull distribution parameters, optimal design values of sample size, sampling interval and control limit coefficient were derived and calculated. Then the cost models under non-uniform and uniform sampling scheme were compared. The results revealed that the model under multiple assignable causes with Taguchi loss function has a lower cost than single assignable cause model and integrated model with non-uniform sampling has a lower cost than that with uniform sampling.

*Keywords*: economic statistical design, $T^2$ control chart, multiple assignable causes, Weibull shock model, Taguchi loss function.

## 1. Introduction

The control charts technique for monitoring the process behavior is one of the basic tools of statistical process control (SPC). These charts are to distinguish between non-random and random variation where non-random variations cause a process to go out of control. Designing a control chart means to find the optimal values for three design parameters, namely, sample size, sampling interval and control limit coefficient which is done under three types of statistical, economic, and economic statistical designs. In the statistical designs, only statistical criteria are considered and in the economic designs, only the cost is important. A better alternative may be the economic statistical designs provided by Saniga (1989) in which both statistical and economic criteria are considered.

Most of the research works, like Duncan (1956) as the pioneer and then followers presented an economic design of $\overline{X}$ control charts with only one assignable cause. However, Duncan (1971), Gibra (1981), Tagaras and Lee (1988), Chung (1991), Chen and Yang (2002), Yang, Su, and Pearn (2010), Ahmed, Sultana, Paul, and Azeem (2014) presented an economic design of $\overline{X}$ control charts with multiple assignable causes.

Economic and economic statistical design needs a probability distribution for the process failure mechanism (PFM). In most of the first research works, Exponential distribution that has a fixed failure rate were used Duncan (1956). Then other distributions such as Weibull, Gamma, Pareto, Generalized exponential, Burr 12 and…, is used as a failure mechanism Banerjee and Rahim (1988); Al-Oraini and Rahim (2002); Kraleti and Kambagowni (2010); Moghadam, Moghadam, Rafie, and Naderi (2016); Heydari, Moghadam, and Eskandari (2016). Banerjee and Rahim (1988) Extended Duncan (1956) by using the more flexible Weibull distribution for single assignable cause model and non-uniform sampling interval by considering the fact that using uniform sampling interval is not logical for the process with increasing failure rate. Then, Chen and Yang (2002) extended the model of Banerjee and Rahim (1988) from a single assignable cause to multiple assignable causes. In their paper, they showed that if the process were affected by several assignable causes, the cost of the model would be reduced in comparison with the model that wrongly assumed only single assignable cause.

With the complexity of products, the need for simultaneous monitoring of multiple quality characteristics appeared and caused the first economic design of the Hotelling $T^2$ control charts developed by Montgomery and Klatt (1972). In most of the research works such as Heikes, Montgomery, and Yeung (1974), Yang and Rahim (2006), Chen, Hsieh, and Chang (2007), Seif, Moghadam, Faraz, and Heuchenne (2011), Bahiraee and Raissi (2014), Faraz, Heuchenne, Saniga, and Costa (2014)the economic design of $T^2$ control chart is presented with single assignable cause Jolayemi and Berrettoni (1989) generalized Duncan (1971) cost model, and produce economic design of $T^2$ control chart with multiple assignable causes.

Due to interactions that parameter variations have on each other, using the concept of loss function for estimating costs of the low-quality product became important. Deming (1982) believed that Taguchi loss function Taguchi, Elsayed, and Hsiang (1989) describe the real world better where the minimum loss is in nominal value and any deviation from the nominal value will increase the amount of loss. Until now, a lot of economic and economic-statistical design developed for control chart by the combination of classic models like Duncan and Lorenzen Vance model by Taguchi loss function Safaei, Kazemzadeh, and Niaki (2012); Al-Ghazi, Al-Shareef, Usher, and Duffuaa (2007); Yang (1998). In economic and economic statistical design, loss cost under control and out of control is calculated with Taguchi loss function in many research works (Serel and Moskowitz (2008); Elsayed and Chen (1994)). In the case of multiple characteristics, Kapur and Cho (1996) developed a multivariate loss function for the multivariate quality characteristics. Chou, Chen, and Liu (2001) used the Taguchi loss function to develop ideas of Montgomery and Klatt (1972) for monitoring multivariate control charts to monitor mean and variance of the process jointly.

The economic statistical design of $T^2$ control charts under Weibull shock model with multiple assignable causes and multivariate Taguchi loss function are not performed yet, thus we present the economic statistical design of $T^2$ control chart with multiple assignable causes under Weibull shock model by considering multivariate Taguchi loss function. The use of the loss function in estimating the costs of producing non-conforming products in the production process makes the cost model more flexible. The loss functions that have been used in this field are univariate, so combining the Multivariate loss function with the cost model in the design of the economic-statistical $T^2$ control chart under Weibull Shock Models, is theoretically an innovation in the literature. Here, based on the Duncan (1971) with multiple assignable causes, sampling design of Banerjee and Rahim (1988) and cost structure of Chen and Yang (2002), an upgraded model is constructed. In our paper, by considering fixed sampling interval (uniform sampling scheme), we calculate the average cost for the cycle and compare our findings with the average cost in the case of non-uniform sampling. To calculate cost functions for uniform and non-uniform sampling schemes, we presented and proved the formulas of

AATS based on multiple assignable causes and ANF in the case of uniform and non-uniform sampling schemes . To construct economic statistical design we used penalty formula and both the statistical properties and optimization of loss cost have been considered simultaneously. The application of this function is more prominent in the industries where the products are economically or qualitatively more important and over estimation or under estimation of the target's value creates various losses.

The structure of this paper is as follows. In the second part, review of $T^2$ control chart is given. In the third part, some performance indicators are defined. Cost model offered with multiple special causes by considering uniform and non-uniform sampling scheme in section four. Improvement of cost model by the use of multivariate Taguchi loss function also presented in section 4. Section 5 includes economic statistical design. Section 6 shows that how to determine input parameters and optimizing cost model based on these input parameters and the comparison between cost models under multiple assignable causes and single assignable cause. The comparison between cost model under multiple assignable causes with uniform and non-uniform schemes also presented in section 6. Finally, the conclusion is presented in section 8.

Table 1: Summarized literature review

| Papers | Assignable Cause | PFM | DesignType | Type of control chart | Objective | Integrated with taguchi loss function |
|---|---|---|---|---|---|---|
| Duncan (1956) | Single | Exponential | Economic | $\overline{X}$ | Cost | No |
| Duncan (1971) | Multiple | Exponential | Economic | $\overline{X}$ | Cost | No |
| Montgomery and Klatt (1972) | Single | Exponential | Economic | $T^2$ | Cost | No |
| Lorenzen and Vance (1986) | Single | Exponential | Economic | $\overline{X}$ | Cost | No |
| Banerjee and Rahim (1988) | Single | Weibull | Economic | $\overline{X}$ | Cost | No |
| Jolayemi and Berrettoni (1989) | Multiple | Exponential | Economic | $T^2$ | Cost | No |
| Rahim and Banerjee (1993) | Single | Increasing hazard rate distribution | Economic | $\overline{X}$ | Cost | No |
| Elsayed and Chen (1994) | Single | Exponential | Economic | $\overline{X}$ | Cost | Yes |
| Alexander, Dillman, Usher, and Damodaran (1995) | Single | Exponential | Economic | $\overline{X}$ | Cost | Yes |
| Zhang and Berardi (1997) | Single | Weibull | Economic- Statistical | $\overline{X}$ | Cost and $\alpha$ and $1-\beta$ | No |
| Chen and Yang (2002) | Multiple | Weibull | Economic | $\overline{X}$ | Cost | No |
| Chou *et al.* (2001) | Single | Exponential | Economic | $\overline{X}$ | Cost and $\alpha$ and $1-\beta$ | Yes |
| Ben-Daya and Duffuaa (2003) | Single | Exponential | Economic | $\overline{X}$ | Cost | Yes |
| Al-Oraini and Rahim (2002) | Single | Gamma | Economic- Statistical | $\overline{X}$ | Cost and $\alpha$ and $1-\beta$ | No |
| Yang and Rahim (2006) | Single | Weibull | Economic- Statistical | $T^2$ | Cost and $\alpha$ and $1-\beta$ | No |
| Yu and Chen (2009) | Multiple | Exponential | Economic- Statistical | $\overline{X}$ | Cost and $\alpha$ and $1-\beta$ | Yes |
| Kraleti and Kambagowni (2010) | Single | Pareto | Economic | $\overline{X}$ | Cost | No |
| Yu, Tsou, Huang, and Wu (2010) | Multiple | Exponential | Economic- Statistical | $\overline{X}$ | Cost and $\alpha$ and $1-\beta$ | No |
| Safaei *et al.* (2012) | Single | Exponential | Multiple Objective ESD | $\overline{X}$ | Cost and $\alpha$ and $1-\beta$ | Yes |
| Heydari *et al.* (2016) | Single | Burr Xll | Economic- Statistical | $\overline{X}$ | Cost and $\alpha$ and $1-\beta$ | No |
| Moghadam *et al.* (2016) | Single | Generalized Exponential | Economic | $\overline{X}$ | Cost | No |
| This paper | Multiple | Weibull | Economic- statistical | $T^2$ | Cost AATS and ANF | Yes |

# 2. $T^2$ control chart overview

Assume that the output of manufacturing process has $p$ correlated quality characteristics and $X$ has a $p$-variate normal distribution with known covariance matrix $\Sigma$ and mean vector $\mu$ (when the process is under control, $\mu$=$\mu_0$). As an expansion of the univariate, Hotelling in 1931 presented the following statistic which, known as $T^2$ Hotelling:

$$T^2 = (\overline{\mathbf{X}} - \boldsymbol{\mu_0})'\boldsymbol{\Sigma}^{-1}(\overline{\mathbf{X}} - \boldsymbol{\mu_0}) \tag{1}$$

where $\overline{\mathbf{X}} = \frac{1}{n}\sum_{i=1}^{n}\mathbf{X}$ is sample mean vector. Assuming that the mean vector and covariance matrix is known, $T^2$ statistic has Chi-Square distribution with $p$ degrees of freedom. Each of the $T^2$ values is compared with the upper $\alpha$ percentage of the Chi-Square distribution $(L = \chi^2_\alpha(p))$ and if the sample values fall below the control limit L, the process is considered in control, otherwise, the process is said to be out of control and the corresponding subgroup(s) is investigated. The occurrence of assignable causes shifts the mean process. In multivariate case, this change is calculated as follows:

$$d_i^2 = (\boldsymbol{\mu_i} - \boldsymbol{\mu_0})'\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu_i} - \boldsymbol{\mu_0}),\ i = 1, 2, \ldots, s \tag{2}$$

In fact, the above equation is Mahalanobis distance between $\mu_0$ and $\mu_i$ vector where $\mu_i$ is $p$ dimensional vector of process mean in the out of control situation due to the $i^{th}$ assignable cause. In $T^2$ control chart, the probability of Type I error is calculated as follows:

$$\alpha = P(T^2 > \chi^2_{\alpha,p}|\boldsymbol{\mu} = \boldsymbol{\mu}_0) \tag{3}$$
$$= \frac{1}{2^{\frac{P}{2}}\Gamma(\frac{P}{2})}\int_L^\infty x^{\frac{P}{2}-1}exp(-\frac{x}{2})dx \tag{4}$$
$$\tag{5}$$

where the $i^{th}$ assignable cause occurs, the process mean shifts. The probability of detecting change in process mean is $1-\beta_i$.

By considering $\beta_i = P(T^2 < \chi^2_{\alpha,p}|\boldsymbol{\mu} = \boldsymbol{\mu}_i), i = 1, 2, ..., s$ as the probability of error Type II, the above probability formula is:

$$\beta_i = exp(-\frac{\eta_i}{2}\Sigma_{j=0}^{\infty}\frac{\eta_i^j}{j!2^{2j}\Gamma(j+\frac{P}{2})}) \tag{6}$$
$$= \int_0^L x^{\frac{P}{2}+j-1}exp(-\frac{x}{2})dx \tag{7}$$

where $\eta_i = nd_i^2, i = 1, 2, ..., s$ is non-central parameter of $\chi^2$ distribution.

## 3. Essential points

Following Banerjee and Rahim (1988), it is expected that the time of being under control until the occurrence of $i^{th}$ assignable cause follows a Weibull distribution with probability density function and hazard rate as below:

$$f_i(t) = \lambda_i k t^{k-1} exp(-\lambda_i t^k).(\quad t > 0, k \geq 1, \lambda_i > 0), i = 1, 2, ..., s. \tag{8}$$

$$r_i(t) = \lambda_i k t^{k-1} \tag{9}$$

where $k$ is shape parameter and $\lambda_i$ is scale parameter. The process is investigated by taking samples of size $n$ from $X$ at time intervals $h_1$, $h_1 + h_2$, $h_1 + h_2 + h_3$ and so on. Where $h_j$ is defined as $j^{th}$ sampling interval and we have $h_1 \geq h_2 \geq h_3, ...$ (The proof of above inequality is given in Appendix B).

Sampling intervals are determined in a way that the probability of shift from a control state is fixed for all intervals when it is in control at the beginning of the interval. In other words, according to Banerjee and Rahim (1988), integrated hazard over each interval should be equal.

$$\int_{\omega_j}^{\omega_{j+1}} r_i(t)\,\mathrm{d}t = \int_{\omega_0}^{h_1} r_i(t)\,\mathrm{d}t, j = 1, 2, .... \tag{10}$$

Accordingly $h_j$ (for more detail see Appendix C) obtained as follows

$$h_j = [j^{\frac{1}{k}} - (j-1)^{\frac{1}{k}}]h_1 \tag{11}$$

For sake of simplicity in deriving required equations, we define $\omega_j$ as below:

$$\omega_j = \Sigma_{i=1}^{j} h_i. \tag{12}$$

It is assumed that the process stops at the time of search and repair and the cost of sampling is negligible. To calculate the average cost per unit of time we need to define the following terms.

1. We define $p_{ij}$ is the conditional probability that $i^{th}$ assignable cause will occur during $j^{th}$ sampling interval , given that $i^{th}$ assignable cause not occur at time $\omega_{j-1}$.

$$p_{ij} = \frac{\int_{\omega_{j-1}}^{\omega_j} f_i(t)\,\mathrm{d}t}{\int_{\omega_{j-1}}^{\infty} f_i(t)\,\mathrm{d}t} = 1 - exp(-\lambda_i(jh_1^k)). \tag{13}$$

let $p_{ij} = p_i$ , for $(i = 1, 2, ..., S), (j = 1, 2, ...)$.

2. We define $q_{ij}$ as the unconditional probability that $i^{th}$ assignable cause will occur during the $j^{th}$ sampling interval. Thus, we have:

$$q_{ij} = \int_{\omega_{j-1}}^{\omega_j} f_i(t), \mathrm{d}t = e^{-\lambda_i \omega_{j-1}^k} - e^{-\lambda_i \omega_j^k} = (1 - p_i)^{j-1} p_i. \tag{14}$$

3. Suppose that $\tau_{ij}$ be the expected duration of the in control period within sampling interval $h_j$, given that $i^{th}$ assignable cause has occurred during this period. Thus, we have:

$$\tau_{ij} = E(T - \omega_{j-1} \mid \omega_{j-1} < T < \omega_j) \tag{15}$$

So the expected $\tau_i$ (the time that process be under control) during any sampling interval is as follows:

$$\tau_i = \sum_{j=1}^{\infty} \tau_{ij} q_{ij} = \sum_{j=1}^{\infty} \int_{\omega_{j-1}}^{\omega_j} (t - \omega_{j-1}) f_i(t)\,\mathrm{d}t = (\frac{1}{\lambda_i})^{\frac{1}{k}} \Gamma(1 + \frac{1}{k}) - h_1 p_i (1 - p_i) A(1 - p_i), \tag{16}$$

where for $|x| < 1$

$$A(X) = \sum_{j=0}^{\infty} (j + 1)^{\frac{1}{k}} X^j. \tag{17}$$

Let $AATS_i$ be the average time between occurrence shift in process mean owing to the $i^{th}$ assignable cause and receiving a right alarm from control chart.

$$AATS_i = h_1 p_i A(1 - p_i) + \frac{\beta_i h_1 p_i [p_i A(1 - p_i) - (1 - \beta_i) A(\beta_i)]}{1 - p_i - \beta_i} - (\frac{1}{\lambda_i})^{\frac{1}{k}} \Gamma(1 + \frac{1}{k}). \tag{18}$$

The concept of $AATS_i$ is presented well in Figure 1. The proof of above formula is presented in Appendix D.

# 4. Cost model structure

## 4.1. Assumptions

In building our model for observing a process by a $T^2$ control chart we make the typical assumptions about the process as follows:

1. The $p$ quality characteristics follow a multivariate normal distribution with mean vector $\mu$ and covariance matrix $\Sigma$.
2. The process is characterized by an in-control state, i.e. $\mu = \mu_0$.
3. The time interval that the process remains in control is a Weibull random variable expecting that the process starts in control state.
4. The occurrences of assignable causes are independent.

5. Multiple assignable causes produce "step changes" in the process mean from $\mu = \mu_0$ to a known $\mu = \mu_1$. This results in a known value of the Mahalanobis distance.

6. In this paper the shift occurred in process mean is noted by $d_i$. Three distributions of uniform, negative-exponential $\frac{1}{2}e^{-\frac{d_i}{2}}$ and half-normal $\frac{1}{\sqrt{2\pi}}e^{-\frac{(\frac{1}{2}d_i)^2}{2}}$ are considered as a prior for $d_i$. Considering these distributions as a prior would cover all values of $d_i$ in a real industry.

7. "Drifting processes" are not a subject of this research. Assignable causes that affect process variability are not addressed; hence it is assumed that the covariance matrix $\Sigma$ is constant over time.

8. Before the shift, the process is considered to be in a state of statistical control (in-control state).

9. The process is not self-correcting. That is, once a transition to an out-of-control state has occurred, the process can be returned to the in-control condition only by management intervention upon appropriate corrective actions.

10. The quality cycle starts with the in-control state and continues until the process is repaired after an out-of-control signal. It is assumed that quality cycle follows a Renewal Reward Process.

11. During the search for an assignable cause, the process is shut down.

### 4.2. Cost function in the case of non-uniform sampling

Every process begins at in control state. At that point, because of the occurrence of one assignable cause, the process will change to out of control state and after detection and repair; it goes back to the control state. This is called quality cycle that develops Renewal Reward Process where the average cost per unit time for cycle $E(A)$ is obtained by the average cost per cycle $E(C)$ divided by the average time per cycle $E(T)$. We assumed that assignable causes affected the process. The occurrence time of any assignable cause follows Weibull distribution.

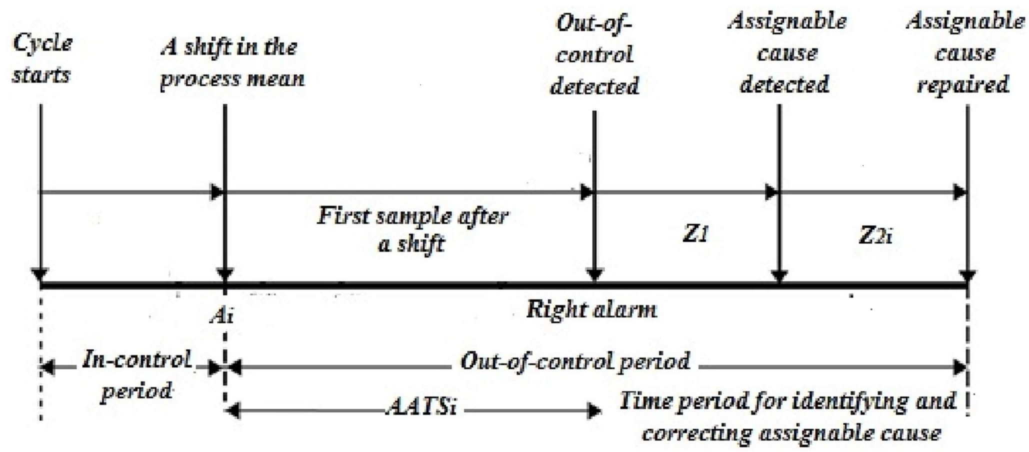


Figure 1: The quality cycle under control and out of control

Here it is assumed that after the occurrence of the $i^{th}$ assignable cause, until the discovery of this deviation, the process will not disturb by other assignable causes. If the time until occurrence of assignable causes noted by $T_1^{'}, T_2^{'}, ..., T_S^{'}$ , then the probability of being under control at time $t$ is:

$$P(T^{'} > t) = P(min(T_1^{'},\ T_2^{'}, ...,\ T_S^{'}) > t) = exp^{-\lambda_0 t^k} \tag{19}$$

where $\lambda_0 = \sum \lambda_i,\ i = 1, 2, \ldots, s.$

Therefore, the time of being in control until the occurrence of multiple assignable causes follows the Weibull distribution.

$$f_0(t) = \lambda_0 k t^{k-1} exp(-\lambda_0 t^k).(\quad t > 0, k \geq 1, \lambda_0 > 0). \tag{20}$$

We consider $P_{0j}$ as the conditional probability that multiple assignable causes $(i = 1, 2, ..., s)$ will occur during $j^{th}$ sampling interval $(j = 1, 2, ...)$ given that multiple assignable causes not occur at time $\omega_{j-1}$. We obtain $p_{0j} = 1 - e^{-}\lambda_0 h_1^k$. Here it is assumed $p_{0j} = p_0$.

The average time that the process is in control is:

$$(\frac{1}{\lambda_0})^{\frac{1}{k}} \Gamma(1 + \frac{1}{k}) + Z_0 ANF \tag{21}$$

In the above formula $ANF$ is the average numbers of false alarm in the quality cycle and is equal to the production of average sample numbers before shift and probability of Type I error $(\alpha)$.

If $A$ is the event of the occurrence of single assignable cause, then average numbers of false alarm calculated as follows:

$$E(Number of samples are taken before shift) = \sum_{j=0}^{\infty} j P(A \in (jh, (j+1)h) \tag{22}$$

$$= \sum_{j=0}^{\infty} j(e^{-\lambda_0 j h_1^k} - e^{-\lambda_0 (j+1) h_1^k}) \tag{23}$$

$$= \frac{e^{-\lambda_0 h_1^k}}{1 - e^{-\lambda_0 h_1^k}}. \tag{24}$$

Therefore, $ANF$ is equal to:

$$ANF = \alpha \frac{1 - p_0}{p_0}. \tag{25}$$

The average time of cycle is:

$$E(T) = (\frac{1}{\lambda_0})^{\frac{1}{k}} \Gamma(1 + \frac{1}{k}) + Z_0 ANF + AATS + Z_1 + \Sigma(\frac{\lambda_i}{\lambda_0}) Z_{2i} \tag{26}$$

where

$$AATS = \sum_{i=1}^{s} \frac{\lambda_i}{\lambda_0} AATS_i. \tag{27}$$

For a better understanding of $E(T)$ one can see Figure 1.

The average cost of cycle is:

$$E(C) = D_0(\frac{1}{\lambda_0})^{\frac{1}{k}} \Gamma(1 + \frac{1}{k}) + YANF + \sum_{i=1}^{s} \frac{\lambda_i}{\lambda_0} D_{1i} AATS_i + \sum_{i=1}^{s} \frac{\lambda_i}{\lambda_0} w_i \tag{28}$$

$$+ (a + bn) \sum_{i=1}^{s} \frac{\lambda_i}{\lambda_0} (\frac{1}{1 - \beta_i}) + (a + bn) Q. \tag{29}$$

(30)

For better understanding of $E(C)$ one can see Figure 1. The way of obtaining $\frac{\lambda_i}{\lambda_0}$ in above formula is presented in Appendix E.

In practice, each process starts from in control state. Then because of occurrence of one assignable cause, it goes to out of control state. It is clear that after repairing and fixing the assignable cause, the process returns to the initial state. This cycle is called quality cycle and its model follows the form of a Renewal Reward Process where the average cost per unit time for the cycle $E(A)$ is calculated by the average cost per cycle $E(C)$ divided by the average time per cycle $E(T)$. In economic design, the purpose is optimizing $E(A)$ without any constraint and finding optimal values for sampling interval, sample size and control limits coefficient.

### 4.3. Cost function in the case of uniform sampling

To evaluate the relative benefits of non-uniform sampling plan in comparison with uniform sampling plan under multiple assignable causes cost model, and by considering fixed sampling interval, we calculate average time and the average cost for the cycle and analyze them. If $h$ is a fixed sampling interval, then we can obtain $E(T)$ as follows:

$$E(T) = (\frac{1}{\lambda_0})^{\frac{1}{k}}\Gamma(1+\frac{1}{k}) + Z_0 ANF + AATS + Z_1 + \Sigma(\frac{\lambda_i}{\lambda_0})Z_{2i}, \tag{31}$$

where

$$AATS = \sum_{i=1}^{s} \frac{\lambda_i}{\lambda_0} AATS_i \tag{32}$$

$$AATS_i = \frac{h}{1-\beta_i} - \tau_i \tag{33}$$

$$\tau_i = (\frac{1}{\lambda_0})^{\frac{1}{k}}\Gamma(1+\frac{1}{k}) - hQ_i \tag{34}$$

$$Q_i = \sum_{j=1}^{\infty} e^{\lambda_i (jh)^k} \tag{35}$$

$$ANF = \alpha Q, Q = \sum_{j=1}^{\infty} e^{\lambda_0 (jh)^k} \tag{36}$$

We also obtain $E(C)$ as fallows:

$$E(C) = D_0(\frac{1}{\lambda_0})^{\frac{1}{k}}\Gamma(1+\frac{1}{k}) + Y ANF + \sum_{i=1}^{s}\frac{\lambda_i}{\lambda_0} D_{1i} AATS_i + \sum_{i=1}^{s}\frac{\lambda_i}{\lambda_0} w_i \tag{37}$$

$$+ (a+bn)\sum_{i=1}^{s}\frac{\lambda_i}{\lambda_0}(\frac{1}{1-\beta_i}) + (a+bn)Q. \tag{38}$$

### 4.4. Improvement of cost function by using Taguchi loss function

To consider the intangible costs, Taguchi loss function is used. In this model, $E(A)$ is used as an economic criterion to assess the measurable costs. To estimate $D_0$ and $D_1$ Taguchi loss function is used. Taguchi, Chowdhury, and Wu (2005) characterized product quality as the loss a product bestows to society from the time the product is delivered and presented the quality loss function as a quality measure. They showed that a quadratic loss function adequately represents economic loss due to the deviation of a quality characteristic from the target value. The Taguchi loss function is presented bellow:

$$L(Y) = C(X-t)^2 \tag{39}$$

In the above formula $L(Y)$, is the loss connected with the value of quality characteristic X, the target value of the quality characteristic is denoted by t, and C is a constant value depending on the width of the specification and the cost at the specification limits.

Based on Eq. (31), Kapur and Cho (1996) improved a multivariate loss function for the multivariate quality characteristics $Y_1, Y_2, ..., Y_P$. The multivariate loss function presented below

$$L(Y_1, Y_2, ..., Y_P) = \sum_{i=1}^{P}\sum_{j=1}^{i} C_{ij}(Y_i - t_i)(Y_j - t_j) \quad (40)$$

where $t_j$ is the target of the $j^{th}$ characteristic, and $C_{ij}$ is a constant depending on the cost at the specification limits and the width of the specification. Chen (1995) discussed completely about determining the values of $C_{ij}$. Specifically, if $Y_i$ and $Y_j$ are independent, then $C_{ij} = 0$. The expected loss per unit of product may be obtained by the use of expectation operator on both sides of Eq. (32).

$$E(L(Y_1, Y_2, ..., Y_P)) = \sum_{i=1}^{P} C_{ii}[(\mu_i - t_i) + \sigma_i^2] + \sum_{i=2}^{P}\sum_{j=1}^{i} C_{ij}[(\mu_i - t_i)(\mu_j - t_j) + \sigma_{ij}] \quad (41)$$

In the above formula $\mu_j$ and $\sigma_j^2$ are the mean and variance of $Y_j$, and the covariance of $Y_j$ and the covariance of $Y_i$ and $Y_j$ is $\sigma_{ij}$.

If we consider Eq (33) when the process under control, ($\mu = \mu_0$), $L_{in}$ is obtained. $Lout, i$ is obtained by considering Eq (33), when the process is out of control $\mu = \mu_1$. We improve Eq (23) and (30) by using $PL_{in}$ instead of $D_0$ and $PLout, i$ instead of $D_{1i}$. In this paper, we consider two quality characteristic. Let $(Y_1, Y_2)$ be two quality characteristic, and assume that when the process is in-control:

$$(Y_1, Y_2) \sim BN(\vec{\mu_0} = (\mu_{Y_1}, \mu_{Y_2}), \Sigma_0), (T_1^{''}, T_2^{''}) = (\mu_{Y_1} - \delta_5\sigma_1, \mu_{Y_2} - \delta_6\sigma_2) \quad (42)$$

where $\sigma_{12} = \rho\sigma_1\sigma_2$ is the covariance of $(Y_1, Y_2)$ and $\rho$ is the coefficient of correlation and $-1 \leq \rho \leq 1$. $\delta_5$ and $\delta_6$ are target shifts, in order to simplify the process we set $\delta_5, \delta_6 \geq 0$.

When the process is out-of-control,

$$(Y_1, Y_2) \sim BN(\vec{\mu_1} = (\mu_{Y_1} + \delta_{1i}, \mu_{Y_2} + \delta_{2i}), \Sigma_0), (T_1^{''}, T_2^{''}) = (\mu_{Y_1} - \delta_5\sigma_1, \mu_{Y_2} - \delta_6\sigma_2) \quad (43)$$

where both $\delta_{1i}$ and $\delta_{2i}$ for $i = 1, 2, ..., s$ are the mean partial shift and $\delta_{1i}, \delta_{2i} \neq 0$.

We use the following bivariate loss function:

$$L(Y_1, Y_2) = K_{11}(Y_1 - T_1^{''})^2 + K_{12}(Y_1 - T_1^{''})(Y_2 - T_2^{''}) + K_{22}(Y_2 - T_2^{''})^2 \quad (44)$$

where $(Y_1, Y_2)$ are quality characteristics, $T_1$ and $T_2$ are target values and $K_{11}, K_{22}$ and $K_{12}$ are constants.

$$E(L(Y_1, Y_2)) = K_{11}E[(Y_1 - T_1^{''})^2] + K_{12}E[(Y_1 - T_1^{''})(Y_2 - T_2^{''})] + K_{22}E[(Y_2 - T_2^{''})^2] \quad (45)$$

$L_{in}$ is obtained by considering Eq (37) when the process is in control and $L_{out,i}$ is obtained by considering Eq (37) .

## 5. Economic statistical design

In the statistical design of control charts, optimal performance of design parameters obtained in terms of statistical criteria. Economic design of control charts based on economic criteria.

In this paper, economic statistical design performed based on minimizing average cost per time and by considering maximum values for the average time between occurrence shift in process mean and receiving a right alarm from control chart ($AATS$) and average numbers of false alarm in the quality cycle ($ANF$). If we note the average cost of a cycle per time by $E(A)$ and the set of economic design parameters of $T^2$ control charts by F, we can show economic statistical design of $T^2$ control charts as follows:

minimize $E_F(A)$

subject to $AATS \leq AATS_U$ and $ANF \leq ANF_U$

where $AATS_u$ and $ANF_u$ are the corresponding bounds of values of $AATS$ and $ANF$.

## 6. Illustrative example

To determine the optimal model parameters, the $R$ package Optim is used through minimizing loss cost. Some of the parameters like cost parameters $(Y, a, b)$ and time parameters $(Z_0, Z_1)$ that are fixed have been determined based on past experience. In this model, $\omega_i$ and $Z_{2i}$ are non-fixed cost and time parameters, respectively. Also, there are Weibull distribution parameters $(\lambda_i, k)$, shift parameter $d_i$ and $(n, h_1, L)$ parameters in the model. In the numerical example we assume:

$$Y = 500, a = 20, b = 4.22, Z_0 = 0.25, Z_1 = 0.25$$

The above parameters are not affected by the occurrence of different assignable causes and the shift created in the mean process. However, $W_i, Z_{2i}, \lambda_i$ and parameters are assumed to be a function of $d_i$. We calculated above parameters below:

a) It is assumed $(Y_1, Y_2)$ be two quality characteristic, and assume that when process is in-control:

$$(Y_1, Y_2) \sim BN(\mu_0 = (2, 3), \Sigma_0) \tag{46}$$

$$(T_1^{''}, T_2^{''}) = (0.32, 1.92) \tag{47}$$

When covariance matrix is:

$$\begin{bmatrix} 4 & 1 \\ 1 & 9 \end{bmatrix}$$

We assume that process is disturbed by ten assignable causes which produce ten shift values in process mean vector. We also assume that when the process is going to out of control, the mean of the process is shifted and we have

$$\mu_i = (2 + \delta_{1i}, 3 + \delta_{2i}) \tag{48}$$

Values of $\delta_{1i}$ and $\delta_{2i}$ and $d_i$ related to them, are listed in Table 2.

As seen before, $L_{in}$ and $L_{out,i}$ is obtained by considering Eq(37) and we have $D_0 = PL_{in}$ and $D_{1i} = PL_{out,i}$. We assume $K_{11} = 1.6, K_{12} = 4, K_{22} = 5, P = 17$. Thus we obtain $D_0 = 946$,

$$D_{1i} = (948, 955.87, 956.624, 957.37, 958.12, 959, 960.02, 960.7, 961.5, 962.24)$$

b) Suppose that $d_i = 0.018$ is a base case. Yang and Rahim (2006) single assignable cause model is compared with our multiple assignable causes model. Base case parameters are also considered for single assignable cause model

$$W = 1100, D_1 = 958.12, Z_2 = 0.75, \lambda = 0.05, d = 0.018$$

Table 2: Model partial shift and shift mean parameters

| $A_i$ | $\delta_{1i}$ | $\delta_{2i}$ | $d_i$ |
|---|---|---|---|
| 1 | 0.02 | 0.03 | 0.036 |
| 2 | 0.01 | 0.02 | 0.022 |
| 3 | 0.009 | 0.019 | 0.021 |
| 4 | 0.008 | 0.018 | 0.019 |
| 5 | 0.007 | 0.017 | 0.018 |
| 6 | 0.006 | 0.014 | 0.015 |
| 7 | 0.005 | 0.009 | 0.01 |
| 8 | 0.004 | 0.008 | 0.008 |
| 9 | 0.003 | 0.007 | 0.007 |
| 10 | 0.002 | 0.006 | 0.006 |

Table 3: Model input parameters

| | | | $PD_i$ | | | | $Z_{2i}$ | | | $W_i$ | | | $\lambda_i$ | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $A_i$ | $d_i$ | $NE$ | $Un$ | $HN$ | $D_{1i}$ | $NE$ | $Un$ | $HN$ | $NE$ | $Un$ | $HN$ | $NE$ | $Un$ | $HN$ |
| 1 | 0.036 | 0.491 | 0.100 | 0.399 | 948 | 0.743 | 0.75 | 0.75 | 1090 | 1100 | 1100 | 0.0154 | 0.0156 | 0.0156 |
| 2 | 0.022 | 0.495 | 0.100 | 0.399 | 955.9 | 0.749 | 0.75 | 0.75 | 1098 | 1100 | 1100 | 0.0155 | 0.0156 | 0.0156 |
| 3 | 0.021 | 0.495 | 0.100 | 0.399 | 956.6 | 0.749 | 0.75 | 0.75 | 1098 | 1100 | 1100 | 0.0155 | 0.0156 | 0.0156 |
| 4 | 0.019 | 0.495 | 0.100 | 0.399 | 957.4 | 0.750 | 0.75 | 0.75 | 1099 | 1100 | 1100 | 0.0155 | 0.0156 | 0.0156 |
| 5 | 0.018 | 0.496 | 0.100 | 0.399 | 958.1 | 0.750 | 0.75 | 0.750 | 1100 | 1100 | 1100 | 0.0155 | 0.0156 | 0.0156 |
| 6 | 0.015 | 0.496 | 0.100 | 0.399 | 959 | 0.751 | 0.75 | 0.75 | 1102 | 1100 | 1100 | 0.0156 | 0.0156 | 0.0156 |
| 7 | 0.01 | 0.498 | 0.100 | 0.399 | 960 | 0.753 | 0.75 | 0.75 | 1104 | 1100 | 1100 | 0.0156 | 0.0156 | 0.0156 |
| 8 | 0.008 | 0.498 | 0.100 | 0.399 | 960.7 | 0.754 | 0.75 | 0.75 | 1106 | 1100 | 1100 | 0.0156 | 0.0156 | 0.0156 |
| 9 | 0.007 | 0.498 | 0.100 | 0.399 | 961.5 | 0.754 | 0.75 | 0.75 | 1106 | 1100 | 1100 | 0.0156 | 0.0156 | 0.0156 |
| 10 | 0.006 | 0.499 | 0.100 | 0.399 | 962.2 | 0.755 | 0.75 | 0.75 | 1107 | 1100 | 1100 | 0.0156 | 0.0156 | 0.0156 |

In this paper, we noted the prior distribution for $d_i$ by $PD_i$. As mentioned earlier three distribution uniform, negative-exponential and half-normal are considered as a prior for $d_i$. The value of Weibull scale parameter $\lambda_i$

is calculated by the use of prior distributions. Other parameter formulas are as below:

$$W_i = (\frac{PD_i}{PD_5}) \times 1100 \tag{49}$$

$$Z_{2i} = (\frac{PD_i}{PD_5}) \times 0.75 \tag{50}$$

$$\lambda_i = (\frac{PD_i}{PD_5}) \times \lambda_5 \tag{51}$$

Input parameters values are listed in Table 3 .

Comparison between optimal values $(n, h_1, L)$ and loss cost for our multiplicity-cause and single-cause model of Yang and Rahim (2006) is presented in Table 4 for different values of the shape parameter Weibull distribution. As seen in Table 4, a single-cause model has a higher loss cost than multiplicity-cause model except in the case of $k = 1$. In other words, when the process is affected by several assignable causes, and wrongly it is assumed that only single assignable cause affected the process, the loss cost will be increased. It should be noted that according to the values obtained in economic design, the upper limit of $AATS$ was considered 1 and the upper limit of $ANF$ were considered 0.5.

In Table 5 comparison between optimal values and loss cost for economic statistical design by considering the non-uniform sampling and uniform sampling scheme for different values of the Weibull distribution shape parameter is given. As it is seen in Table 5 when we use

Table 4: Optimal parameters obtained under multiple assignable causes

| | | Economic design | | | | | | | | Economic Statistical design | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $k$ | $PD$ | $n$ | $h_1$ | $L$ | $\alpha$ | $1-\beta$ | $AATS$ | $ANF$ | $ECT$ | $n$ | $h_1$ | $L$ | $\alpha$ | $1-\beta$ | $AATS$ | $ANF$ | $ECT$ |
| | $YR$ | 12 | 1.2 | 5.47 | 0.065 | 0.9 | 0.747 | 0.262 | 1090.65 | 12 | 1.2 | 5.47 | 0.065 | 0.9 | 0.747 | 0.262 | 1090.65 |
| 1 | $NE$ | 7 | 1.18 | 2.71 | 0.25 | 0.69 | 1.53 | 4.05 | 1127.09 | 97 | 1 | 7.24 | 0.026 | 0.85 | 0.99 | 0.5 | 1427.5 |
| | $Un$ | 6 | 1.05 | 3.55 | 0.17 | 0.72 | 1.52 | 3.95 | 1106.9 | 100 | 1.21 | 7.39 | 0.025 | 0.99 | 1 | 0.5 | 1370.27 |
| | $HN$ | 7 | 1.11 | 3.08 | 0.21 | 0.71 | 1.47 | 3.83 | 1120.29 | 100 | 1.16 | 7.07 | 0.029 | 0.89 | 0.99 | 0.49 | 1385.53 |
| | $YR$ | 11 | 1.7 | 5.18 | 0.075 | 0.89 | 0.428 | 0.16 | 1171.17 | 11 | 1.7 | 5.18 | 0.075 | 0.89 | 0.428 | 0.16 | 1171.17 |
| 1.5 | $NE$ | 6 | 2.27 | 2.2 | 0.33 | 0.72 | 0.6 | 1.71 | 1149 | 17 | 2.65 | 4.14 | 0.12 | 0.71 | 0.99 | 0.5 | 1177.4 |
| | $Un$ | 6 | 2.05 | 3.33 | 0.19 | 0.72 | 0.547 | 1.52 | 1135.32 | 42 | 1.83 | 10.5 | 0.006 | 0.78 | 1 | 0.06 | 1336.54 |
| | $HN$ | 7 | 2.14 | 2.71 | 0.26 | 0.72 | 0.56 | 1.55 | 1142.67 | 44 | 1.69 | 9.69 | 0.008 | 0.75 | 0.99 | 0.07 | 1367.21 |
| | $YR$ | 11 | 1.76 | 5.07 | 0.079 | 0.88 | 0.36 | 0.13 | 1195.13 | 11 | 1.76 | 5.07 | 0.079 | 0.88 | 0.36 | 0.13 | 1195.13 |
| 1.8 | $NE$ | 6 | 2.45 | 1.95 | 0.38 | 0.27 | 0.43 | 1.26 | 1151.94 | 40 | 1.73 | 10.3 | 0.005 | 0.66 | 1 | 0.038 | 1459.54 |
| | $Un$ | 5 | 2.22 | 3.21 | 0.2 | 0.72 | 0.38 | 1.09 | 1140.68 | 10 | 2.88 | 3.74 | 0.15 | 0.77 | 0.55 | 0.5 | 1156.42 |
| | $HN$ | 6 | 2.31 | 2.53 | 0.28 | 0.73 | 0.4 | 1.13 | 1144.94 | 13 | 2.91 | 3.23 | 0.2 | 0.77 | 0.57 | 0.5 | 1160.73 |
| | $YR$ | 11 | 1.76 | 5 | 0.082 | 0.88 | 0.32 | 0.12 | 1204.79 | 11 | 1.76 | 5 | 0.082 | 0.88 | 0.32 | 0.12 | 1204.79 |
| 2 | $NE$ | 4 | 2.49 | 1.79 | 0.41 | 0.75 | 0.36 | 1.07 | 1151.3 | 40 | 2 | 10.5 | 0.005 | 0.66 | 0.99 | 0.024 | 1449.842 |
| | $Un$ | 5 | 2.25 | 3.13 | 0.21 | 0.72 | 0.32 | 0.93 | 1141.86 | 8 | 2.42 | 4.03 | 0.13 | 0.72 | 0.41 | 0.5 | 1148.92 |
| | $HN$ | 6 | 2.34 | 2.42 | 0.3 | 0.73 | 0.338 | 0.96 | 1144.36 | 9 | 2.42 | 3.59 | 0.17 | 0.71 | 0.44 | 0.49 | 1152.69 |
| | $YR$ | 10 | 1.73 | 4.89 | 0.087 | 0.87 | 0.264 | 0.1 | 1215.95 | 10 | 1.73 | 4.89 | 0.087 | 0.87 | 0.264 | 0.1 | 1215.95 |
| 2.5 | $NE$ | 4 | 2.47 | 1.29 | 0.52 | 0.79 | 0.259 | 0.81 | 1146.18 | 6 | 2.83 | 1.33 | 0.51 | 0.72 | 0.33 | 0.51 | 1153 |
| | $Un$ | 5 | 2.21 | 2.93 | 0.23 | 0.72 | 0.22 | 0.69 | 1140.55 | 6 | 2.47 | 2.89 | 0.23 | 0.75 | 0.26 | 0.49 | 1144.09 |
| | $HN$ | 5 | 2.3 | 2.13 | 0.34 | 0.74 | 0.242 | 0.72 | 1139.48 | 7 | 2.4 | 2.58 | 0.27 | 0.74 | 0.28 | 0.49 | 1141.99 |
| | $YR$ | 10 | 1.68 | 4.8 | 0.09 | 0.86 | 0.229 | 0.084 | 1217.35 | 10 | 1.68 | 4.8 | 0.09 | 0.86 | 0.229 | 0.084 | 1217.35 |
| 3 | $NE$ | 4 | 2.35 | 1 | 0.6 | 0.81 | 0.203 | 0.63 | 1139 | 4 | 2.47 | 1 | 0.6 | 0.83 | 0.227 | 0.51 | 1140.36 |
| | $Un$ | 4 | 2.12 | 2.74 | 0.25 | 0.72 | 0.17 | 0.54 | 1136.53 | 5 | 2.63 | 1.24 | 0.53 | 0.86 | 0.2 | 0.5 | 1148.57 |
| | $HN$ | 5 | 2.21 | 1.83 | 0.4 | 0.76 | 0.19 | 0.57 | 1132.74 | 4 | 2.52 | 1.03 | 0.6 | 0.85 | 0.21 | 0.5 | 1136.46 |
| | $YR$ | 10 | 1.57 | 4.67 | 0.097 | 0.86 | 0.18 | 0.68 | 1209.84 | 10 | 1.57 | 4.67 | 0.097 | 0.86 | 0.18 | 0.68 | 1209.84 |
| 4 | $NE$ | 3 | 2.05 | 1 | 0.6 | 0.81 | 0.15 | 0.4 | 1125.9 | 3 | 2.05 | 1 | 0.6 | 0.81 | 0.15 | 0.4 | 1125.9 |
| | $Un$ | 4 | 1.95 | 2.33 | 0.31 | 0.74 | 0.12 | 0.39 | 1126.8 | 4 | 1.95 | 2.33 | 0.31 | 0.74 | 0.12 | 0.39 | 1126.8 |
| | $HN$ | 3 | 2.06 | 1 | 0.61 | 0.83 | 0.13 | 0.43 | 1118.74 | 3 | 2.06 | 1 | 0.61 | 0.83 | 0.13 | 0.43 | 1118.74 |

economic statistical design, the loss cost becomes greater when uniform sampling scheme is used instead of non-uniform sampling scheme

## 7. Conclusions

In this paper, the economic design of $T^2$ control chart with multiple assignable causes under Weibull shock model with Taguchi loss function was presented. The cost model under multiple assignable causes compared with single cause model of Yang and Rahim (2006). The results showed that the model under multiple assignable causes with Taguchi loss function has a lower cost than single assignable cause model. In other words, when the process is affected by several assignable causes, and wrongly it is assumed that only single assignable cause affected the process, the loss cost will be increased.

## Appendix A

### Notations and definitions

$Z_0$: Average time to search for false alarm.

$Z_1$: Average time to discover assignable cause once it is detected by control chart.

$Z_{2i}$: Average time to repair $i^{th}$ assignable cause after it has been discovered.

$D_0$: Average cost per unit of time while the process in control.

$D_{1i}$: Average cost per unit of time while the process is out of control owing to the occurrence of the $i^{th}$ assignable cause ($i = 1, 2, ..., s$).

$L_{in}$: Approximated in control cost obtained by considering modified Taguchi loss function.

$L_{out,i}$: Approximated out of control cost obtained by considering modified Taguchi loss func-

Table 5: Optimal parameters

| | | Economic Statistical design with non-uniform sampling scheme | | | | | | | | Economic Statistical design with uniform sampling scheme | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $k$ | $PD$ | $n$ | $h_1$ | $L$ | $\alpha$ | 1-$\beta$ | $AATS$ | $ANF$ | $ECT$ | $n$ | $h_1$ | $L$ | $\alpha$ | 1-$\beta$ | $AATS$ | $ANF$ | $ECT$ |
| | $YR$ | 12 | 1.2 | 5.47 | 0.065 | 0.9 | 0.747 | 0.262 | 1090.65 | 13 | 1.4 | 6.62 | 0.036 | 0.88 | 0.9 | 0.5 | 1094.34 |
| 1 | $NE$ | 97 | 1 | 7.24 | 0.026 | 0.85 | 0.99 | 0.5 | 1427.5 | 76 | 0.7 | 7.97 | 0.019 | 0.8 | 1 | 0.5 | 1470.08 |
| | $Un$ | 100 | 1.21 | 7.39 | 0.025 | 0.99 | 1 | 0.5 | 1370.27 | 45 | 0.32 | 10.0 | 0.007 | 0.79 | 1 | 0.5 | 1600.9 |
| | $HN$ | 100 | 1.16 | 7.07 | 0.029 | 0.89 | 0.99 | 0.49 | 1385.53 | 72 | 0.79 | 7.85 | 0.019 | 0.83 | 1 | 0.5 | 1401.65 |
| | $YR$ | 11 | 1.7 | 5.18 | 0.075 | 0.89 | 0.428 | 0.16 | 1171.17 | 12 | 0.91 | 5.2 | 0.074 | 0.99 | 0.55 | 0.5 | 1183.24 |
| 1.5 | $NE$ | 17 | 2.65 | 4.14 | 0.12 | 0.71 | 0.99 | 0.5 | 1177.4 | 51 | 0.97 | 5.02 | 0.08 | 0.72 | 1 | 0.5 | 1342 |
| | $Un$ | 42 | 1.83 | 10.1 | 0.006 | 0.78 | 1 | 0.06 | 1336.54 | 48 | 1.15 | 5.04 | 0.08 | 0.88 | 1 | 0.5 | 1319.8 |
| | $HN$ | 44 | 1.69 | 9.69 | 0.008 | 0.75 | 0.99 | 0.07 | 1367.21 | 26 | 0.85 | 4.28 | 0.12 | 0.81 | 0.93 | 0.49 | 1327.63 |
| | $YR$ | 11 | 1.76 | 5.07 | 0.079 | 0.88 | 0.36 | 0.13 | 1195.13 | 10 | 0.78 | 4.79 | 0.091 | 0.88 | 0.5 | 0.5 | 1219.73 |
| 1.8 | $NE$ | 40 | 1.73 | 10.3 | 0.005 | 0.66 | 1 | 0.038 | 1459.54 | 47 | 1.18 | 3.82 | 0.14 | 0.85 | 1 | 0.5 | 1356.5 |
| | $Un$ | 10 | 2.88 | 3.74 | 0.15 | 0.77 | 0.55 | 0.5 | 1156.42 | 44 | 1.32 | 3.89 | 0.14 | 0.89 | 0.99 | 0.5 | 1363.43 |
| | $HN$ | 13 | 2.91 | 3.23 | 0.2 | 0.77 | 0.57 | 0.5 | 1160.73 | 45 | 1.22 | 3.82 | 0.15 | 0.88 | 0.96 | 0.5 | 1353.81 |
| | $YR$ | 11 | 1.76 | 5 | 0.082 | 0.88 | 0.32 | 0.12 | 1204.79 | 11 | 0.68 | 4.72 | 0.094 | 0.9 | 0.42 | 0.5 | 1230.94 |
| 2 | $NE$ | 40 | 2 | 10.4 | 0.005 | 0.66 | 0.99 | 0.024 | 1449.84 | 47 | 1.29 | 3.22 | 0.2 | 0.87 | 0.98 | 0.5 | 1377 |
| | $Un$ | 8 | 2.42 | 4.03 | 0.13 | 0.72 | 0.41 | 0.5 | 1148.92 | 24 | 1 | 4.11 | 0.13 | 0.84 | 0.99 | 0.5 | 1333.48 |
| | $HN$ | 9 | 2.42 | 3.59 | 0.17 | 0.71 | 0.44 | 0.49 | 1152.69 | 26 | 0.84 | 4.26 | 0.12 | 0.81 | 0.93 | 0.5 | 1327.63 |
| | $YR$ | 10 | 1.73 | 4.89 | 0.087 | 0.87 | 0.264 | 0.1 | 1215.95 | 10 | 0.7 | 3.97 | 0.13 | 0.9 | 0.42 | 0.5 | 1270.02 |
| 2.5 | $NE$ | 6 | 2.83 | 1.33 | 0.51 | 0.72 | 0.33 | 0.51 | 1153 | 47 | 143 | 2.22 | 0.33 | 0.91 | 0.92 | 0.5 | 1429.82 |
| | $Un$ | 6 | 2.47 | 2.89 | 0.23 | 0.75 | 0.26 | 0.49 | 1144.09 | 11 | 0.49 | 4.98 | 0.082 | 0.72 | 0.91 | 0.5 | 1327.62 |
| | $HN$ | 7 | 2.4 | 2.58 | 0.27 | 0.74 | 0.28 | 0.49 | 1141.99 | 17 | 0.96 | 3.29 | 0.19 | 0.8 | 0.97 | 0.5 | 1353.43 |
| | $YR$ | 10 | 1.68 | 4.8 | 0.09 | 0.86 | 0.229 | 0.084 | 1217.35 | 10 | 0.79 | 3.25 | 0.2 | 0.94 | 0.44 | 0.5 | 1302.67 |
| 3 | $NE$ | 4 | 2.47 | 1 | 0.6 | 0.83 | 0.227 | 0.51 | 1140.36 | 49 | 1.23 | 2.1 | 0.35 | 0.92 | 0.77 | 0.5 | 1449.74 |
| | $Un$ | 5 | 2.63 | 1.24 | 0.53 | 0.86 | 0.2 | 0.5 | 1148.57 | 12 | 0.62 | 4 | 0.13 | 0.78 | 0.79 | 0.5 | 1352.8 |
| | $HN$ | 4 | 2.52 | 1.03 | 0.6 | 0.85 | 0.21 | 0.5 | 1136.46 | 47 | 1.25 | 2.13 | 0.34 | 0.93 | 0.76 | 0.5 | 1463.84 |
| | $YR$ | 10 | 1.57 | 4.67 | 0.097 | 0.86 | 0.18 | 0.68 | 1209.84 | 13 | 2.5 | 1 | 0.6 | 0.99 | 0.94 | 0.086 | 1507.75 |
| 4 | $NE$ | 3 | 2.05 | 1 | 0.6 | 0.81 | 0.15 | 0.4 | 1125.9 | 51 | 1.01 | 2.01 | 0.37 | 0.92 | 0.62 | 0.5 | 1473.23 |
| | $Un$ | 4 | 1.95 | 2.33 | 0.31 | 0.74 | 0.12 | 0.39 | 1126.8 | 47 | 1.05 | 2.1 | 0.35 | 0.94 | 0.61 | 0.5 | 1520.34 |
| | $HN$ | 3 | 2.06 | 1 | 0.61 | 0.83 | 0.13 | 0.43 | 1118.74 | 49 | 1.02 | 2.04 | 0.36 | 0.84 | 0.6 | 0.5 | 1487.96 |

tion.

$Y$: The average cost per false alarm when the process is under control.

$W_i$: Cost to locate and repair $i^{th}$ assignable cause.

$a$: Fixed sample cost.

$b$: Unit sample cost.

$P$: Production rate.

$\Delta$: Tolerance rate.

$A$: The cost to society for manufacturing a product out of specification.

$\sigma$: Standard deviation of the process.

$\lambda_i$: Weibull Scale parameter.

$k$: Weibull Shape parameter.

$n$: Sample size.

$h_1$: Sampling interval.

$L$: Control limits coefficient.

$\rho$: Average correlation factor within samples.

# Appendix B

According to Banerjee and Rahim (1988) and Regarding of $h_j$ we have:

$$h_j = [j^{\frac{1}{k}} - (j-1)^{\frac{1}{k}}]h_1$$

We show $h_j$ is decreasing based of $j$ :

$$\begin{aligned}\frac{\partial}{\partial j}h_j &= [\frac{1}{k}j^{\frac{1}{k}-1} - \frac{1}{k}(j-1)^{\frac{1}{k}-1}]h_1 \\ &= [j^{\frac{1}{k}-1} - (j-1)^{\frac{1}{k}-1}]\frac{h_1}{k}\end{aligned}$$

$h_1$ is sampling interval and positive. In the above formula, $\frac{1}{k}$ is positive, because the shape parameter of Weibull distribution $K \geq 1$. Hence $j^{\frac{1}{k}-1} - (j-1)^{\frac{1}{k}-1} \leq 0$, therefore we have: $\frac{\partial}{\partial j}h_j \leq 0$. Because of above reason, $h_j$ is decreasing based of $j$.

# Appendix C

Recall that the $X \sim N(\mu, V)$, with $V = \sigma^2 R$ is the process variance and $R$ is the correlation matrix.

$$\begin{aligned}Var(\overline{X}) &= \frac{1}{n^2}[\sum_{i=1}^{n} Var(X_i) + \sum_{i \neq j} Cov(X_i, X_j)] \\ &= \frac{1}{n^2}[n\sigma^2 + \sum_{i \neq j} V_{ij}] \\ &= \frac{1}{n^2}[n\sigma^2 + \sigma^2\{\sum_{i \neq j} r_{ij}\}]\end{aligned}$$

Let $\rho = \frac{\sum_{i \neq j} r_{ij}}{n(n-1)}$. Then

$$\begin{aligned}Var(\overline{X}) &= \frac{1}{n^2}[n\sigma^2 + \sigma^2 n(n-1)\rho] \\ &= \frac{\sigma^2}{n}[1 + (n-1)\rho]\end{aligned}$$

# Appendix D

According to Banerjee and Rahim (1988) and Regarding equation (7), $\omega_j$ and $h_j$ can be obtained as follows:

$$\int_{\omega_j}^{\omega_{j+1}} r_i(t)\,\mathrm{d}t = \int_{\omega_0}^{h_1} r_i(t)\,\mathrm{d}t, j = 1, 2, \ldots$$

$$\begin{aligned}\int_{\omega_j}^{\omega_{j+1}} \lambda_i k t^{k-1}(t)\,\mathrm{d}t &= \int_{\omega_0}^{h_1} \lambda_i k t^{k-1}(t)\,\mathrm{d}t \\ &= \omega_{j+1}^k - \omega_j^k = h_1^k\end{aligned}$$

$$If\ j = 1 : \omega_2^k = \omega_1^k + h_1^k \Rightarrow \omega_2 = 2^{\frac{1}{k}} h_1$$

$$If\ j = 2 : \omega_3^k = \omega_2^k + h_1^k \Rightarrow \omega_3 = 3^{\frac{1}{k}} h_1$$

$$\vdots$$

Therefore $\omega_j = j^{\frac{1}{k}} h_1$ and

$$h_j = \omega_j - \omega_{j-1}$$
$$= [j^{\frac{1}{k}} - (j-1)^{\frac{1}{k}}]h_1, j = 1, 2, ....$$

# Appendix E

Let $AATS_i$ be the average time between occurrence shifts in process mean owing to $i^{th}$ assignable cause and receiving true alarm from control chart, Banerjee and Rahim (1988).

$$\begin{aligned}
AATS_i &= \sum_{j=1}^{\infty}[\sum_{k=1}^{\infty} q_{ij}[\omega_{k+j-1} - \omega_{j-1}]\beta_i^{k-1}(1-\beta_i)] - \tau_i \\
&= \sum_{j=1}^{\infty}[\sum_{k=1}^{\infty}(1-p_i)^{j-1}p_i[\omega_{k+j-1} - \omega_{j-1}]\beta_i^{k-1}(1-\beta_i)] - \tau_i \\
&= (1-\beta_i)p_i\sum_{j=1}^{\infty}[\sum_{k=1}^{\infty}[\underbrace{(1-p_i)^{j-1}\omega_{k+j-1}\beta_i^{k-1}} - \underbrace{(1-p_i)^{j-1}\omega_{j-1}\beta_i^{k-1}}]] - \tau_i
\end{aligned}$$

For the first part, we have:

$$\begin{aligned}
I &= \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}(1-p_i)^{j-1}\omega_{k+j-1}\beta_i^{k-1} \\
&= \sum_{l=1}^{\infty}\sum_{j=1}^{l}(1-p_i)^{j-1}\omega_l\beta_i^{l-j} \\
&= \sum_{l=1}^{\infty}\omega_l\beta_i^l\sum_{j=1}^{l}(1-p_i)^{-1}(\frac{1-p_i}{\beta_i})^j \\
&= (\frac{1}{1-p_i})\sum_{l=1}^{\infty}\omega_l\beta_i^l(\frac{1-p_i}{\beta_i-1+p_i}) - (\frac{1}{1-p_i})\sum_{l=1}^{\infty}\omega_l(\frac{(1-p_i)^{l+1}}{\beta_i-1+p_i}) \\
&= (\frac{1}{p_i+\beta_i-1})(\sum_{l=1}^{\infty}\omega_l\beta_i^l - \sum_{l=1}^{\infty}\omega_l(1-p_i)^l) \\
&= (\frac{h_1}{p_i+\beta_i-1})(\beta_i A(\beta_i) - (1-p_i)A(1-p_i))
\end{aligned}$$

For the second part, we have:

$$\begin{aligned}
II &= \sum_{j=1}^{\infty}\sum_{k=1}^{\infty}(1-p_i)^{j-1}\omega_{j-1}\beta_i^{k-1} \\
&= \sum_{j=1}^{\infty}(1-p_i)^{j-1}\omega_{j-1}\sum_{k=1}^{\infty}\beta_i^{k-1} \\
&= \frac{h_1(1-p_i)}{1-\beta_i}A(1-p_i)
\end{aligned}$$

By substituting and simplifying, final formula is obtained.

# Appendix F

If $B_i$ is the event of the occurrence of $i^{th}$ assignable cause, then

$$P(B_i) = P(T_i < Y),$$

where $Y = min(T_1, T_2, ..., T_{i-1}, T_{i+1}, ..., T_s)$
Let $\lambda' = \sum_{i \neq j}$ and $\lambda_0 = \sum \lambda_j, j = 1, 2, ..., s$, then

$$\begin{aligned}
P(B_i) &= P(T_i < Y) \\
&= \int_0^{+\infty} P(T_i < Y | Y = y) f_Y(y) \mathrm{d}y \\
&= \int_0^{+\infty} P(T_i < Y) f_Y(y) \mathrm{d}y \\
&= \int_0^{+\infty} (1 - e^{-\lambda_i y^k}) \lambda' k y^{k-1} e^{-\lambda y^k} \mathrm{d}y \\
&= \int_0^{+\infty} \lambda' k y^{k-1} e^{-\lambda y^k} \mathrm{d}y \\
&- \int_0^{+\infty} e^{-\lambda_i y^k} \lambda' k y^{k-1} e^{-\lambda y^k} \mathrm{d}y \\
&= 1 - \lambda' \int_0^{+\infty} k y^{k-1} e^{-y^k(\lambda_i + \lambda')} \mathrm{d}y \\
&= 1 - \frac{\lambda'}{\lambda_i + \lambda'} \\
&= \frac{\lambda_i}{\lambda_i + \lambda'} \\
&= \frac{\lambda_i}{\lambda_0}
\end{aligned}$$

## References


Ahmed I, Sultana I, Paul SK, Azeem A (2014). "Performance Evaluation of Control Chart for Multiple Assignable Causes Using Genetic Algorithm." *The International Journal of Advanced Manufacturing Technology*, **70**, 1889–1902. doi:10.1007/s00170-013-5412-0.

Al-Ghazi A, Al-Shareef K, Usher JS, Duffuaa SO (2007). "Integration of Taguchi's Loss Function in the Economic Design of Control Charts with Increasing Failure Rate and Early Replacement." In *In 2007 IEEE International Conference on Industrial Engineering and Engineering Management*, pp. 338–346. doi:10.1108/02656710310476552.

Al-Oraini HA, Rahim MA (2002). "Economic Statistical Design of Control Charts for Systems with Gamma ($\lambda$, 2) In-control Times." *Computers and Industrial Engineering*, **43**(3), 645–654. doi:10.1080/02664760320000035430.

Alexander SM, Dillman MA, Usher JS, Damodaran B (1995). "Economic Design of Control Charts Using the Taguchi Loss Function." *Computers and Industrial Engineering*, **28**(3), 671–679.

Bahiraee E, Raissi S (2014). "Economic Design of Hotelling's Control Chart on the Presence of Fixed Sampling Rate and Exponentially Assignable Causes." *Journal of Industrial Engineering International*, **10**(4), 229–238. doi:10.1007/s40092-014-0062-x.

Banerjee PK, Rahim MA (1988). "Economic Design of $\overline{X}$-control Charts under Weibull Shock Models." *Technometrics*, **3**(4), 407–414. doi:10.2307/1269803.

Ben-Daya M, Duffuaa SO (2003). "Integration of Taguchi's Loss Function Approach in the Economic Design of $\overline{X}$-chart." *International Journal of Quality & Reliability Management*, **20**(5), 607–619. doi:10.1108/02656710310476552.

Chen GZ (1995). *A Study of an Application on the Multi-characteristic Quality Loss Function*. Master's thesis, Providence University.

Chen YK, Hsieh KL, Chang CC (2007). "Economic Design of the VSSI Control Charts for Correlated Data." *International Journal of Production Economics*, **107**(2), 528–539. doi:10.1016/j.ijpe.2006.10.008.

Chen YS, Yang YM (2002). "Economic Design of Control Charts with Weibull In-control Times when There Are Multiple Assignable Causes." *International Journal of Production Economics*, **77**(1), 17–23. doi:0.1016/S0925-5273(01)00196-7.

Chou CY, Chen CH, Liu HR (2001). "Economic Design of Control Charts for Non-normally Correlated Data." *International Journal of Production Research*, **39**(9), 1931–1941. doi:10.1080/00207540110034913.

Chung KJ (1991). "Economic Designs of Attribute Control Charts for Multiple Assignable Causes." *Optimization*, **22**(5), 775–786. doi:10.1080/02331939108843719.

Deming WE (1982). *Out of the Crisis, Mass.* Institute of Technology.

Duncan AJ (1956). "The Economic Design of $\overline{X}$ Charts Used to Maintain Current Control of a Process." *Journal of the American Statistical Association*, **51**(274), 228–242. doi:10.1080/01621459.1956.10501322.

Duncan AJ (1971). "The Economic Design of $\overline{X}$-charts when There Is a Multiplicity of Assignable Causes." *Journal of the American Statistical Association*, **66**(4), 107–121. doi:10.1080/01621459.1971.10482230.

Elsayed EA, Chen A (1994). "An Economic Design of Control Chart Using Quadratic Loss Gunction." *The International Journal of Production Research*, **32**(4), 837–887. doi:10.1080/00207549408956976.

Faraz A, Heuchenne C, Saniga E, Costa AF (2014). "Double-objective Economic Statistical Design of the VP Control Chart: Wald's Identity Approach." *Journal of Statistical Computation and Simulation*, **84**(10), 2123–2137. doi:10.1080/00949655.2013.784315.

Gibra IN (1981). "Economic Design of Attribute Control Charts for Multiple Assignable Causes." *Journal of Quality Technology*, **13**(2), 93–99. doi:10.1080/00224065.1981.11980997.

Heikes RG, Montgomery DC, Yeung JY (1974). "Alternative Process Models in the Economic Design of Control Charts." *AIIE Transactions*, **6**(1), 55–61. doi:10.1080/05695557408974933.

Heydari AA, Moghadam MB, Eskandari F (2016). "Economic and Economic Statistical Designs of $\overline{X}$ Control Charts under Burr XII Shock Model." (in press).

Jolayemi JK, Berrettoni JN (1989). "An Optimal Design of Multivariate Control Charts in the Presence of Multiple Assignable Causes." *Applied Mathematics and Computation*, **32**(1), 17–33. doi:10.1016/0096-3003(89)90044-1.

Kapur KC, Cho BR (1996). "Economic Design of the Specification Region for Multiple Quality Characteristics." *AIIE Transactions*, **28**(3), 237–248. doi:10.1080/07408179608966270.

Kraleti SR, Kambagowni VS (2010). "Optimal Design of Control Chart with Pareto in Control Times." *The International Journal of Advanced Manufacturing Technology*, **48**(9-12), 829–837. doi:10.1007/s0017000-9-2348-5.

Lorenzen TJ, Vance LC (1986). "The Economic Design of $\overline{X}$ Control Charts: a Unified Approach." *Technometrics*, **28**(1), 3–10. doi:10.1080/00401706.1986.10488092\#.XHlSf-gzbIU.

Moghadam MB, Moghadam MG, Rafie S, Naderi H (2016). "A Generalized Version of Banerjee and Rahim Model in Economic Design of $\overline{X}$ Control Chart under Generalized Exponential Shock Model Using Non-uniform Sampling Scheme." (in press).

Montgomery DC, Klatt PJ (1972). "Minimum Cost Multivariate Quality Control Tests." *AIIE Transactions*, **4**(2), 103–110. doi:10.1080/05695557208974836.

Rahim MA, Banerjee PK (1993). "A Generalized Model for the Economic Design of Control Charts for Production Systems with Increasing Failure Rate and Early Replacement." *Naval Research Logistics (NRL)*, **40**(6), 787–809.

Safaei AS, Kazemzadeh RB, Niaki STA (2012). "Multi-objective Economic Statistical Design of $\overline{X}$-control Chart Considering Taguchi Loss Function." *The International Journal of Advanced Manufacturing Technology*, **59**(9-12), 1091–1101. doi:10.1007/s00170-011-3550-9.

Saniga EM (1989). "Economic Statistical Control-chart Designs with an Application to and R Charts." *Technometrics*, **31**(3), 313–320. doi:10.1080/00401706.1989.10488554.

Seif A, Moghadam MB, Faraz A, Heuchenne C (2011). "Statistical Merits and Economic Evaluation of Control Charts with the vssc Scheme." *Arabian Journal for Science and Engineering*, **36**(7), 1461–1470.

Serel DA, Moskowitz H (2008). "Joint Economic Design of EWMA Control Charts for Mean and Variance." *European Journal of Operational Research*, **184**(1), 157–168. doi:10.1016/j.ejor.2006.09.084.

Tagaras G, Lee HL (1988). "Economic Design of Control Charts with Different Control Limits for Different Assignable Causes." *Management Science*, **34**(11), 1347–1366. doi:10.1287/mnsc.34.11.1347.

Taguchi G, Chowdhury S, Wu Y (2005). *Taguchi's Quality Engineering Handbook*. John Wiley and Sons.

Taguchi G, Elsayed EA, Hsiang TC (1989). *Quality Engineering in Production Systems*. McGraw-Hill College.

Yang SF (1998). "Economic Statistical Design of S Control Charts Using Taguchi Loss Function." *International Journal of Quality & Reliability Management*, **15**(3), 259–272. doi:10.1108/02656719810209446.

Yang SF, Rahim MA (2006). "Multivariate Extension to the Economic Design of Control Chart under Weibull Shock Model." *Journal of Statistical Computation and Simulation*, **76**(12), 1035–1047. doi:10.1080/10629360600569188.

Yang YM, Su CY, Pearn WL (2010). "Economic Design of Control Charts for Continuous Flow Process with Multiple Assignable Causes." *International Journal of Production Economics*, **128**(1). doi:10.1016/j.ijpe.2010.04.048.

Yu FJ, Chen HK (2009). "Economic-Statistical Design of $\overline{X}$ Control Charts Using Taguchi Loss Functions." *IFAC Proceedings Volumes*, **42**(4), 1719–1723. doi:10.3182/200906033RU2001.0098.

Yu FJ, Tsou CS, Huang KI, Wu Z (2010). "An Economic-Statistical Design of control Charts with Multiple Assignable Causes." **17**(4), 327–338.

Zhang G, Berardi V (1997). "Economic Statistical Design of Control Charts for Systems with Weibull In-control Times." *Computers and Industrial Engineering*, **32**(3), 575–586. doi:10.1016/S0360-8352(96)00314\-2.

**Affiliation:**

M.H. Naderi
Department of statistics
Allameh Tabataba'i University
Tehran,Iran.
E-mail: h.naderi@atu.ac.ir

A. Seif
Department of statistics
Bu-Ali Sina University
Hamedan, Iran.
E-mail: erfan.seif@gmail.com

M. Bameni Moghadam
Department of statistics
Allameh Tabataba'i University
Tehran,Iran.
E-mail: Bamenimoghadam@atu.ac.ir

Austrian Journal of Statistics http://www.ajs.or.at/
published by the Austrian Society of Statistics http://www.osg.or.at/
Volume 50 *Submitted:* 2018-08-13
July 2021 *Accepted:* 2019-03-15