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On a Generalized Alpha Skew Laplace Distribution: Properties and Applications

Jondeep Das
Department of Statistics,
Dibrugarh University,
Dibrugarh, India

Partha Jyoti Hazarika
Department of Statistics,
Dibrugarh University,
Dibrugarh, India

Dimpal Pathak
Department of Statistics,
Dibrugarh University,
Dibrugarh, India

Piotr Sulewski
Institute of Exact and Technical Sciences,
Pomeranian University, Poland

Abstract

This study introduces a novel extension of the alpha skew Laplace distribution (Harandi and Alamatsaz 2013), designed to model datasets with both unimodal and bimodal characteristics effectively. A detailed examination of its statistical properties is conducted, including moments, moment-generating functions, cumulative distribution functions, and characterization results based on conditional distributions. The study also develops a location-scale generalization, enhancing the scope of the distribution for broader applications. Besides, parameter estimation is performed using maximum likelihood methods, supported by a comprehensive analysis of the Fisher Information Matrix, ensuring robust and precise inference.

The utility of the new distribution is validated through simulation studies which confirms the asymptotic consistency of its parameter estimates under varying sample sizes. Real-life applications involving white cell count data of Australian athletes and failure times of aircraft windshields demonstrate its superior performance over some established models. The newly introduced model consistently achieves lower Akaike and Bayesian Information Criteria (AIC/BIC) values, confirming its efficacy in fitting complex datasets. Additionally, likelihood ratio tests provide statistical evidence supporting its distinct advantages over nested models. This distribution represents a significant advancement in the field of probability and statistical modeling, offering a robust and versatile tool for analyzing diverse datasets with asymmetric or bimodal traits.

Keywords: Laplace distribution, skew Laplace distribution, skewness, AIC, bimodal distribution.

1. Introduction

Since Azzalini (1985) introduced the skew normal distribution, numerous researchers have been consistently investigating the skew normal (SN) model from various perspectives. The probability density function (PDF) of the SN model provided by Azzalini (1985) was given by

$$\phi(x;\lambda) = 2\phi(x)\Phi(\lambda x); \quad x \in R, \lambda \in R,$$

where, $\phi(.)$ and $\Phi(.)$ are the PDF and cumulative distribution function (CDF) of standard normal distribution respectively. In the existing literature, numerous studies are present that explore various generalizations, modifications, and extensions of Azzalini's skew normal model introduced in 1985. Later on, Azzalini (1986) elaborated some extension of prior findings, incorporating additional results. Concurrently, Henze (1986) provided a probabilistic representation of the skew normal distribution (Azzalini 1985). Additionally, skew generalized normal distribution Arellano-Valle, Gómez, and Quintana (2004), extended skew generalized normal distribution (Choudhury and Matin 2011) , epsilon skew normal distribution Mudholkar and Hutson (2000) , Balakrishnan skew normal distribution Sharafi and Behboodian (2008) etc. were some of the noticeable extensions of skew normal Azzalini (1985) distribution. Conversely, Aryal and Nadarajah (2005) presented the skew Laplace (SLa) distribution by employing the framework of the SN model introduced by Azzalini (1985). This involved substituting the CDF and PDF of the standard normal distribution with those of the standard Laplace distribution. The PDF of skew Laplace distribution was given by

$$f(x;\lambda) = \begin{cases} \frac{1}{2}e^{x(\lambda+1)}, & x \le 0 \\ \left(e^{-x} - \frac{1}{2}e^{-x(\lambda+1)}\right), & x > 0 \end{cases}$$

where, λ is the asymmetry parameter with positive support. Likewise, the skew logistic distribution, which was introduced by Wahed and Ali (2001), follows a skewed model framework based on the skew normal distribution (Azzalini 1985). Nadarajah (2009) also discussed some important statistical properties of the skew logistic distribution. Besides these, recently some new families of unimodal asymmetric distribution are introduced in the literature which includes Sine generalized family of distributions (Oramulu, Alsadat, Kumar, Bahloul, and Obulezi 2024), Novel two-parameter quadratic exponential distribution (Bousseba, Zeghdoudi, Sapkota, Tashkandy, Bakr, Kumar, and Gemeay 2024), A new unit distribution (Karakaya, Rajitha, Sağlam, Tashkandy, Bakr, Muse, Kumar, Hussam, and Gemeay 2024), A New 3-Parameter Bounded Beta Distribution (Althubyani, El-Bar, Fawzy, and Gemeay 2022), A New Power Topp–Leone distribution (Atchadé, N'bouké, Djibril, Shahzadi, Hussam, Aldallal, Alshanbari, Gemeay, and El-Bagoury 2023), Generalized Gudermannian Distribution (Rasekhi, Saber, Hamedani, El-Raouf, Aldallal, and Gemeay 2022), Half Logistic Modified Kies Exponential distribution (Alghamdi, Shrahili, Hassan, Gemeay, Elbatal, and Elgarhy 2023), Flexible Extension of Reduced Kies Distribution (Almuqrin, Gemeay, El-Raouf, Kilai, Aldallal, and Hussam 2022) etc.

In addition to examining different skewed distributions capable of fitting unimodal datasets, there are some families of distribution designed specifically for modeling datasets exhibiting bimodal behavior in both symmetric and asymmetric cases. Bimodal distributions have been extensively applied across various fields, highlighting their significance in statistical analysis. For example, in agricultural research, Sarma, Rao, and Rao (1990) employed bimodal distributions to study seed length data. Similarly, in medical research, Wang, Wen, Symmans, Pusztai, and Coombes (2009) explored gene expression patterns in cancer data using bimodal frameworks. da Braga, Cordeiro, Ortega, and da Cruz (2016) applied an odd log-logistic skew-normal model to analyze temperature and soybean production data, uncovering distinct bimodal features. Famoye, Lee, and Eugene (2004) contributed to understanding reproductive processes by examining egg diameter data using a specialized distribution. Additionally, Fan, Gao, Zhang, Adams, Kutsop, Bierson, Liu, Yang, Young, Cheng *et al.* (2022) investigated the atmospheric haze on Pluto, identifying a bimodal distribution in their findings. So, for the past few years, a continuous study is undergo to develop different distributions fo fitting bimodal data. Kim (2005) introduced a category of two-piece skew normal distribution that extends the skew normal distribution proposed by Azzalini in 1985 to accommodate bimodality. This class was further extended by Arnold, Gómez, and Salinas (2009). On a separate note, Elal-Olivero (2010) introduced another innovative distribution family designed to fit both bimodal and unimodal data. This skew distribution was named as Alpha Skew Normal (ASN) distribution with the PDF

$$f(x;\alpha) = \frac{(1-\alpha x)^2+1}{2+\alpha^2}\phi(x), \quad x \in R, \alpha \in R.$$

Employing a similar approach, Hazarika and Chakraborty (2014) introduced the alpha skew logistic distribution. Similarly, Harandi and Alamatsaz (2013) applied the same concept to introduce the alpha skew Laplace (ASLa) distribution. The PDF of Alpha Skew Laplace distribution was given by

$$f(x;\alpha) = \left(\frac{(1-\alpha x)^2+1}{4(1+\alpha^2)}\right)\left(\frac{e^{-|x|}}{2}\right), \quad x \in R, \alpha \in R. \tag{1}$$

In addition to the mentioned contributions, several families of probability distributions were introduced using the mechanism proposed by Balakrishnan (Arnold, Beaver, Azzalini, Balakrishnan, Bhaumik, Dey, Cuadras, Sarabia, Arnold, and Beaver 2002). These include the Balakrishnan alpha skew normal (Hazarika, Shah, and Chakraborty 2020), Balakrishnan alpha skew logistic (Shah, Chakraborty, and Hazarika 2020a), Balakrishnan Alpha Skew Laplace (Shah, Hazarika, and Chakraborty 2020c) distributions, Generalized Balakrishnan alpha skew normal (Shah, Hazarika, Chakraborty, and Hamedani 2024) etc. which were designed to fit both unimodal and bimodal datasets. Conversely, examples of research exploring the bimodal behavior of datasets include the log alpha skew normal distribution (Venegas, Bolfarine, Gallardo, Vergara-Fernández, and Gómez 2016), Balakrishnan log alpha skew normal distribution (Shah, Chakraborty, Hazarika, and Ali 2020b), Balakrishnan Alpha skew generalized t distribution (Pathak, Shah, Hazarika, Chakraborty, and Das 2023b), Bimodal skew-symmetric normal distribution (Hassan and El-Bassiouni 2016), log bimodal skew normal distribution (Bolfarine, Gómez, and Rivas 2011), and bimodal Tanh skew normal distribution (Das, Hazarika, Chakraborty, Pathak, Hamedani, and Karamikabir 2024). Sharafi, Sajjadnia, and Behboodian (2017) introduced a novel generalization of the alpha skew normal distribution proposed by (Elal-Olivero 2010). This generalization encompasses normal, skew normal (Azzalini 1985), alpha skew normal (Elal-Olivero 2010) distribution and normal distribution as three specific cases. Terming the generalized alpha skew normal (GASN) distribution, it proved to be a superior fitting model compared to the alpha skew normal (Elal-Olivero 2010) distribution, particularly when dealing with datasets involving up to two modes. The density function of the distribution was given as

$$f(x;\alpha) = \frac{(1-\alpha x)^2+1}{2+\alpha^2}\phi(x)\Phi(\lambda x), \quad x \in Z, \alpha \in R.$$

Subsequently, some another novel families of probability distribution were introduced in the literature, enabling the fitting of data with more than two modes. Prominent outcomes from research endeavors in these directions include the alpha beta skew normal distribution Shafiei, Doostparast, and Jamalizadeh (2016), alpha beta skew logistic distribution (Esmaeili, Lak, Alizadeh *et al.* 2020), generalized alpha beta skew normal distribution (Shah, Hazarika, Chakraborty, and Ali 2023), Some flexible classes of trimodal and multimodal distribution (Martínez-Flórez, Tovar-Falón, and Elal-Olivero 2022), tri-modal skew logistic distribution (Pathak, Hazarika, Chakraborty, Das, and Hamedani 2023a), and flexible alpha skew normal distribution (Das, Pathak, Hazarika, Chakraborty, and Hamedani 2023), among others.

This article introduces a new family of skewed distributions that accommodates both unimodal and bimodal datasets for fitting, employing the mechanism of the generalized alpha skew normal distribution proposed by Sharafi *et al.* (2017). In this context, the PDF and CDF of the Laplace distribution are employed instead of the standard normal distribution. This results in a new distribution that serves as a generalization of existing families, including alpha skew Laplace (Harandi and Alamatsaz 2013), skew Laplace (Aryal and Nadarajah 2005), and the Laplace distribution. Additionally, the superiority of the proposed model is assessed by comparing it with several existing sub-models found in the extensive literature, employing two real-life datasets.

The rest of the article is been organized as follows: Section 2 introduces a new generalization of the alpha skew Laplace distribution (Harandi and Alamatsaz 2013), including plots of the density function for various parameter values and some special cases. Section 3 discusses important statistical properties of the new distribution, while Section 4 provides its characterizations. Section 5 focuses on parameter estimation for the proposed distribution and includes the location-scale extension and the Fisher information matrix. Simulation results are presented in Section 6. Section 7 covers the real-life application of the proposed model using two real-world datasets. Finally, Section 8 presents hypothesis testing for the new distribution, and Section 9 concludes the article.

2. A new generalization of alpha skew Laplace distribution

In this section, a new generalization of the alpha skew Laplace distribution (Harandi and Alamatsaz 2013) is presented, along with its significant statistical characteristics and graphical presentation.

Definition 2.1. A random variable is said to follow Generalized Alpha Skew Laplace distribution if its probability density function (PDF) is given as

$$f(x;\alpha,\lambda) = \begin{cases} \left(\dfrac{(1-\alpha x)^2+1}{2C}\right) e^{-(1+|\lambda|)|x|}, & \lambda x \leq 0 \\[2ex] \left(\dfrac{(1-\alpha x)^2+1}{C}\right) e^{-|x|}\left(1-\dfrac{1}{2}e^{-\lambda x}\right), & \lambda x \geq 0 \end{cases} \tag{2}$$

where, $\alpha \in R, x \in R$ and C is the normalizing constant which is given as

$$C = 2\left(1+\alpha^2-\frac{\alpha\lambda(2+\lambda)}{(1+\lambda)^2}\right).$$

The density function of the newly proposed distribution is denoted as $GASLa(\alpha,\lambda)$. Again, borrowing the concept of Gupta, Chang, and Huang (2002), PDF of the new distribution also can be rewritten as

$$f(x;\alpha,\lambda) = \left(\frac{(1-\alpha x)^2+1}{C}\right)\left(\frac{e^{-|x|}\left(1+Sign(\lambda x)(1-e^{-|\lambda x|})\right)}{2}\right). \tag{3}$$

where, α and λ are shape parameters and $Sign(\lambda x)$ is an indicator function that gives -1, 0, or 1 depending on whether x is negative, zero, or positive.

Throughout the rest of the article, unless otherwise stated, it is assumed that $\lambda > 0$ as for $\lambda < 0$ can be obtained using the fact that $-X \sim GASLa(-\alpha,-\lambda)$.

Some special cases of $GASLa(\alpha,\lambda)$ distribution are obtained as

i. If $\alpha = 0$ then, $X \sim SLa(\lambda)$ (Aryal and Nadarajah 2005).

ii. If $\lambda = 0$ then, $X \sim ASLa(\lambda)$ (Harandi and Alamatsaz 2013).

iii. If $\alpha = 0, \lambda = 0$ then, $X \sim La(0, 1)$.

iv. If $X \sim GASLa(\alpha, \lambda)$ then, $-X \sim GASLa(-\alpha, -\lambda)$.

Furthermore, plot of the density function of $GASLa(\alpha, \lambda)$ distribution for different values of parameters is shown in Figure 1. It is observed that, depending upon the choices of the parameters the density function of the $GASLa(\alpha, \lambda)$ distribution contains at most two modes. The Figure 1 (a) illustrates that the $GASLa(\alpha, \lambda)$ distribution possesses unimodality when $\alpha = 0$. On the other hand from the Figure 1 (b) it is clear that for some fixed value of λ if value of α increases then the PDF exhibits high peak to the right tail. Similarly, the PDF may exhibit high peak to the right tail for some other increasing value of α with fixed λ (Figure 1 (c)). Moreover, Figure 1 (d) shows that for some fixed value of α, if λ is increased then the PDF shows higher peak towards the right tail.

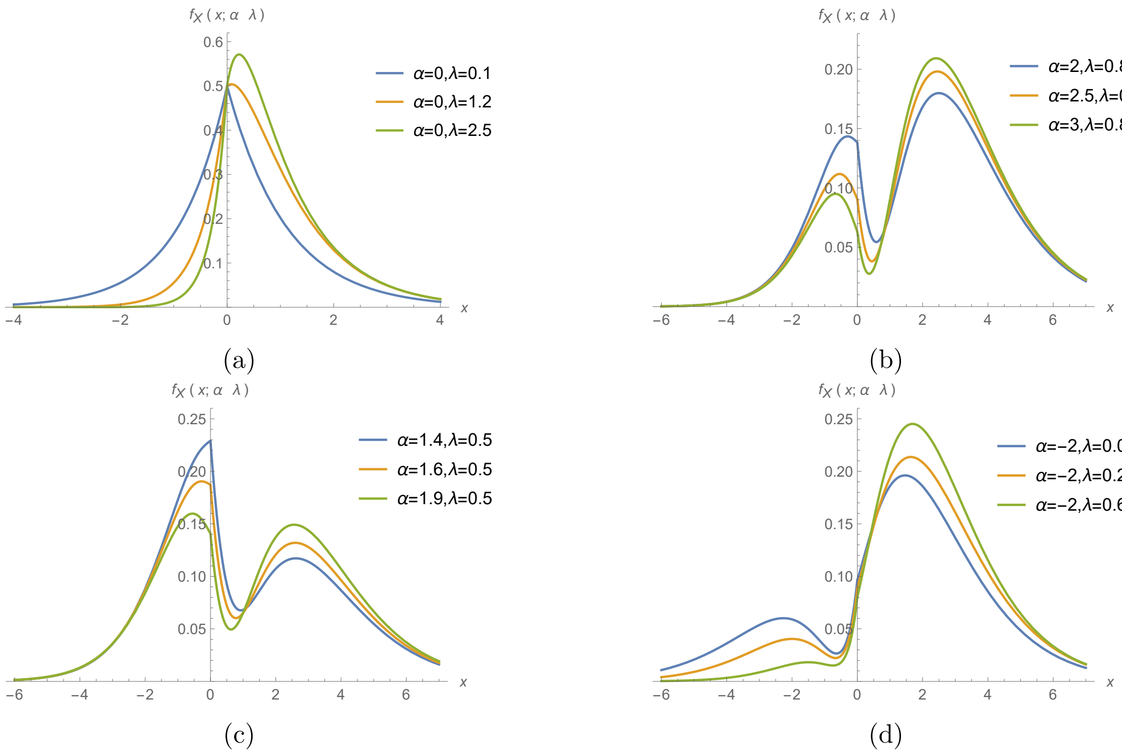


Figure 1: Density function of $GASLa(\alpha, \lambda)$ with different values of α and λ

3. Statistical properties of introduced distributions

This section is dedicated to the primary statistical properties of the $GASLa(\alpha, \lambda)$ distribution, including its distribution function, order moments, moment generating function, moments of order statistics, skewness and kurtosis, quantiles, and pseudo-random generator.

3.1. Cumulative distribution function (CDF)

Theorem 3.1. *The CDF of generalized alpha skew Laplace (GASLa) distribution is given by*

$$F(x;\alpha,\lambda)=\begin{cases}\dfrac{e^{C_1x}}{CC_1}\left[1-\dfrac{\alpha(C_1x-1)}{C_1}+\dfrac{\alpha^2\left[(C_1x-1)^2+1\right]}{2C_1^2}\right], & x<0,\\[2ex] \dfrac{1}{2C}\Bigg[4(1-e^{-x})+\dfrac{2e^{-C_1x}}{C_1}-4\alpha\Big(1-e^{-x}(1+x)\Big) & \\ \quad+\dfrac{2\alpha}{C_1^2}\Big(2-e^{-C_1x}(1+C_1x)\Big)+2\alpha^2\Big[2-e^{-x}\Big\{(x+1)^2+1\Big\}\Big] & \\ \quad+\dfrac{\alpha^2e^{-C_1x}}{C_1^3}\Big((C_1x+1)^2+1\Big)\Bigg], & x\geq 0\end{cases} \tag{4}$$

where, $C_1=1+\lambda$.

Proof. From (2), for $x<0$, we have

$$\begin{aligned}F(x;\alpha,\lambda)=P(X\leq x)&=\int_{-\infty}^{x}\frac{(1-\alpha x)^2+1}{2C}e^{(1+\lambda)x}dx\\&=\frac{1}{2C}(2I_1-2\alpha I_2+\alpha^2I_3),\end{aligned}$$

where,

$$\begin{aligned}I_1&=\int_{-\infty}^{x}e^{(1+\lambda)x}dx=\frac{e^{(1+\lambda)x}}{1+\lambda}=\frac{e^{C_1x}}{C_1},\\I_2&=\int_{-\infty}^{x}xe^{(1+\lambda)x}dx=\frac{e^{x(1+\lambda)(-1+x+x\lambda)}}{(1+\lambda)^2}=\frac{e^{C_1x}\left[C_1x-1\right]}{C_1^2}\\I_3&=\int_{-\infty}^{x}x^2e^{(1+\lambda)x}dx=\frac{e^{x(1+\lambda)}\left[2+x(1+\lambda)(-2+x+x\lambda)\right]}{(1+\lambda)^3}=\frac{e^{C_1x}\left[(C_1x-1)^2+1\right]}{C_1^3}\end{aligned}$$

Based on the results of the integrations, we obtain,

$$\begin{aligned}F(x;\alpha,\lambda)&=\frac{1}{2C}\left[2\frac{e^{C_1x}}{C_1}-2\alpha\frac{e^{C_1x}\left[C_1x-1\right]}{C_1^2}+\alpha^2\frac{e^{C_1x}\left[(C_1x-1)^2+1\right]}{C_1^3}\right]\\&=\frac{e^{C_1x}}{CC_1}\left[1-\frac{\alpha(C_1x-1)}{C_1}+\frac{\alpha^2\left[(C_1x-1)^2+1\right]}{2C_1^2}\right]\end{aligned}\tag{5}$$

Again for $x\geq 0$, we have

$$\begin{aligned}F(x;\alpha,\lambda)&=\int_{-\infty}^{0}\frac{(1-\alpha x)^2+1}{2C}e^{(1+\lambda)x}dx+\int_{0}^{x}\frac{(1-\alpha x)^2+1}{2C}e^{-x}(2-e^{-\lambda x})dx\\&=I_4+I_5.\end{aligned}\tag{6}$$

Based on (5), we obatin the first summand of (6) as

$$I_4=F(0;\alpha,\lambda)=\frac{1}{CC_1}\left[1+\frac{\alpha}{C_1}+\frac{\alpha^2}{C_1^2}\right]$$

The second summand of (6) is given by

$$I_5=\frac{1}{2C}\int_0^x\frac{2-2\alpha x+\alpha^2x^2}{(2e^{-x}-e^{-xC_1})^{-1}}dx=\frac{1}{2C}\left[4I_6-2I_7-4\alpha I_8+2\alpha I_9+2\alpha^2I_{10}-\alpha^2I_{11}\right],$$

where,

$$I_6=\int_0^x e^{-x}dx=1-e^{-x}, I_7=\int_0^x e^{-xC_1}dx=\frac{1-e^{-C_1x}}{C_1},$$
$$I_8=\int_0^x xe^{-x}dx=1-e^{-x}(1+x), I_9=\int_0^x xe^{-xC_1}dx=\frac{1-e^{-C_1x}(1+C_1x)}{C_1^2}$$
$$I_{10}=\int_0^x x^2e^{-x}dx=2-e^{-x}\left[(x+1)^2+1\right], I_{11}=\int_0^x x^2e^{-xC_1}dx=\frac{2-e^{-C_1x}(1+(C_1x+1)^2)}{C_1^2}$$

Based on the results of simple integration verified in Mathcad, we obtain

$$I_5=\frac{1}{2C}\left[4(1-e^{-x})-2\frac{1-e^{-C_1x}}{C_1}-4\alpha(1-e^{-x}(1+x))+2\alpha\frac{1-e^{-C_1x}(1+C_1x)}{C_1^2}\right.$$
$$\left.+2\alpha^2\left(2-e^{-x}\left[(x+1)^2+1\right]\right)-\alpha^2\frac{2-e^{-C_1x}(1+(C_1x+1)^2)}{C_1^2}\right].$$

Hence, after some simplifications, (6) can be obtained as

$$F(x;\alpha,\lambda)=\frac{1}{2C}\left[4(1-e^{-x})+\frac{2e^{-C_1x}}{C_1}-4\alpha\Big(1-e^{-x}(1+x)\Big)\right.$$
$$+\frac{2\alpha}{C_1^2}\Big(2-e^{-C_1x}(1+C_1x)\Big)+2\alpha^2\Big[2-e^{-x}\Big\{(x+1)^2+1\Big\}\Big]$$
$$\left.+\frac{\alpha^2e^{-C_1x}}{C_1^3}\Big((C_1x+1)^2+1\Big)\right]$$

Hence, the final results of CDF for $GASLa(\alpha,\lambda)$ distribution is obtained as

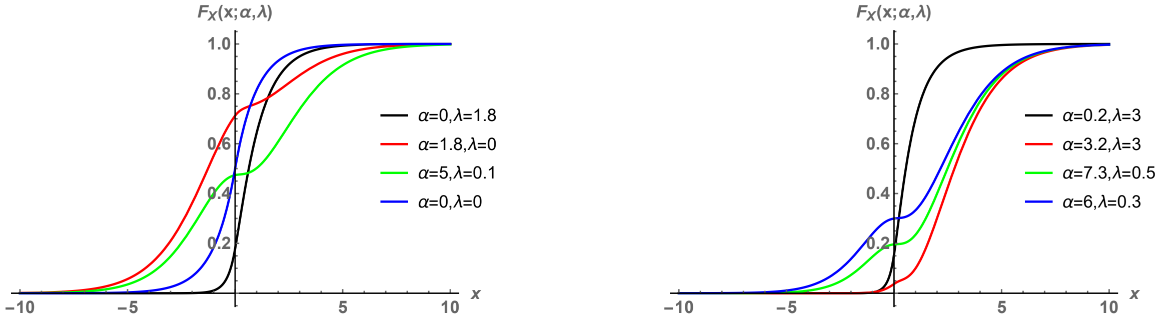
$$F(x;\alpha,\lambda)=\begin{cases}\dfrac{e^{C_1x}}{CC_1}\left[1-\dfrac{\alpha(C_1x-1)}{C_1}+\dfrac{\alpha^2\left[(C_1x-1)^2+1\right]}{2C_1^2}\right], & x<0,\\[2ex] \dfrac{1}{2C}\left[4(1-e^{-x})+\dfrac{2e^{-C_1x}}{C_1}-4\alpha\Big(1-e^{-x}(1+x)\Big)\right.\\ \quad+\dfrac{2\alpha}{C_1^2}\Big(2-e^{-C_1x}(1+C_1x)\Big)+2\alpha^2\Big[2-e^{-x}\Big\{(x+1)^2+1\Big\}\Big]\\ \quad\left.+\dfrac{\alpha^2e^{-C_1x}}{C_1^3}\Big((C_1x+1)^2+1\Big)\right], & x\geq 0\end{cases}$$

□

The CDF of $GASLa(\alpha,\lambda)$ for different selected parameters are plotted in Figure 2. The presence of bimodality in the new distribution also can be observed in Figure 2.

The $GASLa(\alpha,\lambda)$ can be used to departure from the Gaussian distribution with PDF $\phi(x;0,1)$. The similarity measure between our proposal and the $N(0,1)$ is given by (Sulewski 2020). $M(\alpha,\lambda)=\int_{-\infty}^{\infty} min\Big[f(x;\alpha,\lambda),\phi(x;0,1)\Big]$.

Numerical examination in Mathcad gives $M_max(0.22,0.303)=0.891$, so the $GASLa(0.22,0.303)$ is the $N(0,1)$ in 89.1%. Exemplary $M(\alpha,\lambda)$ values are presented in Table 1.



(a) (b)

Figure 2: CDFs of the $GASLa(\alpha,\lambda)$ for selected parameter values

Table 1: Similarity measure $M(\alpha,\lambda)$ of the $GASLa(\alpha,\lambda)$ to $N(0,1)$. Rows I show values of α when $\lambda = 0.303$, rows II – values of λ when $\alpha = 0.22$.

I	0.220	-0.028	-0.171	-0.294	-0.403	-0.521	-0.656
II	0.303	0.714	1.007	1.400	1.978	2.933	4.828
$M(a,b,1.638)$	0.891	0.850	0.800	0.750	0.700	0.650	0.600

3.2. Order moments

Theorem 3.2. *Let $X \sim GASLa(\alpha,\lambda)$, then n^{th} order moment of X is given by*

$$E(X^n) = \begin{cases} \frac{1}{C}\left[2\Gamma(n+1) - 2\alpha\Gamma(n+2)\left\{1 - \frac{1}{C_1^{n+2}}\right\} + \alpha^2\Gamma(n+3)\right]; & n \text{ is even} \\ \frac{1}{C}\left[2\Gamma(n+1)\left\{1 - \frac{1}{C_1^{n+1}}\right\} - 2\alpha\Gamma(n+2)\alpha^2\Gamma(n+3)\left\{1 - \frac{1}{C_1^{n+3}}\right\}\right]; & n \text{ is odd} \end{cases} \tag{7}$$

Proof. **Case I.:** When n is even:

$$\begin{aligned} E(X^n) &= \frac{1}{C}\left[\int_{-\infty}^{0} x^n((1-\alpha x)^2+1)\frac{1}{2}e^{C_1 x}\,dx \right. \\ &\qquad \left. + \int_0^{\infty} x^n((1-\alpha x)^2+1)\{e^{-x} - \frac{1}{2}e^{-C_1 x}\}dx\right] \\ &= \frac{1}{C}\left[2\left\{\frac{1}{2}\int_{-\infty}^{0} x^n e^{C_1 x}\,dx + \int_0^{\infty} x^n\left(e^{-x} - \frac{1}{2}e^{-C_1 x}\right)dx\right\} \right. \\ &\quad - 2\alpha\left\{\frac{1}{2}\int_{-\infty}^{0} x^{n+1}e^{C_1 x}\,dx + \int_0^{\infty} x^{n+1}\left(e^{-x} - \frac{1}{2}e^{-C_1 x}\right)dx\right\} \\ &\quad \left. + \alpha^2\left\{\frac{1}{2}\int_{-\infty}^{0} x^{n+2}e^{C_1 x}\,dx + \int_0^{\infty} x^{n+2}\left(e^{-x} - \frac{1}{2}e^{-C_1 x}\right)\right\}dx\}\right] \\ &= \frac{1}{C}\left[2I_{12} - 2\alpha I_{13} + \alpha^2 I_{14}\right] \end{aligned}$$

It is observed that the integrals I_{12}, I_{13} and I_{14} are same as the n^{th}, $(n+1)^{th}$ and $(n+2)^{th}$ order moments of the skew Laplace distribution of Aryal and Nadarajah (2005). Thus using the value involved in the literature the even-order moments of $GASLa(\alpha,\lambda)$

distribution can be written as

$$E(X^n) = \frac{1}{C}\left[2\Gamma(n+1) - 2\alpha\Gamma(n+2)\left\{1 - \frac{1}{C_1^{n+2}}\right\} + \alpha^2\Gamma(n+3)\right] \tag{8}$$

Case 2: When n is odd:

Similarly, the odd order moments of $GASLa(\alpha, \lambda)$ becomes

$$E(X^n) = \frac{1}{C}\left[2\Gamma(n+1)\left\{1 - \frac{1}{C_1^{n+1}}\right\} - 2\alpha\Gamma(n+2) + \alpha^2\Gamma(n+3)\left\{1 - \frac{1}{C_1^{n+3}}\right\}\right] \tag{9}$$

□

Remark 1. *From the (8) and (9), the first four moments of $GASLa(\alpha, \lambda)$ distribution can be derived as*

$$E(X) = \frac{1}{C}\left[2\left\{1 - \frac{1}{C_1^2}\right\} - 4\alpha + 6\alpha^2\left\{1 - \frac{1}{C_1^4}\right\}\right],$$

$$E(X^2) = \frac{1}{C}\left[4 - 12\alpha\left\{1 - \frac{1}{C_1^4}\right\} + 24\alpha^2\right],$$

$$E(X^3) = \frac{1}{C}\left[12\left\{1 - \frac{1}{C_1^4}\right\} - 48\alpha + 120\alpha^2\left\{1 - \frac{1}{C_1^{n+6}}\right\}\right],$$

$$E(X^4) = \frac{1}{C}\left[48 - 240\alpha\left\{1 - \frac{1}{C_1^6}\right\} + 720\alpha^2\right].$$

3.3. Moment generating function (MGF)

Theorem 3.3. *The moment generating function of $GASLa(\alpha, \lambda)$ distribution is obtained as*

$$M(t) = \frac{1}{C}\left[\left\{\frac{1}{C_1+t} + \frac{2}{1-t} - \frac{1}{C_1-t}\right\} + \alpha\left\{\frac{1}{(C_1+t)^2} - \frac{2}{(1-t)^2} + \frac{1}{(C_1-t)^2}\right\}\right.$$
$$\left. + \alpha^2\left\{\frac{1}{(C_1+t)^3} + \frac{2}{(1-t)^3} - \frac{1}{(C_1-t)^3}\right\}\right] \tag{10}$$

Proof. Using the PDF of (2), MGF can be defined as

$$M(t) = \frac{1}{C}\left[\int_{-\infty}^{0} e^{tx}((1-\alpha x)^2+1)\frac{1}{2}e^{C_1 x}\,dx\right.$$
$$\left. + \int_0^{\infty} e^{tx}\Big((1-\alpha x)^2+1\Big)\left\{e^{-x} - \frac{1}{2}e^{-C_1 x}\right\}dx\right]$$
$$= \frac{1}{C}\Big[I_{15} + I_{16}\Big]$$

By using (3.351.1) and (3.326.2) in Gradshteyn and Ryzhik (2000), the integrals I_{15} and I_{16} can be written as

$$I_{15} = \frac{1}{C_1+t} + \alpha\frac{1}{(C_1+t)^2} + \alpha^2\frac{1}{(C_1+t)^3}$$

and,

$$I_{16} = \left[\frac{2}{1-t} - \frac{1}{C_1 - t}\right] - \alpha\left[\frac{2}{(1-t)^2} - \frac{1}{(C_1-t)^2}\right] + \alpha^2\left[\frac{2}{(1-t)^3} - \frac{1}{(C_1-t)^3}\right]$$

Substituting the results of I_{15} and I_{16}, the final expression for the MGF of $GASLa(\alpha, \lambda)$ distribution is obtained as

$$M(t) = \frac{1}{C}\left[\left\{\frac{1}{C_1+t} + \frac{2}{1-t} - \frac{1}{C_1-t}\right\} + \alpha\left\{\frac{1}{(C_1+t)^2} - \frac{2}{(1-t)^2} + \frac{1}{(C_1-t)^2}\right\} + \alpha^2\left\{\frac{1}{(C_1+t)^3} + \frac{2}{(1-t)^3} - \frac{1}{(C_1-t)^3}\right\}\right]$$

□

Remark 2. *Substituting* (it) *instead of* t *in equation (10), characteristic function of* $GASLa(\alpha, \lambda)$ *distribution can be written as*

$$\phi(t) = \frac{1}{C}\left[\left\{\frac{1}{C_1+it} + \frac{2}{1-it} - \frac{1}{C_1-it}\right\} + \alpha\left\{\frac{1}{(C_1+t)^2} - \frac{2}{(1-it)^2} + \frac{1}{(C_1-it)^2}\right\} + \alpha^2\left\{\frac{1}{(C_1+it)^3} + \frac{2}{(1-it)^3} - \frac{1}{(C_1-it)^3}\right\}\right]$$

Remark 3. *Considering* log *on both sides of the equation (10), the cumulative generating function of can be obtained as*

$$K(t) = -\log C + \log\left[\left\{\frac{1}{C_1+t} + \frac{2}{1-t} - \frac{1}{C_1-t}\right\} + \alpha\left\{\frac{1}{(C_1+t)^2} - \frac{2}{(1-t)^2} + \frac{1}{(C_1-t)^2}\right\} + \alpha^2\left\{\frac{1}{(C_1+t)^3} + \frac{2}{(1-t)^3} - \frac{1}{(C_1-t)^3}\right\}\right].$$

3.4. Mean deviation

Theorem 3.4. *The mean deviation of* $GASLa(\alpha, \lambda)$ *distribution about the mean* μ *is obtained as*

$$\mu(x) = 2\mu F(\mu) - \frac{e^{C_1\mu}}{CC_1^2}\left[\frac{\alpha^2}{C_1^2}\left\{(C_1\mu-1)^3 + 3C_1\mu - 5\right\} - \frac{2\alpha}{C_1}\left\{(C_1\mu-1)^2+1\right\} + 2\{C_1\mu - 1\}\right]. \tag{11}$$

Proof. The extent of dispersion within a population is partially measured by the total deviations from both the mean and the median. These deviations, known as the mean deviation about the mean and the mean deviation about the median, are defined as follows:

$$\delta_1(X) = \int_{-\infty}^{\infty} |x-\mu| f(x)dx, \qquad \delta_2(X) = \int_{-\infty}^{\infty} |x-M| f(x)dx,$$

respectively, where $\mu = E(X)$ and M denotes the median. So, $\delta_1(X)$ can be calculated as

$$\begin{aligned}\delta_1(X) &= \int_{-\infty}^{\infty} |x-\mu| f(x)dx \\ &= 2\mu F(\mu) - 2\int_{-\infty}^{\mu} x f(x)dx \\ &= 2\mu F(\mu) - 2I_{17}\end{aligned}$$

where, $\mu = E(X)$ and $F(u)$ can be calculated from (4). Now,

$$\begin{aligned}I_{17} &= \int_{-\infty}^{\mu} x f(x)dx \\ &= \frac{1}{2C}\int_{-\infty}^{\mu} x\left((1-\alpha x)^2+1\right)e^{C_1 x}dx \\ &= \frac{1}{2C}\left[\alpha^2\int_{-\infty}^{\mu} x^3 e^{C_1 x}dx - 2\alpha\int_{-\infty}^{\mu} x^2 e^{C_1 x}dx + 2\int_{-\infty}^{\mu} x e^{C_1 x}dx\right],\end{aligned}$$

after some simple computation, I_{17} can be calculated as

$$I_{17} = \frac{e^{C_1\mu}}{2CC_1^2}\left[\frac{\alpha^2}{C_1^2}\left\{(C_1\mu-1)^3+3C_1\mu-5\right\} - \frac{2\alpha}{C_1}\left\{(C_1\mu-1)^2+1\right\} + 2\{C_1\mu-1\}\right]$$

Hence, the final expression of mean deviation about mean for $GALSa(\alpha,\lambda)$ distribution is obtained as

$$\begin{aligned}\mu(x) = 2\mu F(\mu) - \frac{e^{C_1\mu}}{CC_1^2}&\left[\frac{\alpha^2}{C_1^2}\left\{(C_1\mu-1)^3+3C_1\mu-5\right\} - \frac{2\alpha}{C_1}\left\{(C_1\mu-1)^2+1\right\}\right. \\ &\left. + 2\{C_1\mu-1\}\right].\end{aligned}$$

□

Remark 4. *Replacing μ by median M in (11), mean deviation about median of $GASLa(\alpha,\lambda)$ distribution may be obtained.*

3.5. Mode

This subsection provides the graphical method of Behboodian (1970) which is applied to find the mode of $GASLa(\alpha,\lambda)$ distribution.

Theorem 3.5. *The $GASLa(\alpha,\lambda)$ distribution has at most two modes.*

Proof. Let, $X \in GASLa(\alpha,\lambda)$, then by (1) and (2) we get,

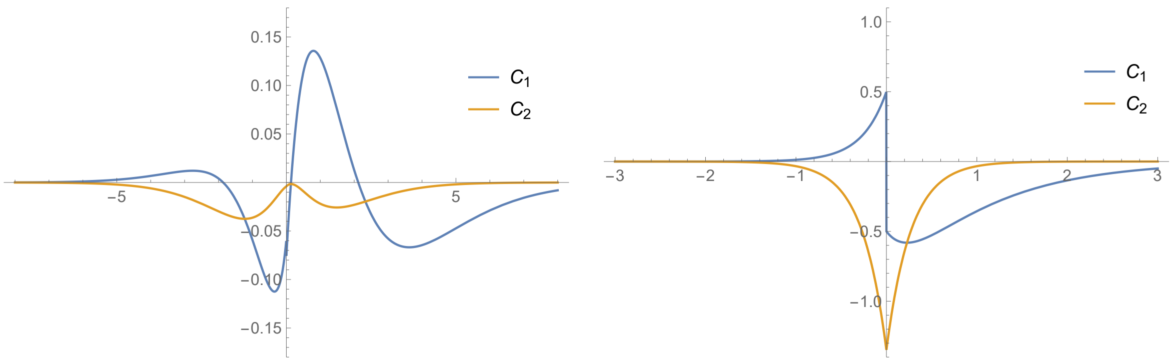
$$f(x;\alpha,\lambda) = \frac{4(1+\alpha^2)}{C(\alpha,\lambda)} f(x;\alpha)G(\lambda x), \tag{12}$$

where, $f(x;\alpha)$ is the PDF of alpha skew Laplace distribution given by Harandi and Alamatsaz (2013) and $f(x;\alpha,\lambda)$ is the PDF of newly proposed generalized alpha skew Laplace distribution respectively. $G(\cdot)$ is the CDF of the standard Laplace distribution. Then,

$$f'(x;\alpha,\lambda) = \frac{4(1+\alpha^2)}{C(\alpha,\lambda)}\left[f'(x;\alpha)G(\lambda x) + \lambda f(x;\alpha)g(\lambda x)\right] \tag{13}$$

where $g(.)$ is the PDF of the standard Laplace distribution. Now to show that (12) has at most two modes, it should be proved that (13) has one or three answers. For this, the graphical method (Behboodian 1970) is applied here. Now,

$$f'(x;\alpha,\lambda) = F_1(x) - F_2(x), \tag{14}$$



(a) (b)

Figure 3: (a) The Plot of D_1 and D_2 for $\alpha = 7.8,\ \lambda = 0.46$; (b) The Plot of D_1 and D_2 for $\alpha = -0.01,\ \lambda = 2.7$

where

$$F_1(x) = \frac{4(1+\alpha^2)}{C(\alpha,\lambda)} f'(x;\alpha)G(\lambda x), \tag{15}$$

and

$$F_2(x) = -\frac{4(1+\alpha^2)}{C(\alpha,\lambda)} \lambda f(x;\alpha)g(\lambda x). \tag{16}$$

Now, setting, $f'(x;\alpha,\lambda) = 0$, it is obtained that

$$F_1(x) = F_2(x). \tag{17}$$

Since (2) vanishes out of (-8,8), therefore the curves of $D_1 : y = F_1(x)$ and $D_2 : y = F_2(x)$ for $\alpha = 7.8,\ \lambda = 0.46$ and for $\alpha = -0.01,\ \lambda = 2.7$ considering $x \in (-8, 8)$ have been drawn respectively. From Figure 3 it can be seen that these two curves have at least one and at most three intersection points where the values of x of these points are the roots of (24). As, $\lim_{x->\pm\infty} f(x;\alpha,\lambda) = 0$, therefore, if (17) possesses one answer then it should be the mode of (12) and if (17) possesses three answers, then (12) should have two modes. Hence, it may be concluded that, the $GASLa(\alpha,\lambda)$ distribution has at least one and at most two modes. □

3.6. Moments of order statistics

Theorem 3.6. *Let $X_{i,n}$ be the i^{th} order statistic $(X_{1,n} \leq X_{2,n} \leq ... \leq X_{n,n})$ in a sample size n from the $GASLa(\alpha,\lambda)$ and C is the normalizing constant. The k^{th} moment of the i^{th} order statistic $X_{i,n}$ is defined as*

$$\alpha_{k,i,n} = E(X_{i,n}^k) = \int_{-\infty}^{\infty} x^k f_{i,n}(x;\alpha,\lambda)dx,$$

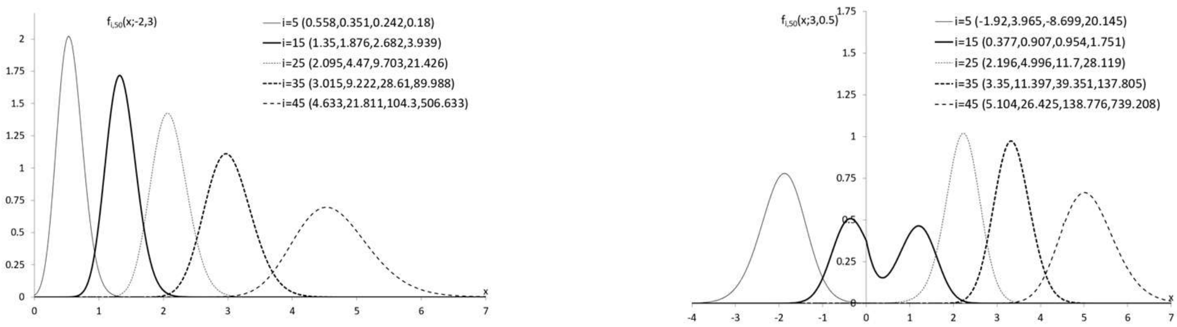
where,

$$f_{i,n}(x;\alpha,\lambda) = \binom{n}{i-1} \frac{(n-i+1)}{F(x;\alpha,\lambda)^{1-i}\left[1-F(x;\alpha,\lambda)\right]^{i-n}} \frac{\left[(1-\alpha x)^2+1\right]\left[1+sign(\lambda x)\left(1-e^{-|\lambda x|}\right)\right]}{2Ce^{|x|}}$$

Proof. The proof based on (3), and the definition of the PDF order statistics

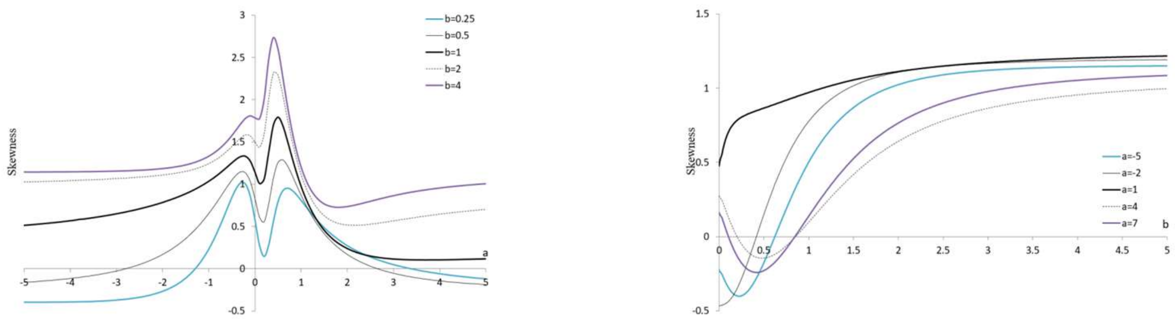
$$f_{i,n}(x;\alpha,\lambda) = \frac{n!}{(i-1)!(n-i)!} f(x;\alpha,\lambda)F(x;\alpha,\lambda)^{i-1}\left[1-F(x;\alpha,\lambda)\right]^{n-i}$$

is easy to perform. □



(a) (b)

Figure 4: PDF of the $X_{10i-5,50}(i = 1, 2, ...5)$ of the $GASLa(\alpha, \lambda)$ distribution



(a) (b)

Figure 5: Skewness of $GASLa(\alpha, \lambda)$

Figure 4 shows PDF of the $X_{10i-5,50}(i = 1, 2, ..., 5)$ of the $GASLa(-2, 3)$ (left) and $GASLa(3, 0.5)$ (right), as well as $\alpha_{k,i,n}(k = 1, 2, ...4)$ in brackets, respectively. The modal value x_m of the unimodal curve increases with the i value. The $f_{i,50}(x_m; a, b)$ value is the highest for $i = 5$ (Figure 4, left) and for $i = 25$ (Figure 4, right). The values of $\alpha_{k,i,n}$ increases with the i value. Note that the curve of $f_{15,50}(x; 3, 0.5)$ is bimodal.

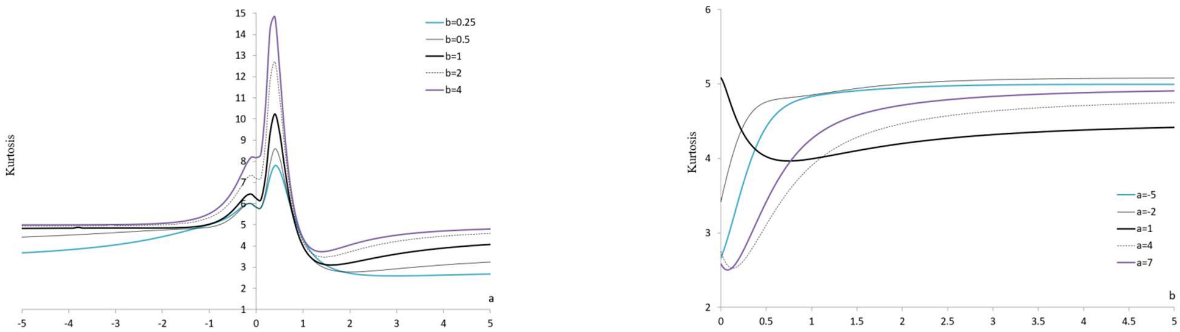
3.7. Skewness and kurtosis

Based on the order (non-central) moments $\alpha_k = E(X^k)(k = 1, 2, ..., 4)$ given by Remark 1 (see subsection 1) and using their relationship with the central moments $\mu_k = \sum_{i=0}^{k}(-1)^i\binom{k}{i}\alpha_{k-i}\alpha_1^i$, we can easily calculate the skewness γ_1 and kurtosis γ_2 of the $GASLa(\alpha, \lambda)$, obviously.

Figure 5 shows γ_1 as a function of α for selected λ values (left) and γ_1 as a function of λ for selected α values (right). As the value of λ increases, bimodality disappears (left). For initial λ values, $\gamma_1(\lambda)$ is monotonical function. For the initial λ values, $\gamma_1(\lambda)$ is a monotonic function. Larger λ values do not affect skewness (right).

Figure 6 shows γ_2 as a function of α for selected λ values (left) and γ_2 as a function of λ for selected α values (right). The $\gamma_2(\alpha)$ is the bimodal function. The highest $\gamma_2(\alpha)$ values are in the interval $(0, 1)$. The higher the λ values, the higher the $\gamma_2(\alpha)$ values. For the initial λ values, $\gamma_2(\lambda)$ is a monotonic function. Larger λ values do not affect kurtosis (right). Figures 3 and 4 are characterized by similar shapes.

We calculate γ_1 and γ_2 for 10^5 randomly values of $\alpha = Unif(-10, 10)$ and $\lambda = Unif(0, 10)$. Figure 7 presents a set of points (γ_1, γ_2) located in a rectangle $(-1, 4.5) \times (1, 21.25)$. Symbol MP denotes the Malakhov parabola $\gamma_2 = \gamma_1^2 + 1$. We obtain very interesting shape with $\gamma_1 \in (-0.544, 2.994), \gamma_2 \in (2.5, 17.222)$.



(a) (b)

Figure 6: Kurtosis of $GASLa(\alpha, \lambda)$

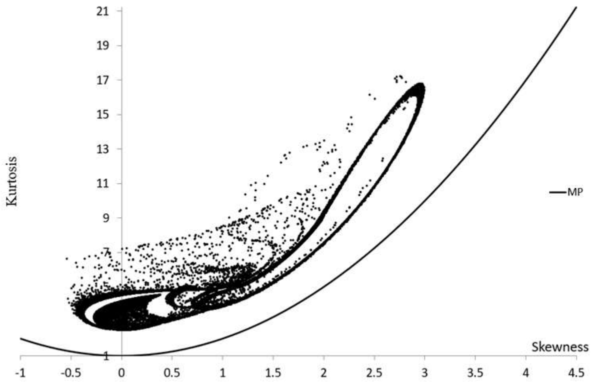


Figure 7: Skewness and Kurtosis of $GASLa(\alpha, \lambda)$

3.8. Quantiles and pseudo-random generator

Theorem 3.7. *Let $X \sim GASLa(\alpha, \lambda)$. The $p^{th}(0 < p < 1)$ quantile x_p is a solution of the equation*

$$F(x_p; \alpha, \lambda) - p = 0 \tag{18}$$

Proof. The proof is based on the quantile definition. □

Theorem 3.8. *Let $R \sim Unif(0, 1)$. The pseudo-random number generator of X:*

- *is a solution of the equation*

$$F(X; \alpha, \lambda) - R = 0 \tag{19}$$

- *can be obtained using the following acceptance-rejection algorithm (Von Neumann 1951):*

  Step 1: Generate a random variable $Z \sim La(0, 1)$.

  Step 2: Generate a random variable $W \sim ASLa(\alpha)$.

  Step 3: If $\lambda W > Z$ then $X = W$ else go to Step 1.

Proof. The proof (19) follows from (18). □

The R codes (R Core Team 2021) of the own functions for computing CDF, PDF, moments, quantile, mode, skewness, kurtosis, PDF and moments of order moments and pseudo-random number generator are provided in Appendix.

3.9. Shannon entropy

Theorem 3.9. *Let $f(x;\alpha,\lambda)$ be the PDF (2). The Shannon entropy S is given by (Shannon 1948).*

$$S(\alpha,\lambda) = -\int_{-\infty}^{\infty} f(x;\alpha,\lambda)\ln\left[f(x;\alpha,\lambda)\right]dx. \tag{20}$$

Proof. The proof is easy to make. □

Figure 8 shows the Shannon entropy as a function of α for selected λ values (left) and as a function of λ for selected α values (right). If $0 < \alpha < 1$ then curves have a bathtub shape. The higher the λ values, the lower the S values (left). If $a = const$, the curves have a unimodal shape with modes close to 0 (right).

4. Characterization results

This section is responsible for the characterization results of $GASLa(\alpha,\lambda)$ distribution based on conditional distribution.

4.1. Characterizations based on conditional distribution

Theorem 4.1. *If $W \sim ASLa(\alpha)$ and $Z \sim La(0,1)$ are independent then,*

$$W|\{\lambda W > Z\} \sim GASLa(\alpha,\lambda).$$

Proof. Let,

$$X = W|\{\lambda W > Z\}.$$

So,

$$\begin{aligned} P(X \le x) &= P(W \le x|\lambda W > Z) \\ &= \frac{P(W \le x, \lambda W > Z)}{P(\lambda W > Z)}. \end{aligned}$$

Now, using the PDF of alpha skew Laplace (Harandi and Alamatsaz 2013) we have,

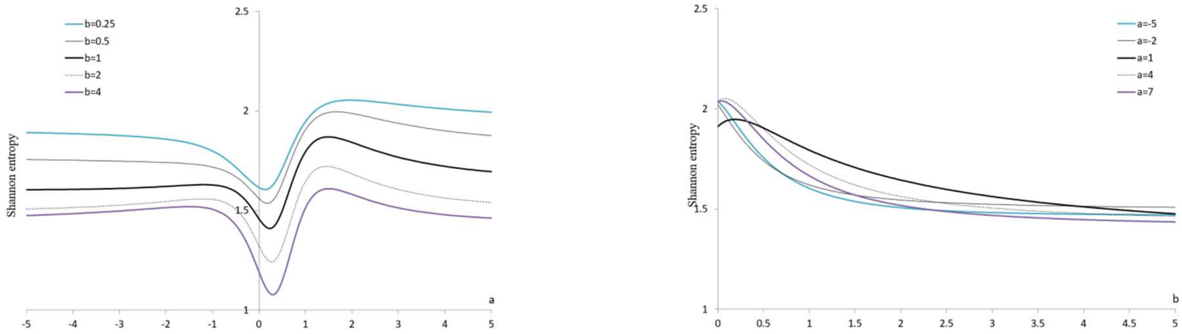
$$P(W \le x, \lambda W > Z) = \int_{-\infty}^{x}\left(\frac{(1-\alpha x)^2+1}{4(1+\alpha^2)}\right)\left(\frac{e^{-|x|}}{2}\right)\left(\frac{1+sign(\lambda x)(1-e^{-|\lambda x|}}{2}\right)dx$$

and

$$\begin{aligned} P(\lambda W > Z) &= \int_{-\infty}^{\infty}\left(\frac{(1-\alpha x)^2+1}{4(1+\alpha^2)}\right)\left(\frac{e^{-|x|}}{2}\right)\left(\frac{1+sign(\lambda x)(1-e^{-|\lambda x|}}{2}\right)dx \\ &= \frac{C}{4(1+\alpha^2)}. \end{aligned}$$

Therefore,

$$\begin{aligned} P(X \le x) &= \frac{\int_{-\infty}^{x}\left(\frac{(1-\alpha x)^2+1}{4(1+\alpha^2)}\right)\left(\frac{e^{-|x|}}{2}\right)\left(\frac{1+sign(\lambda x)(1-e^{-|\lambda x|}}{2}\right)dx}{\frac{C}{4(1+\alpha^2)}} \\ &= \int_{-\infty}^{x}\left(\frac{(1-\alpha x)^2+1}{C}\right)\left(\frac{e^{-|x|}}{2}\right)\left(\frac{1+sign(\lambda x)(1-e^{-|\lambda x|}}{2}\right)dx \end{aligned}$$



(a) (b)

Figure 8: Shannon entropy of $GASLa(\alpha, \lambda)$

and hence the density function of $X = W|\{\lambda W > Z\}$ has been obtained as

$$f(x) = \left(\frac{(1-\alpha x)^2 + 1}{C}\right)\left(\frac{e^{-|x|}}{2}\right)\left(\frac{1 + sign(\lambda x)(1 - e^{-|\lambda x|})}{2}\right)$$

So,

$$W|\{\lambda W > Z\} \sim GASLa(\alpha, \lambda).$$

□

5. Parameter estimation

5.1. Location and scale extension

Let $X \sim GASLa(\alpha, \lambda)$, then $Y = \mu + \beta X$ is said as the location and scale extension of X considering location parameter μ and scale parameter β. Thus a location-scale generalized alpha skew Laplace distribution is obtained which is denoted as $Y \sim GASLa(\mu, \beta, \alpha, \lambda)$ and the PDF of the distribution is given as

$$f(x; \mu, \beta, \alpha, \lambda) = \frac{\left[\left\{1 - \alpha\left(\frac{y-\mu}{\beta}\right)\right\}^2 + 1\right]\left[1 + sign\left(\frac{\lambda(y-\mu)}{\beta}\right)\left(1 - e^{-\left|\left(\frac{\lambda(y-\mu)}{\beta}\right)\right|}\right)\right]}{2\beta C e^{\left|\frac{y-\mu}{\beta}\right|}} \tag{21}$$

where, $y \in R$, $\mu \in R$, $\alpha \in R$, $\beta > 0$ and $\lambda > 0$.

5.2. Maximum likelihood estimation

Let us consider that, $y_1, y_2, ..., y_n$ be a random sample of size n drawn from $GASLa(\mu, \beta, \alpha, \lambda)$ distribution. Then the log-likelihood function for the set of parameters $\theta = (\mu, \beta, \alpha, \lambda)$ is given by

$$\begin{aligned} l(\theta) = &\sum_{i=1}^{n} \log\left[\left(1 - \alpha\left(\frac{y-\mu}{\beta}\right)\right)^2 + 1\right] - n\log 2 - n\log\beta - n\log C - \sum_{i=1}^{n}\frac{|y-\mu|}{\beta} \\ &+ \log\left[\left(1 + sign\left(\frac{\lambda(y-\mu)}{\beta}\right)\left(1 - \exp\left(-\frac{|y-\mu|}{\beta}\lambda\right)\right)\right)\right] \end{aligned} \tag{22}$$

Now, the equation (22) is differentiated concerning the set of parameters and the results are obtained as

$$\frac{\partial l(\theta)}{\partial \mu} = \sum_{i=1}^{n} \frac{2\alpha(\beta+\alpha\mu-\alpha y)}{\beta^2\left(1+A(y;\alpha,\mu,\beta)\right)} + \sum_{i=1|}^{n} \frac{|y-\mu|'}{\beta} - \frac{\lambda|y-\mu|'\phi(\lambda y)}{\beta\Phi(\lambda y)},$$

$$\frac{\partial l(\theta)}{\partial \beta} = \sum_{i=1}^{n} \frac{2\alpha(y-\mu)(\beta+\alpha\mu-\alpha y)}{\beta^3 A(y;\alpha,\mu,\beta)} + \sum_{i=1|}^{n} \frac{|y-\mu|}{\beta^2} - \frac{n}{\beta} - \frac{\lambda|y-\mu|\phi(\lambda y)}{\beta^2\Phi\lambda y},$$

$$\frac{\partial l(\theta)}{\partial \alpha} = \frac{n\left(2\alpha-1+\frac{1}{(1+\lambda)^2}\right)}{1+\alpha\left(\alpha-1+\frac{1}{(1+\lambda)^2}\right)} + \sum_{i=1}^{n} \frac{2(y-\mu)(\beta+\alpha\mu-\alpha y)}{\beta^2 A(y;\alpha,\mu\beta)},$$

$$\frac{\partial l(\theta)}{\partial \lambda} = -\frac{|y-\mu|\phi(\lambda y)}{\beta\Phi(\lambda y)} - \frac{2n\alpha|\lambda|}{(1+|\lambda|)\left(1+\alpha^2+\frac{(1+(\alpha-1)\alpha)(\lambda+2sign[\lambda])}{sign[\lambda]^2}\right)},$$

where, $A(y;\alpha,\mu,\beta) = \left(1+\frac{(\beta+\alpha\mu-\alpha y)^2}{\beta^2}\right)$ and $\phi(\cdot)$ and $\Phi(\cdot)$ are the PDF and CDF of Laplace distribution respectively with an additional parameter λ. It can be observed that solving the complicated system of linear equations is not possible analytically. Therefore, the numerical approximation method is employed using R software to obtain the maximum likelihood estimator for the parameters by optimizing for the set of parameters, $\theta = (\mu, \beta, \alpha, \lambda)$.

5.3. Fisher information matrix

Theorem 5.1. *Let* $\ln[f(x;a,b)]$ *be given by (21). The Fisher Information Matrix* $I_{i,j}^{a,b}(i,j = 1.2)$ *of the* $GASLa(\alpha,\lambda)$ *distribution is given by*

$$I_{i,j}^{\alpha,\lambda} = \begin{bmatrix} E\left\{\frac{d^2\ln[f(x;\alpha,\lambda)]}{d\alpha^2}\right\} & E\left\{\frac{d}{d\alpha}\left(\frac{d\ln[f(x;\alpha,\lambda)]}{d\lambda}\right)\right\} \\ E\left\{\frac{d}{d\lambda}\left(\frac{d\ln[f(x;\alpha,\lambda)]}{d\alpha}\right)\right\} & E\left\{\frac{d^2\ln[f(x;\alpha,\lambda)]}{d\lambda^2}\right\} \end{bmatrix},$$

where, $I_{1,2}^{\alpha,\lambda} = I_{2,1}^{\alpha,\lambda}$*, obviously.*

Proof. The proof follows from the definition of the Fisher Information Matrix. □

In distribution theory, there are papers with more or less complicated FIM formulas, but it is difficult to find a numerical analysis. The $I_{i,j}^{\alpha,\lambda}(i,j=1,2)$ values for parameter values from Figures 1 are:

$$I_{i,j}^{0,0.1} = \begin{bmatrix} -1.97 & 1.503 \\ 1.503 & -1.232 \end{bmatrix}, I_{i,j}^{0,1.2} = \begin{bmatrix} -1.371 & 0.188 \\ 0.188 & -0.11 \end{bmatrix}, I_{i,j}^{0,2.5} = \begin{bmatrix} -1.157 & 0.047 \\ 0.047 & -0.026 \end{bmatrix},$$

$$I_{i,j}^{2,0.8} = \begin{bmatrix} -0.138 & -0.078 \\ -0.078 & -0.54 \end{bmatrix}, I_{i,j}^{2.5,0.8} = \begin{bmatrix} -0.058 & -0.059 \\ -0.059 & -0.517 \end{bmatrix}, I_{i,j}^{3,0.8} = \begin{bmatrix} -0.027 & -0.044 \\ -0.044 & -0.496 \end{bmatrix},$$

$$I_{i,j}^{1.4,0.5} = \begin{bmatrix} -0.414 & -0.119 \\ -0.119 & -1.145 \end{bmatrix}, I_{i,j}^{1.6,0.5} = \begin{bmatrix} -0.278 & -0.130 \\ -0.130 & -1.166 \end{bmatrix}, I_{i,j}^{1.9,0.5} = \begin{bmatrix} -0.156 & -0.122 \\ -0.122 & -1.177 \end{bmatrix},$$

$$I_{i,j}^{-2,0.05} = \begin{bmatrix} -0.082 & -0.193 \\ -0.193 & -5.614 \end{bmatrix}, I_{i,j}^{-2,0.2} = \begin{bmatrix} -0.068 & -0.110 \\ -0.110 & -1.925 \end{bmatrix}, I_{i,j}^{-2,0.6} = \begin{bmatrix} -0.052 & -0.038 \\ -0.038 & -0.316 \end{bmatrix}$$

Positive $I_{i,j}^{\alpha,\lambda}$ values occur very rarely. If λ increases ($\alpha = 0$) then $I_{1,2}^{\alpha,\lambda} = I_{2,1}^{\alpha,\lambda)}$ increase and $I_{2,2}^{\alpha,\lambda}$ decreases. If α increases ($\lambda = 0.5, 0.8$) then $I_{1,2}^{\alpha,\lambda} = I_{2,1}^{\alpha,\lambda}$ and $I_{1,1}^{\alpha,\lambda}$ increase.

6. Simulation study

A simulation study is conducted to assess how well the maximum likelihood estimates perform in estimating the parameters of the model. The acceptance-rejection algorithm given in Theorem 3.8 is employed for generating a set of random variable. The process is replicated 1000 times along with the three different generated samples of size $n = 100, 300$ and 500 and finally, the MLEs are estimated for each generated sample using the GenSA package (GenSA-package, Version − 1.0.3) in R software. The estimated statistics are presented in terms of biases and mean square errors (MSEs) of the estimates and the formulas are given by

$$Bias(\hat{\theta}) = E(\hat{\theta}) - \theta \;\; and \;\; MSE(\hat{\theta}) = V(\hat{\theta}) + Bias(\hat{\theta})^2 \;\; Where, \; \hat{\theta} = (\hat{\mu}, \hat{\beta}, \hat{\alpha}, \hat{\lambda})$$

Tables 2 and 3, demonstrate effective performance in accurately estimating the model parameters. Moreover, with a larger sample size, the bias and mean-square error of the Maximum Likelihood Estimates (MLEs) decrease as anticipated. Consequently, it can be inferred that the asymptotic consistency of maximum likelihood estimators holds for moderate and large sample sizes.

Table 2: Simulation results

					$\mu = 0, \quad \beta = 1$					
			μ		β		λ		α	
α	λ	n	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
-2	1.5	100	-0.1658	0.0987	-0.1098	0.0879	0.0745	0.0517	0.0695	0.0531
		300	0.0875	0.0657	0.0987	0.0487	-0.0548	0.0413	-0.0407	0.0421
		500	0.0658	0.0365	-0.0540	0.0178	-0.0444	0.0140	0.0231	0.0146
	2	100	-0.0641	0.0507	-0.0789	0.0580	-0.0727	0.0597	-0.0755	0.0501
		300	-0.0207	0.0410	0.0549	0.0492	-0.0566	0.0431	0.0525	0.0410
		500	-0.0231	0.0146	0.0479	0.0260	-0.0500	0.0248	-0.0454	0.0109
0	1.5	100	0.0980	0.0564	-0.0721	0.0511	0.0645	0.0587	0.1008	0.0904
		300	-0.0561	0.0246	-0.0541	0.0402	-0.0499	0.0420	0.0805	0.0631
		500	0.0152	0.0098	-0.0473	0.0198	0.0233	0.0240	-0.0650	0.0315
	2	100	-0.1008	0.0987	0.0690	0.0540	0.0725	0.0510	-0.0738	0.0509
		300	-0.0635	0.0317	-0.0417	0.0421	-0.0540	0.0492	0.0577	0.0412
		500	-0.0218	0.0095	0.0239	0.0316	0.0498	0.0199	-0.0420	0.0149
2	1.5	100	0.0698	0.0507	-0.0540	0.0500	-0.0981	0.0570	0.0755	0.0858
		300	-0.0529	0.0446	-0.0234	0.0473	-0.0524	0.0232	0.0918	0.0460
		500	0.0125	0.0088	-0.0239	0.0126	-0.0156	0.0190	-0.0541	0.0138
	2	100	-0.0801	0.0530	0.0697	0.0555	0.0987	0.0900	-0.1758	0.0967
		300	0.0509	0.0156	-0.0487	0.0401	-0.0543	0.0531	-0.0826	0.0653
		500	0.0109	0.0093	0.0241	0.0156	-0.0315	0.0305	0.0631	0.0301

7. Real life applications

In this section, two real life data sets are used to demonstrate the applicability of $GASLa(\alpha, \lambda)$ distribution. For this we fit Laplace distribution $La(\mu, \beta)$, skew Laplace distribution (Aryal and Nadarajah 2005) $SLa(\lambda, \mu, \beta)$, alpha skew Laplace distribution (Harandi and Alamatsaz 2013) $ASLa(\alpha, \mu, \beta)$, alpha skew normal distribution (Elal-Olivero 2010) $ASN(\alpha, \mu, \beta)$ and alpha skew logistic distribution (Hazarika and Chakraborty 2014) $ASL(\alpha, \mu, \beta)$ and having fitted the models are compared with our newly introduced model. Utilizing the GenSA package within the R software, the fitted model values are determined through maximum likelihood methods. To assess and compare the models, both the Akaike Information Criterion (AIC) and the Bayesian Information Criterion (BIC) are taken into account.

7.1. White cells count (WCC) dataset

For this illustration, a data set of white cells count (WCC) of 202 Australian athletes, given in Cook and Weisberg (1994) is considered. Table 4 obtains the maximum likelihood estimate

Table 3: Simulation result

			$\mu=0,\quad\beta=1$							
			μ		β		λ		α	
α	λ	n	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
-2	1.5	100	-0.0750	0.0560	0.0576	0.0521	0.0540	0.0451	0.0665	0.0548
		300	-0.0636	0.0477	-0.0281	0.0409	-0.0466	0.0329	-0.0487	0.0412
		500	0.0600	0.0341	-0.0263	0.0201	0.0223	0.0231	0.0211	0.0111
	2	100	-0.0549	0.0503	-0.0670	0.0511	0.0655	0.0533	0.0665	0.0479
		300	-0.0299	0.0470	0.0440	0.0413	-0.0591	0.0416	-0.0487	0.0332
		500	-0.0230	0.0220	0.0219	0.0118	-0.0339	0.0350	0.0355	0.0219
0	1.5	100	0.0565	0.0513	0.0547	0.0454	-0.0689	0.0509	-0.0580	0.0468
		300	-0.0290	0.0434	-0.0413	0.0327	0.0441	0.0443	-0.0466	0.0364
		500	-0.0242	0.0221	0.0221	0.0200	-0.0207	0.0153	0.0240	0.0234
	2	100	-0.0660	0.0540	-0.1026	0.0847	0.0640	0.0503	-0.1000	0.0840
		300	-0.0449	0.0466	0.0690	0.0483	-0.0378	0.0456	-0.0499	0.0498
		500	0.0216	0.0178	-0.0297	0.0155	-0.0230	0.0221	0.0214	0.0197
2	1.5	100	-0.0669	0.0540	0.0655	0.0542	0.0509	0.0521	0.0759	0.0569
		300	0.0447	0.0419	-0.0510	0.0411	-0.0268	0.0410	-0.0630	0.0435
		500	0.0210	0.0117	0.0354	0.0356	-0.0260	0.0231	-0.0610	0.0314
	2	100	-0.0660	0.0459	0.0549	0.0419	-0.0549	0.0453	-0.0534	0.0514
		300	-0.0484	0.0302	-0.0487	0.0307	-0.0460	0.0330	0.0245	0.0436
		500	0.0359	0.0210	0.0223	0.0216	0.0243	0.0237	-0.0210	0.0221

Table 4: MLE's, log-likelihood, AIC and BIC for white cells count (WCC) of 202 Australian athletes

Distributions	μ	β	α	λ	*logL*	*AIC*	*BIC*
$La(\mu,\beta)$	6.844	1.380	–	–	-407.142	818.284	824.901
$SLa(\mu,\beta,\lambda)$	6.400	1.287	–	0.762	-400.366	806.732	816.657
$ASLa(\mu,\beta,\alpha)$	6.400	1.263	-0.265	–	-400.992	807.984	817.909
$ASN(\mu,\beta,\alpha)$	8.194	1.684	0.874	–	-398.393	802.786	812.711
$ASL(\mu,\beta,\alpha)$	6.413	0.948	-0.210	–	-398.943	803.886	813.811
$GASLa(\alpha,\lambda,\mu,\beta)$	4.300	0.933	2.402	3.977	-394.685	797.370	810.603

of the fitted models with the value of log-likelihood, AIC and BIC. Besides, the behaviour of the fitted models are represented in Figure 9. behavior From Table 4 it is observed that the value of AIC and BIC of $GASLa(\alpha,\lambda)$ distribution is less than that of the AIC and BIC value of other distributions while Figure 9 shows the good fit of $GASLa(\alpha,\lambda)$ distribution. Therefore, it is possible to draw the conclusion that the $GASLa(\alpha,\lambda)$ distribution is yet another alternative for modeling bimodal data set.

7.2. Failure times of Aircraft windshield dataset

For the illustration, another data set on failure times of 84 Aircraft Windshield (El-Bassiouny, Abdo, and Shahen (2015)) is considered. here, Table 5 obtains the maximum likelihood estimate of the fitted models with the value of log-likelihood, AIC and BIC while the behaviour of the fitted models are shown in Figure 10.

The Table 5 describes that the value of AIC and BIC of $GASLa(\alpha,\lambda)$ distribution are less than that of the value of other distributions under consideration. Figure 10 portraits the good fit of the $GASLa(\alpha,\lambda)$ distribution. Hence, from this illustration also it may be concluded that the $GASLa(\alpha,\lambda)$ distribution is better fitted for modelling bimodal data set.

8. Hypothesis testing

Here it is been observed that $La(\mu,\beta)$, $SLa(\mu,\beta,\lambda)$, $ASLa(\mu,\beta,\alpha)$ and $GASLa(\mu,\beta,\alpha,\lambda)$

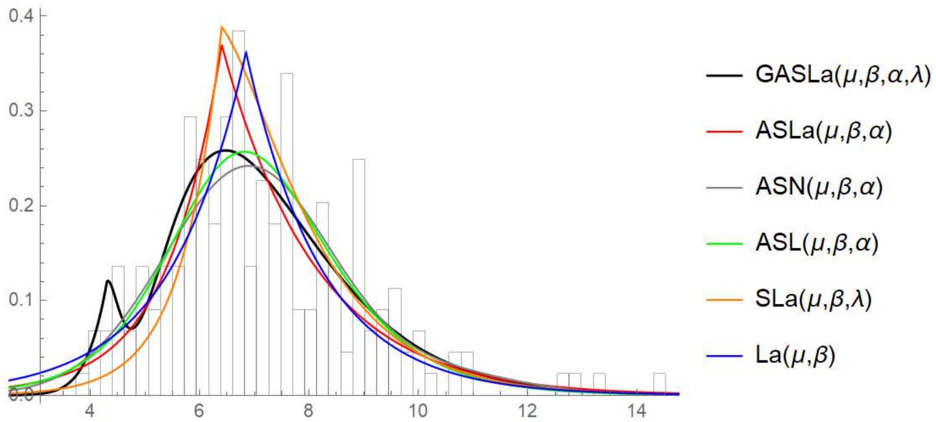


Figure 9: Plots of observed and expected densities of some distributions for white cells count (WCC) of 202 Australian athletes

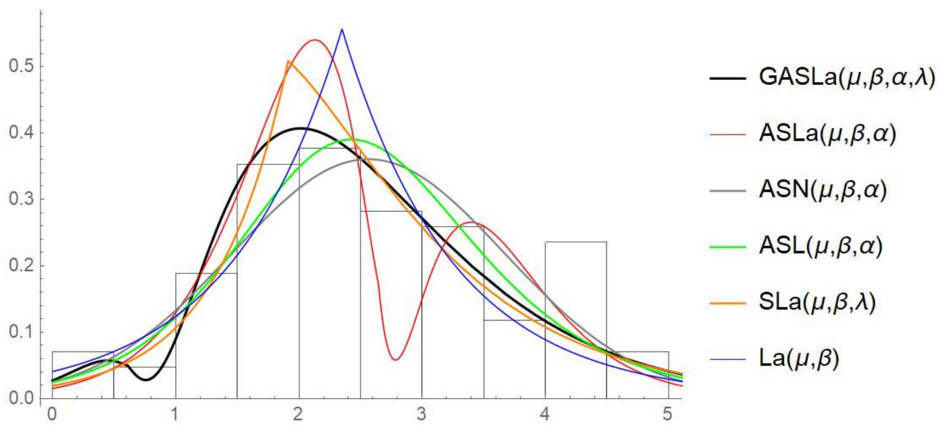


Figure 10: Plots of observed and expected densities of some distributions for failure times of 84 Aircraft Windshield

Table 5: MLE's, log-likelihood, AIC and BIC for failure times of 84 Aircraft Windshield

Distributions	μ	β	α	λ	*logL*	*AIC*	*BIC*
$La(\mu,\beta)$	2.3492	0.8973	–	–	-133.120	270.240	275.101
$SLa(\lambda,\mu,\beta)$	1.9120	0.9829	–	0.6853	-129.150	264.300	271.592
$ASLa(\alpha,\mu,\beta)$	2.6460	0.3275	2.8157	–	-130.248	266.496	273.788
$ASN(\mu,\beta,\alpha)$	2.702	1.106	0.126	–	-129.223	264.446	271.738
$ASL(\mu,\beta,\alpha)$	2.206	0.629	-0.173	–	-131.011	268.022	275.314
$GASLa(\alpha,\lambda,\mu,\beta)$	0.6109	0.6332	4.3526	2.3611	-126.187	260.374	270.097

are nested models. Therefore, to discriminate between them, the likelihood ratio (LR) test is used. The test statistics as well as the null hypothesis of the test are as follows:

(i) to discriminate $ASLa(\mu,\beta,\alpha)$ from $GASLa(\mu,\beta,\alpha,\lambda)$ distribution, the null hypothesis $H_0 : \lambda = 0$ have to test against the alternative hypothesis $H_1 : \lambda \neq 0$. The test statistic is

$$-2\log(LR) = -2[\log L(\hat{\mu}_1, \hat{\beta}_1, \hat{\alpha}_1, \lambda = 0|x) - \log L(\hat{\mu}_2, \hat{\beta}_2, \hat{\alpha}_2, \hat{\lambda}_2)] \sim \chi^2_1,$$

where $(\hat{\mu}_1, \hat{\beta}_1, \hat{\lambda}_1)$ and $(\hat{\mu}_2, \hat{\beta}_2, \hat{\lambda}_2, \hat{\alpha}_2)$ are the MLEs of $ASLa(\mu,\beta,\alpha)$ and $GASLa(\mu,\beta,\alpha,\lambda)$ PDFs, respectively; and $r = 1$ (difference between the numbers of parameters).

(ii) To discriminate $SLa(\mu,\beta,\lambda)$ from $GASLa(\mu,\beta,\alpha,\lambda)$ distribution, the null hypothesis $H_0 : \alpha = 0$ have to test against the alternative hypothesis $H_1 : \alpha \neq 0$. The test statistic is

$$-2\log(LR) = -2[\log L(\hat{\mu}_1, \hat{\beta}_1, \hat{\lambda}_1, \alpha = 0|x) - \log L(\hat{\mu}_2, \hat{\beta}_2, \hat{\alpha}_2, \hat{\lambda}_2)] \sim \chi^2_1,$$

where $(\hat{\mu}_1, \hat{\beta}_1, \hat{\lambda}_1)$ and $(\hat{\mu}_2, \hat{\beta}_2, \hat{\lambda}_2, \hat{\alpha}_2)$ are the MLEs' of $SLa(\mu,\beta,\lambda)$ and $GASLa(\mu,\beta,\alpha,\lambda)$ PDFs, respectively; and $r = 1$ (difference between the numbers of parameters).

(iii) To discriminate $La(\mu,\beta)$ from $GASLa(\mu,\beta,\alpha,\lambda)$ distribution, the null hypothesis $H_0 : \alpha = 0, \lambda = 0$ have to test against the alternative hypothesis $H_1 : \alpha \neq 0, \lambda \neq 0$. The test statistic is

$$-2\log(LR) = -2[\log L(\hat{\mu}_1, \hat{\beta}_1, \alpha = 0, \lambda = 0|x) - \log L(\hat{\mu}_2, \hat{\beta}_2, \hat{\alpha}_2, \hat{\lambda}_2)] \sim \chi^2_2,$$

where $(\hat{\mu}_1, \hat{\beta}_1$ and $(\hat{\mu}_2, \hat{\beta}_2, \hat{\lambda}_2, \hat{\alpha}_2)$ are the MLEs' of $SLa(\mu,\beta)$ and $GASLa(\mu,\beta,\alpha,\lambda)$ PDFs, respectively; and $r = 2$ (difference between the numbers of parameters).

Table 6: LRT for different hypotheses for the data set I, II and III

Hypothesis	LRT statistic		d.f.	Critical Values at 5 %
	Dataset I	Dataset II		
$H_0 : \alpha = 0$ Vs $H_1 : \alpha \neq 0$	12.614	8.122	1	3.841
$H_0 : \lambda = 0$ Vs $H_1 : \lambda \neq 0$	11.362	5.962	1	3.841
$H_0 : \alpha = 0, \lambda = 0$ Vs $H_1 : \alpha \neq 0, \lambda \neq 0$	24.914	13.866	2	5.991

From Table 6, it is observed that, for the entire null hypothesis, the value of LR test statistics is greater than that of the tabulated critical value at 5% level of significance. Consequently, it can be stated that the collected data originates from the $GASLa(\alpha,\lambda)$ distribution.

9. Conclusion and future scope of the study

This research presents the Generalized Alpha Skew Laplace (GASLa) distribution as a significant contribution to the domain of skew probability distributions, capable of addressing the limitations of existing models by effectively capturing both unimodal and bimodal characteristics. The theoretical derivation of its properties reveals an advanced structure, encompassing key metrics such as moments, skewness, kurtosis, and entropy, alongside practical tools like pseudo-random number generation and mode analysis. The development of a location-scale extension further enhances its applicability, enabling it to adapt to various data scenarios.

Through simulation studies, the GASLa distribution demonstrates the consistency of parameter estimation, supported by decrease in bias and mean square error as sample sizes increase. Further applications to real-life datasets, such as the white cell counts of Australian athletes and aircraft windshield failure times, showcase its ability to outperform competing distributions in terms of AIC and BIC. The likelihood ratio tests further validate the GASLa model's superiority and highlight its distinct advantages over nested models.

In addition to its strong empirical and theoretical performance, the GASLa distribution provides a versatile framework for extending statistical modeling into areas requiring higher flexibility, such as modeling skewed or multimodal data. Future research directions could involve extending this model to handle multimodal datasets with more than two modes, and developing applications in emerging fields like machine learning and data science. Furthermore, there is a scope of studying the logarithmic extension of the new distribution along with its properties and applications. Besides, the new distribution can be extended using the Balakrishnan mechanism (Arnold *et al.* 2002) and the superiority of those can be compared with the newly models in terms of some model selection criterion.

Appendix A

Let, $\mu \in R, \beta > 0, \alpha \in R, \beta \in R, \lambda \in R$. The PDFs of the models used in Section 6 are

$$f_{La}(x;\mu,\beta) = \frac{1}{2\beta}\exp\left(\frac{-|x-\mu|}{\beta}\right), \quad f_{SLa}(x;\mu,\beta,\lambda) = 2f_{La}(x;\mu,\beta)F_{La}(x;\mu,\beta)$$

$$f_{ASL}(x;\mu,\beta,\alpha) = \frac{\left(1-\frac{\alpha(x-\mu)}{\beta}\right)^2+1}{\frac{\pi^2\alpha^2}{3}+2}\,\frac{e^{-\frac{x-\mu}{\beta}}}{\beta\left(e^{-\frac{x-\mu}{\beta}}+1\right)^2}$$

$$f_{ASN}(x;\mu,\beta,\alpha) = \frac{\left(1-\frac{\alpha(x-\mu)}{\beta}\right)^2+1}{\alpha^2+2}\,\frac{e^{-\frac{(x-\mu)^2}{2\beta^2}}}{\sqrt{2\pi}\beta},$$

$$f_{ASLa}(x;\mu,\beta,\alpha) = \frac{\left(1-\frac{\alpha(x-\mu)}{\beta}\right)^2+1}{4\left(\alpha^2+1\right)\beta}\,\frac{1}{2\beta}\exp\left(\frac{-|x-\mu|}{\beta}\right)$$

where, $F(\cdot)$ and $f(\cdot)$ are the CDF and PDF of the particular distributions respectively.

Appendix B

R codes for PDF, CDF, quantile, mode, k-th order moment, skewness, excess kurtosis, PDF of order statistics, moments of order statistics and pseudo-random number generator are available at https://github.com/PiotrSule/GASLa.

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Affiliation:

Jondeep Das
Department of Statistics,
Dibrugarh University, Dibrugarh, India
E-mail: jondeepdas98@gmail.com

Partha Jyoti Hazarika
Department of Statistics,
Dibrugarh University, Dibrugarh, India
E-mail: parthajhazarika@gmail.com

Dimpal Pathak
Department of Statistics,
Dibrugarh University, Dibrugarh, India
E-mail: dimpalpathak20@gmail.com

Piotr Sulewski
Institute of Exact and Technical Sciences,
Pomeranian University, Poland
E-mail: piotr.sulewski@apsl.edu.pl

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