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A New Extended Rayleigh Distribution with Applications of COVID-19 Data

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Abstract

In this manuscript, we proposed a new extension of the Rayleigh distribution named as MTI Rayleigh distribution (MTIRD) that provides better fits than the Rayleigh distribution and some of its known generalizations. Various properties of the proposed distribution, including moments, moment generating function, hazard rate, conditional moments, Bonferroni and Lorenz curve, mean residual life, mean waiting time, Renyi entropy and order statistics are derived. Maximum likelihood estimation procedure is employed to estimate the unknown parameters. An extensive simulation study is carried out to illustrate the behaviour of MLEs on the basis of Mean Square Errors. The flexibility of the new distribution is assessed by applying it to two real data sets of COVID-19 mortality. The comparative behaviour of MTIRD with Rayleigh distribution, Power Rayleigh distribution, Transmuted Rayleigh distribution, Exponentiated Rayleigh distribution, Weibull Rayleigh distribution provided the evidence that it outperforms the other competing distributions based on Akaike Information Criterion, Bayesian Information Criterion, Akaike Information criterion Corrected, and other goodness of fit measures.

Keywords: MTI transformation, Rayleigh distribution, MTI Rayleigh distribution, moments, conditional moments, Renyi entropy, maximum likelihood estimation.

1. Introduction

One of the key objectives in statistics is to develop effective statistical models to represent natural phenomena using well known probability distributions. These distributions are used to model events in nature that are inherently uncertain and risky. Given the complexity of natural phenomena, traditional distributions often fall short in accurately modeling them. As a result, generalized probability distributions have been extended and modified to better fit such data. By introducing additional parameters to existing distributions, these extensions have improved the ability to accurately represent natural phenomena and describe the distribution's tail. Many extensions of the Rayleigh probability distribution have been developed due to its significant role in describing various natural phenomena. The Rayleigh probability density function, attributed to Lord Rayleigh (1842–1919), is concerned with describing skewed data see Rayleigh (1880). Many researchers consider one scale parameter Rayleigh, like Diebolt and Robert (1994) discussed deviation and distance measure in economic, which

can be applied in another natural phenomena data. Kundu and Raqab (2005) provided a generalization of the Rayleigh probability distribution and estimated its unknown parameters using several different methods. Voda (2007) used the conservative technique to derive a new generalization of the Rayleigh probability distribution. Dey (2009) presented Bayesian estimates of the Rayleigh probability distribution parameters using the linex loss functions and square error loss function. Merovci (2013) used the square ordinal transformation method in developing the transmuted Rayleigh probability distribution. Merovci and Elbatal (2015) presented a Weibull- Rayleigh probability distribution. Mahmoud and Ghazal (2017) discussed the estimation of the exponentiated Rayleigh parameters based on type II censored data. Ateeq, Qasim, and Alvi (2019) derived the Rayleigh-Rayleigh distribution (RRD) using the Transformed Transformer technique. Almetwally, Almongy, and ElSherpieny (2019) used the maximum likelihood and maximum product spacing estimates for generalized Rayleigh distribution based on the adaptive type-II progressive censoring schemes. Al-Babtain (2020) proposed a new extension of the Rayleigh distribution with a two parameter called type I half logistic Rayleigh distribution. Bhat, Ahmad, Almetwally, Yehia, Alsadat, and Tolba (2023) proposed a new extension of the odd Lindley power Rayleigh distribution, studied its properties, and evaluated parameter estimation techniques using both classical and Bayesian methods. Bhat and Ahmad (2020) introduced a new generalization of the Rayleigh distribution using the power transformation technique. Mir and Ahmad (2024) recently introduced the Sine power Rayleigh distribution and explored its properties and applications.
The main aim of this paper is to propose and study a new lifetime model called the MTI Rayleigh distribution (MTIRD) based on the method of MTI. The primary purpose of the new model is that the additional parameter provides several desirable properties and greater flexibility in the form of the hazard and density functions. Moreover, the proposed model outperforms some well-established models when applied to two real data sets of COVID-19 mortality. The remaining Sections of this article are structured as follows. Section 2 provides an introduction to the MTI method. The MTIRD is introduced in Section 3, followed by a discussion of its reliability analysis and statistical properties in Sections 4 and 5, respectively. Section 6 focuses on the estimation of unknown parameters using the maximum likelihood approach. The simulation study and applicability of the model are discussed in Sections 7 and 8, respectively. Finally, some conclusions are provided in Section 9.

2. MTI transformation method

The CDF and PDF of the MTI Transformation Method proposed by Lone, Dar, and Jan (2022) are defined by

$$F_{MTI}(x) = \frac{\alpha F(x)}{\alpha - log\alpha \ \bar{F}(x)} \ ; \qquad x \in \mathbb{R}, \alpha \in \mathbb{R}^{+}. \tag{1}$$

where, $\bar{F}(x) = 1 - F(x)$.

$$f_{MTI}(x) = \frac{\alpha(\alpha - log\alpha) f(x)}{(\alpha - log\alpha \ \bar{F}(x))^2} \ ; \qquad x \in \mathbb{R}, \alpha \in \mathbb{R}^{+}. \tag{2}$$

Where $F(x)$ and $f(x)$ in Eq. (1) and Eq. (2) above are the CDF and PDF of the base line distribution respectively.

3. MTI Rayleigh distribution (MTIRD)

The Rayleigh distribution (RD), named after Lord Rayleigh (1880) is prominent lifetime probability model concerned with describing skewed data. The probability density function (PDF) associated with random variable $X > 0$ having RD with scale parameter θ is given by

$$f(x;\theta) = \frac{x}{\theta^2} e^{-\frac{x^2}{2\theta^2}} \ ; \quad x > 0, \theta > 0. \tag{3}$$

and the corresponding cumulative distribution function (CDF) is given as

$$F(x;\theta) = 1 - e^{-\frac{x^2}{2\theta^2}} \ ; \quad x > 0, \theta > 0. \tag{4}$$

Here we introduce, MTI method. Considering $F(x;\theta)$ be the CDF of Rayleigh distribution. Then the CDF of MTIRD can be obtained by inserting Eq. (4) in Eq. (1) and is given by

$$F(x;\alpha,\theta) = \frac{\alpha(1 - e^{-\frac{x^2}{2\theta^2}})}{\alpha - log\alpha \ e^{-\frac{x^2}{2\theta^2}}} \ ; \quad x > 0, \alpha \neq 1, \alpha, \theta > 0. \tag{5}$$

The corresponding PDF of MTIRD is obtained as

$$f(x;\alpha,\theta) = \frac{x}{\theta^2} \frac{\alpha(\alpha - log\alpha) \ e^{-\frac{x^2}{2\theta^2}}}{(\alpha - log\alpha \ e^{-\frac{x^2}{2\theta^2}})^2} \ ; x > 0, \alpha \neq 1, \alpha, \theta > 0. \tag{6}$$

Fig 1 visually illustrates the PDF of the MTIRD for different values of the parameters α and θ. This depiction highlights the versatility of the MTIRD, showing PDFs that can be right-skewed, symmetric, or even exhibit an increasing density function. These variations demonstrate the model's adaptability in capturing diverse data patterns related to lifetime distributions.

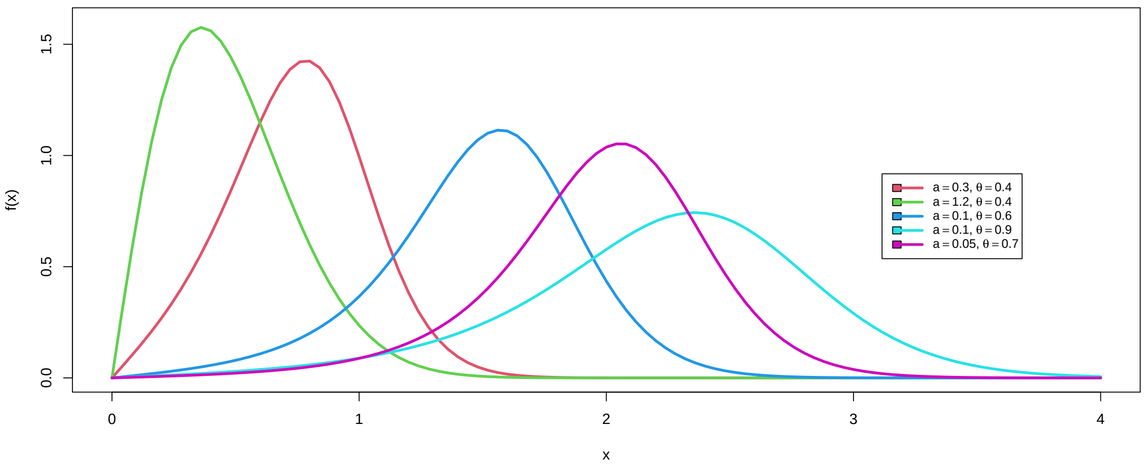

Figure 1: Plots of the pdf of the MTIRD

4. Reliability analysis of the MTI Rayleigh distribution (MTIRD)

This Section presents the derivation of key reliability measures for the MTIRD, including the survival function, hazard rate, reverse hazard rate, cumulative hazard function, and Mills ratio.

4.1. Survival function

The reliability function is defined as the probability that beyond a certain time period system will function. The reliability function or the survival function of MTIRD is given as

$$R(x;\alpha,\theta) = 1 - F(x;\alpha,\theta) = \frac{(\alpha - log\alpha) \ e^{-\frac{x^2}{2\theta^2}}}{\alpha - log\alpha \ e^{-\frac{x^2}{2\theta^2}}}. \tag{7}$$

4.2. Hazard rate

The hazard rate, also known as the failure rate, is a measure of the instantaneous rate of failure at any given time, given that the subject has survived up to that time. The hazard rate provides insights into the likelihood of failure over time. The hazard rate for MTIRD is obtained as

$$h(x;\alpha,\theta)=\frac{f(x;\alpha,\theta)}{R(x;\alpha,\theta)}=\frac{\alpha\ x}{\theta^2(\alpha-log\alpha\ e^{-\frac{x^2}{2\theta^2}})}. \tag{8}$$

Figure 2 depicts graphs of the hazard rate of the MTIRD for different parameter values. Figure 2 suggests that the proposed distribution is quite flexible in nature and can exhibit variety of shapes.

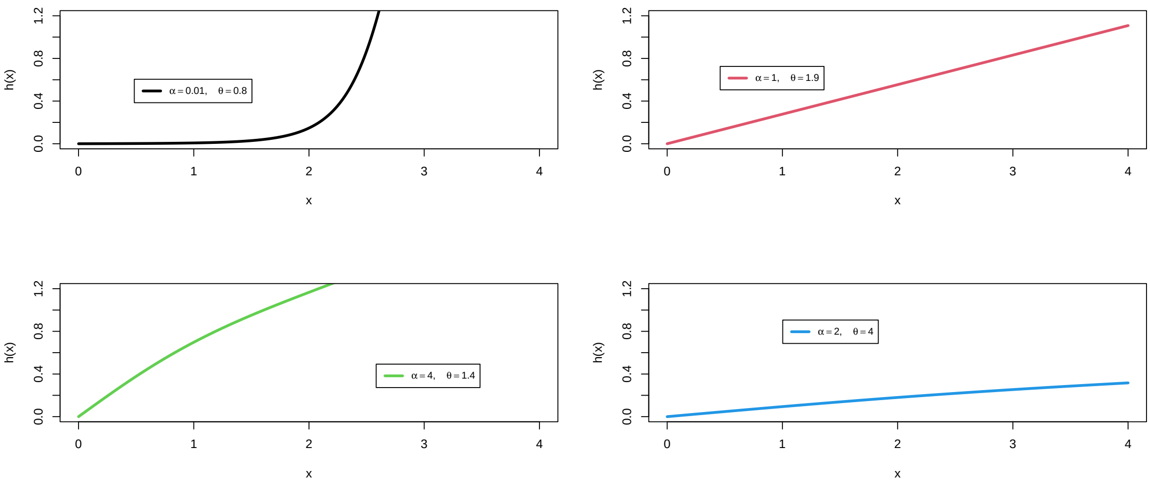

Figure 2: Plots of the hazard rate of the model

4.3. Reverse hazard rate

The reverse hazard rate quantifies the rate of failure at a specific time, given that the failure happens after that time. It is defined as the ratio between the life probability density to its distribution function. The reverse hazard rate for MTIRD is obtained as

$$h_r(x;\alpha,\theta)=\frac{f(x;\alpha,\theta)}{F(x;\alpha,\theta)}=\frac{x\ (\alpha-log\alpha)\ e^{-\frac{x^2}{2\theta^2}}}{\theta^2(\alpha-log\alpha\ e^{-\frac{x^2}{2\theta^2}})(1-e^{-\frac{x^2}{2\theta^2}})}. \tag{9}$$

4.4. Cumulative hazard function

The cumulative hazard rate is the integral of the hazard rate over time, representing the total accumulated risk of failure up to a specified time and is given for MTIRD as

$$\Lambda(x;\alpha,\theta)=-\log R(x;\alpha,\theta)=\log\left\{\frac{\alpha-\log\alpha\ e^{-\frac{x^2}{2\theta^2}}}{(\alpha-\log\alpha)\ e^{-\frac{x^2}{2\theta^2}}}\right\}. \tag{10}$$

4.5. Mills ratio

The Mills ratio is used in reliability analysis, especially when handling censored or truncated data, to adjust the estimates of distribution parameters and understand the behaviour of failure times. It is defined as the ratio of the cumulative distribution function to the reliability function and is given for MTIRD as

$$M.R = \frac{F(x;\alpha,\theta)}{R(x;\alpha,\theta)} = \frac{\alpha(1-e^{-\frac{x^2}{2\theta^2}})}{\alpha - \log\alpha \ e^{-\frac{x^2}{2\theta^2}}}. \tag{11}$$

5. Statistical properties of the MTIRD

This Section is presented to analyze the primary distributional and statistical properties of the proposed model.

5.1. Quantile function

Theorem 1. *If $X \sim MTIRD(\alpha,\theta)$, then the quantile function of X is given as*

$$x = \left[-2\theta^2 \log\left(\frac{\alpha(u-1)}{u\log\alpha - \alpha}\right)\right]^{\frac{1}{2}}. \tag{12}$$

where U is a uniform random variable, $0 < u < 1$.

Proof. Let $F(x;\alpha,\theta) = u$. The quantile function of the MTIRD can be obtained as follows.

$$\frac{\alpha(1-e^{-\frac{x^2}{2\theta^2}})}{\alpha - \log\alpha \ e^{-\frac{x^2}{2\theta^2}}} = u$$

$$e^{-\frac{x^2}{2\theta^2}} = \frac{(u-1)\alpha}{u \ \log\alpha - \alpha}$$

Taking the logarithm on both sides and simplifying further, we obtain the required quantile function as

$$x = \left[-2\theta^2 \log\left(\frac{\alpha(u-1)}{u\log\alpha - \alpha}\right)\right]^{\frac{1}{2}}. \tag{13}$$

The first quartile (Q_1), median (Q_2), and third quartile (Q_3) can be derived by setting $u = \frac{1}{4}, \frac{1}{2}$ and $\frac{3}{4}$ in Eq. (13) respectively. □

5.2. Moments

Theorem 2. *If $X \sim MTIRD(\alpha,\theta)$, then the r^{th} moment of the MTIRD about origin is given as*

$$\mu_r' = \frac{(\alpha - \log\alpha)}{\alpha} \sum_{j=0}^{\infty} \left(\frac{\log\alpha}{\alpha}\right)^j \Gamma\left(\frac{r}{2}+1\right) \left(\frac{2\theta^2}{j+1}\right)^{\frac{r}{2}}. \tag{14}$$

Proof. The r^{th} moment of the MTIRD is obtained by using the following series representation.

$$(1-u)^{-2} = \sum_{a=0}^{\infty} (a+1)u^a \ ; \qquad |u| < 1, \tag{15}$$

The r^{th} moment of X can be obtained as

$$\mu_r' = E(X^r) = \int_0^{\infty} x^r f(x;\alpha,\theta)dx.$$

$$= \int_0^{\infty} x^r \frac{x}{\theta^2} \frac{\alpha(\alpha-\log\alpha)e^{-\frac{x^2}{2\theta^2}}}{(\alpha - \log\alpha \ e^{-\frac{x^2}{2\theta^2}})^2} dx. \tag{16}$$

By substituting $e^{-\frac{x^2}{2\theta^2}} = y$ in (16), we get

$$\mu_r' = \frac{(\alpha - \log\alpha)}{\alpha} \sum_{j=0}^{\infty} (j+1) \left(\frac{\log\alpha}{\alpha}\right)^j \left(\int_0^1 \left(-2\theta^2 \log y\right)^{\frac{r}{2}} dy \right). \tag{17}$$

Again, substituting $(-2\theta^2 \log y) = x$ in Eq. (17), we get the final expression as

$$\mu_r' = \frac{(\alpha - \log\alpha)}{\alpha} \sum_{j=0}^{\infty} \left(\frac{\log\alpha}{\alpha}\right)^j \Gamma\left(\frac{r}{2}+1\right) \left(\frac{2\theta^2}{j+1}\right)^{\frac{r}{2}}. \tag{18}$$

setting $r = 1$ in Eq. (18), the mean of the model is computed as

$$\mu_1' = E(X) = \frac{(\alpha - \log\alpha)}{\alpha} \sum_{j=0}^{\infty} \left(\frac{\log\alpha}{\alpha}\right)^j \Gamma\left(\frac{3}{2}+1\right) \left(\frac{2\theta^2}{j+1}\right)^{\frac{1}{2}}. \tag{19}$$

Similarly, substituting $r = 2, 3, 4$ in Eq. (18) the Second , Third and Fourth moments about origin of the MTIRD are obtained respectively. □

5.3. Moment generating function of MTIRD

The moments of a distribution are represented by the moment generating function (MGF). The following theorem provides the MGF for the MTIRD .

Theorem 3. *If $X \sim MTIRD(\alpha, \theta)$, then the moment generating function, $M_X(t)$, is*

$$M_X(t) = \frac{(\alpha - log\alpha)}{\alpha} \sum_{r=0}^{\infty} \sum_{j=0}^{\infty} \frac{t^r}{r!} \left(\frac{log\alpha}{\alpha}\right)^j \left(\frac{2\theta^2}{j+1}\right)^{\frac{r}{2}} \Gamma(\frac{r}{2}+1). \tag{20}$$

Proof. The moment generating function of the MTIRD is defined as

$$M_X(t) = \int_0^{\infty} e^{tx} f(x; \alpha, \theta) dx. \tag{21}$$

Using the series representation of e^{tx}, we have

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} E(X^r) \tag{22}$$

Substituting the value of Eq. (18) in Eq. (22), we get

$$M_X(t) = \frac{(\alpha - log\alpha)}{\alpha} \sum_{r=0}^{\infty} \sum_{j=0}^{\infty} \frac{t^r}{r!} \left(\frac{log\alpha}{\alpha}\right)^j \left(\frac{2\theta^2}{j+1}\right)^{\frac{r}{2}} \Gamma(\frac{r}{2}+1). \tag{23}$$

□

5.4. Conditional moments and associated measures

In this Section, the expression for conditional moments is derived. First, we will introduce an important lemma that will be applied in the next subsection.

Lemma 1. *Let us suppose a random variable $X \sim MTIRD(\alpha, \theta)$ with PDF given in equation (6), and let $\varphi_r(z) = \int_0^z x^r f(x; \alpha, \theta)\, dx$ denote the r^{th} incomplete moment, then we have*

$$\varphi_r(z) = \frac{(\alpha - \log\alpha)}{\alpha} \sum_{j=0}^{\infty} \left(\frac{\log\alpha}{\alpha}\right)^j \left(\frac{2\theta^2}{j+1}\right)^{\frac{r}{2}} \gamma\left(\frac{r}{2}+1, \frac{(j+1)z^2}{2\theta^2}\right). \tag{24}$$

where $\gamma(a, b) = \int_0^b z^{a-1} e^{-z} dz$ denotes the lower incomplete gamma function.

Proof. Using the PDF of MTIRD given in Eq. (6), we have

$$\varphi_r(z) = \int_0^z x^r f(x;\alpha,\theta)dx = \alpha\,(\alpha - \log\alpha) \sum_{j=0}^{\infty} \left(\frac{\log\alpha}{\alpha}\right)^j \left(\frac{2\theta^2}{j+1}\right)^{\frac{r}{2}} \int_0^z x^r \frac{x}{\theta^2} \frac{e^{-\frac{x^2}{2\theta^2}}}{(\alpha - log\alpha\ e^{-\frac{x^2}{2\theta^2}})^2} dx. \quad (25)$$

On Simplification, we obtain r^{th} incomplete moment, then we have

$$\varphi_r(z) = \frac{(\alpha - \log\alpha)}{\alpha} \sum_{j=0}^{\infty} \left(\frac{\log\alpha}{\alpha}\right)^j \left(\frac{2\theta^2}{j+1}\right)^{\frac{r}{2}} \gamma\left(\frac{r}{2}+1, \frac{(j+1)z^2}{2\theta^2}\right). \quad (26)$$

Setting r=1 in Eq. (26) will yield first incomplete moment as given by

$$\varphi_1(z) = \frac{(\alpha - \log\alpha)}{\alpha} \sum_{j=0}^{\infty} \left(\frac{\log\alpha}{\alpha}\right)^j \left(\frac{2\theta^2}{j+1}\right)^{\frac{1}{2}} \gamma\left(\frac{3}{2}, \frac{(j+1)z^2}{2\theta^2}\right). \quad (27)$$

□

Lorenz and Bonferroni inequality curves

The Lorenz and Bonferroni inequality curves are an important application of the first incomplete moment. For a given probability distribution, they are defined by

$$L_p = \frac{1}{E(X)} \int_0^t x f(x;\alpha,\theta)\,dx = \frac{\varphi_1(t)}{E(X)}.$$

$$L_p = \frac{\sum_{j=0}^{\infty} \left(\frac{\log\alpha}{\alpha}\right)^j \left(\frac{1}{j+1}\right)^{\frac{1}{2}} \gamma\left(\frac{3}{2}, \frac{(j+1)t^2}{2\theta^2}\right)}{\sum_{j=0}^{\infty} \left(\frac{log\alpha}{\alpha}\right)^j \left(\frac{1}{j+1}\right)^{\frac{1}{2}} \Gamma(\frac{3}{2})}.$$

Similarly,

$$B_P = \frac{1}{pE(X)} \int_0^t x f(x;\alpha,\theta)\,dx = \frac{\varphi_1(t)}{pE(X)}.$$

$$B_P = \frac{\sum_{j=0}^{\infty} \left(\frac{\log\alpha}{\alpha}\right)^j \left(\frac{1}{j+1}\right)^{\frac{1}{2}} \gamma\left(\frac{3}{2}, \frac{(j+1)t^2}{2\theta^2}\right)}{p\sum_{j=0}^{\infty} \left(\frac{log\alpha}{\alpha}\right)^j \left(\frac{1}{j+1}\right)^{\frac{1}{2}} \Gamma(\frac{3}{2})}.$$

r^{th} conditional moment and r^{th} reversed conditional moment of MTIRD

The r^{th} conditional moment of the MTIRD is calculated by

$$E\left[X^r | x > t\right] = \frac{1}{R(t)} \int_t^{\infty} x^r\, f(x;\alpha,\theta)\,dx = \frac{1}{R(t)} \left[E(X^r) - \varphi_r(t)\right].$$

Inserting the value of Eq.s (7), (18) and (26), we obtain

$$E\left[X^r | x > t\right] = \frac{(\alpha - log\alpha\ e^{-\frac{t^2}{2\theta^2}}) \sum_{j=0}^{\infty} \left(\frac{\log\alpha}{\alpha}\right)^j \left(\frac{2\theta^2}{j+1}\right)^{\frac{r}{2}} \left[\Gamma\left(\frac{r}{2}+1\right) - \gamma\left(\left(\frac{r}{2}+1\right), \frac{(j+1)t^2}{2\theta^2}\right)\right]}{\alpha\ e^{-\frac{t^2}{2\theta^2}}}.$$

Similarly, the r^{th} reversed conditional moment of the MTIRD is defined by

$$E\left[X^r | x \le t\right] = \frac{1}{F(t)} \int_0^t x^r f(x;\alpha,\theta)\,dx = \frac{\varphi_r(t)}{F(t)}.$$

$$E\left[X^r|x\leq t\right]=\frac{(\alpha-log\alpha)\,(\alpha-log\alpha\;e^{-\frac{t^2}{2\theta^2}})\sum_{j=0}^{\infty}\left(\frac{\log\alpha}{\alpha}\right)^j\left(\frac{2\theta^2}{j+1}\right)^{\frac{r}{2}}\gamma\left(\left(\frac{r}{2}+1\right),\frac{(j+1)t^2}{2\theta^2}\right)}{\alpha^2\;(1-e^{-\frac{t^2}{2\theta^2}})}.$$

Mean residual life (MRL) and mean waiting time (MWT)

Suppose X is a continuous random variable having survival function R(x), the mean residual life function, say $\mu(t)$ is defned as the expected life of an item afer it has reached a certain age t, is given by

$$\mu(t)=\frac{1}{R(t)}\left[E(t)-\int_0^t xf(x;\alpha,\theta)\,dx\right]-t=\frac{1}{R(t)}\left[E(t)-\varphi_1(t)\right]-t.$$

After inserting the value of equation (7), (19) and (27), we obtain the required expression for MRL as

$$\mu(t)=\frac{(\alpha-log\alpha\;e^{-\frac{t^2}{2\theta^2}})\sum_{j=0}^{\infty}\left(\frac{\log\alpha}{\alpha}\right)^j\left(\frac{2\theta^2}{j+1}\right)^{\frac{1}{2}}\left[\Gamma\left(\frac{3}{2}\right)-\gamma\left(\frac{3}{2},\frac{(j+1)t^2}{2\theta^2}\right)\right]}{\alpha\;e^{-\frac{t^2}{2\theta^2}}}-t.$$

The mean waiting time is very important to analyse the actual time of failure of an already failed item. It represents the amount of time that has passed since an object failed, assuming that the failure occurred within the interval [0, t]. The mean waiting time say $\overline{\mu}(t)$, is defined by

$$\overline{\mu}(t)=t-\frac{1}{F(t)}\int_0^t xf(x;\alpha,\theta)\,dx=t-\frac{\varphi_1(t)}{F(t)}.$$

$$\overline{\mu}(t)=t-\frac{(\alpha-log\alpha)\,(\alpha-log\alpha\;e^{-\frac{t^2}{2\theta^2}})\sum_{j=0}^{\infty}\left(\frac{\log\alpha}{\alpha}\right)^j\left(\frac{2\theta^2}{j+1}\right)^{\frac{1}{2}}\gamma\left(\frac{3}{2},\frac{(j+1)t^2}{2\theta^2}\right)}{\alpha^2\;(1-e^{-\frac{t^2}{2\theta^2}})}.$$

5.5. Renyi entropy

Theorem 4. *If $X\sim MTIRD(\alpha,\theta)$, then the Renyi entropy of the MTIRD is given as*

$$RE_X(u)=\frac{1}{1-u}\log\left\{\left(\frac{(\alpha-\log\alpha)}{\alpha}\right)^u\left(\frac{2}{\theta^2}\right)^{\frac{u-1}{2}}\sum_{a=0}^{\infty}\binom{2u}{a}\left(\frac{\log\alpha}{\alpha}\right)^a\frac{\Gamma\left(\frac{u+1}{2}\right)}{(u+a)^{\frac{u+1}{2}}}\right\}.$$

Proof. The Renyi entropy, say $RE_X(u)$ of MTIRD can be defined as

$$RE_X(u)=\frac{1}{1-u}log\left(\int_{-\infty}^{\infty}f^u(x,\alpha,\theta)dx\right);\quad u>0,\quad u\neq1. \tag{28}$$

Substituting Eq. (6) in Eq. (28), we get

$$RE_X(u)=\frac{1}{1-u}log\left\{\left(\frac{\alpha(\alpha-log\alpha)}{\theta^2}\right)^u\int_0^{\infty}\frac{x^u\;e^{-\frac{ux^2}{2\theta^2}}}{\left(\alpha-log\alpha\;e^{-\frac{x^2}{2\theta^2}}\right)^{2u}}dx\right\}. \tag{29}$$

By using the same procedure as in the Eq. (18), we get the final expression for Renyi entropy as

$$RE_X(u)=\frac{1}{1-u}\log\left\{\left(\frac{(\alpha-\log\alpha)}{\alpha}\right)^u\left(\frac{2}{\theta^2}\right)^{\frac{u-1}{2}}\sum_{a=0}^{\infty}\binom{2u}{a}\left(\frac{\log\alpha}{\alpha}\right)^a\frac{\Gamma\left(\frac{u+1}{2}\right)}{(u+a)^{\frac{u+1}{2}}}\right\}.$$

□

5.6. Order statistics of MTIRD

In real-world applications incorporating data from life testing studies, order statistics is very important. Let $x_{(r;n)}$ be the r^{th} order statistics with the random sample $x_{(1)}, x_{(2)}, x_{(3)}, ...x_{(n)}$ derived from the MTIRD having the PDF $f(x;\alpha,\theta)$ and CDF $F(x;\alpha,\theta)$. Therefore, the PDF and CDF of $x_{(r;n)}$ say $f_{(r;n)}(x)$ and $F_{(r;n)}(x)$ are respectively defined as

$$f_{(r;n)}(x) = \frac{1}{B(n, n-r+1)} \left[F(x;\alpha,\theta)\right]^{r-1} \left[1 - F(x;\alpha,\theta)\right]^{n-r} f(x;\alpha,\theta). \tag{30}$$

$$F_{(r;n)}(x) = \sum_{j=r}^{n} \binom{n}{j} \left[F(x;\alpha,\theta)\right]^{j} \left[1 - F(x;\alpha,\theta)\right]^{n-j}. \tag{31}$$

Using Eq. (6) and Eq. (5) in Eq. (30) and Eq. (31), the PDF and CDF of r^{th} ordered statistics for the MTIRD are derived and are expressed as

$$f_{r:n}(x) = \frac{x\alpha^{r}(1-e^{-\frac{x^2}{2\theta^2}})^{r-1}\left((\alpha - log\alpha)e^{-\frac{x^2}{2\theta^2}}\right)^{n-r+1}}{\theta^2 \; B(r, n-r+1)\left(\alpha - log\alpha \; e^{-\frac{x^2}{2\theta^2}}\right)^{n+1}}.$$

where $B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ is the beta function.

$$F_{(r;n)}(x) = \sum_{j=r}^{n} \binom{n}{j} \frac{\left(\alpha(1-e^{-\frac{x^2}{2\theta^2}})\right)^{j}\left((\alpha - log\alpha)e^{-\frac{x^2}{2\theta^2}}\right)^{n-j}}{\left(\alpha - log\alpha \; e^{-\frac{x^2}{2\theta^2}}\right)^{n}}.$$

5.7. Stress strength reliability

Theorem 5. *If $X_1 \sim MTIRD(\alpha_1, \theta)$ and $X_2 \sim MTIRD(\alpha_2, \theta)$, where X_1 and X_2 are independent strength and stress rv's respectively, then, the stress strength reliability $P(X_1 > X_2)$, of the MTIRD is*

$$SSR = \frac{\alpha_1 - log\alpha_1}{\alpha_1} \sum_{j=0}^{\infty}\sum_{k=0}^{\infty} \left(\frac{log\alpha_1}{\alpha_1}\right)^{j} \left(\frac{log\alpha_2}{\alpha_2}\right)^{k} \frac{(j+1)}{(j+k+1)(j+k+2)}.$$

Proof. The stress strength reliability $P(X_1 > X_2)$, of the MTIRD, say SSR, can be obtained as

$$SSR = \int_0^{\infty} f_1(x) F_2(x) dx. \tag{32}$$

Using Eq.s (5) and (6) in (32), the stress-strength reliability SSR, can be obtained as

$$SSR = \frac{\alpha_1 - log\alpha_1}{\theta^2\alpha_1} \int_0^{\infty} x e^{-\frac{x^2}{2\theta^2}} (1 - e^{-\frac{x^2}{2\theta^2}}) \left(1 - \frac{log\alpha_1}{\alpha_1} e^{-\frac{x^2}{2\theta^2}}\right)^{-2} \left(1 - \frac{log\alpha_2}{\alpha_2} e^{-\frac{x^2}{2\theta^2}}\right)^{-1} dx.$$

Using the following series representations.

$$(1-u)^{-2} = \sum_{a=0}^{\infty} (a+1) u^{a} \; ; \qquad |u| < 1 \tag{33}$$

$$(1-u)^{-1} = \sum_{a=0}^{\infty} u^{a} \; ; \qquad |u| < 1 \tag{34}$$

Therefore, SSR for MTIRD is

$$SSR = \frac{\alpha_1 - log\alpha_1}{\alpha_1} \sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \left(\frac{log\alpha_1}{\alpha_1}\right)^j \left(\frac{log\alpha_2}{\alpha_2}\right)^k \frac{(j+1)}{(j+k+1)(j+k+2)}$$

□

6. Estimation

This Section covers the maximum likelihood estimation method for determining the unknown parameters, α and θ, of the MTIRD.

6.1. Maximum likelihood estimation

Let $x_1, x_2, ..., x_n$ be a random sample from MTIRD with parameters $\alpha, \theta > 0$, then the logarithm of the likelihood function of MTIRD is given by

$$l = nlog(\alpha(\alpha - log\alpha)) - 2nlog\theta + \sum_{a=1}^{n} logx_a - \frac{1}{2\theta^2} \sum_{a=1}^{n} x_a^2 - 2 \sum_{a=1}^{n} log\left(\alpha - log\alpha\, e^{-\frac{x_a^2}{2\theta^2}}\right) \quad (35)$$

The MLEs of α and θ are achieved by partially differentiating (35) w.r.t. the corresponding parameters and equating to zero, we have

$$\frac{\partial l}{\partial \alpha} = \frac{n}{\alpha} + \frac{n(\alpha - 1)}{\alpha(\alpha - log\alpha)} - 2 \sum_{a=1}^{n} \left[\frac{\alpha - e^{-\frac{x_a^2}{2\theta^2}}}{\alpha\left(\alpha - log\alpha\, e^{-\frac{x_a^2}{2\theta^2}}\right)}\right] = 0 \quad (36)$$

$$\frac{\partial l}{\partial \theta} = \frac{1}{\theta^3} \sum_{a=1}^{n} x_a^2 - \frac{2n}{\theta} + \frac{2\, log\alpha}{\theta^3} \sum_{a=1}^{n} \left[\frac{x_a^2\, e^{-\frac{x_a^2}{2\theta^2}}}{\alpha - log\alpha\, e^{-\frac{x_a^2}{2\theta^2}}}\right] = 0 \quad (37)$$

The equations 36 and 37 are nonlinear and do not have closed-form solutions. To estimate the parameters, we will use R software to solve these equations.

7. Simulation study

In this Section, we conduct a comprehensive simulation study using R software to investigate the behaviour of the maximum likelihood estimators (MLEs) for various sample sizes. We generate random samples of sizes 25, 75, 150, 300, and 500 from the MTIRD and repeat the process 1000 times. The parameters are selected as $(0.80, 1.20)$, $(2.20, 0.60)$, $(1.10, 1.40)$, and $(0.85, 1.65)$, corresponding to the standard order (α, θ). For each scenario, we compute the average MLE values, bias, and empirical mean squared errors (MSEs). The results are presented in Table 1, showing that the estimates are stable and closely aligned with the true parameter values. Additionally, in all scenarios, the MSE decreases as the sample size increases, demonstrating the consistency of the estimators.

Table 1: MLE, bias, and MSE for the parameters α and θ

Sample size	Parameters		MLE		Bias		MSE	
n	α	θ	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\theta}$
25	0.80	1.20	0.96958	1.16150	0.49330	0.16669	0.56503	0.04146
75			0.85777	1.18731	0.25483	0.10494	0.16435	0.01625
150			0.83338	1.19771	0.16984	0.07212	0.06240	0.00799
300			0.81900	1.20106	0.11141	0.05086	0.02245	0.00411
500			0.80445	1.20036	0.08000	0.03817	0.01101	0.00236
25	2.20	0.60	1.67451	0.54241	0.96343	0.08365	1.35510	0.01097
75			1.86974	0.56769	0.78377	0.04812	0.89743	0.00380
150			2.02239	0.57958	0.65956	0.03370	0.65546	0.00194
300			2.06690	0.58487	0.60704	0.02449	0.52038	0.00104
500			2.14409	0.58889	0.53913	0.01846	0.40804	0.00058
25	1.10	1.40	1.30378	1.32234	0.69652	0.1878	0.79249	0.05403
75			1.30118	1.37959	0.49945	0.11822	0.52381	0.02097
150			1.23121	1.39097	0.35321	0.08716	0.29101	0.01176
300			1.16881	1.39418	0.24227	0.06502	0.14984	0.00645
500			1.14516	1.39883	0.18504	0.05250	0.08117	0.00435
25	0.85	1.65	1.02575	1.59788	0.50055	0.21574	0.56888	0.06876
75			0.93688	1.63750	0.29140	0.14030	0.22318	0.03059
150			0.88617	1.64282	0.18817	0.10221	0.08429	0.01626
300			0.87552	1.64944	0.13080	0.07323	0.03479	0.00844
500			0.86288	1.65128	00.09599	0.05664	0.01667	0.00517

8. Applications to COVID-19 data

This Section concentrates on the application of the proposed model to real-life data sets. Using information measures such as the Akaike Information Criterion (AIC), Bayesian Information Criterion (BIC), and Akaike Information Criterion Corrected (AICc), as well as goodness-of-fit statistics like the Kolmogorov–Smirnov statistics along with the corresponding p-value, the potentiality of the proposed model is determined by comparing its performance with several other introduced models, namely Power Rayleigh Distribution (PRD) Bhat and Ahmad (2020) , Transmuted Rayleigh Distribution (TRD) Merovci (2013), Exponentiated Rayleigh Distribution (ERD) Surles and Padgett (2001), Weibull Rayleigh Distribution (WRD) Ahmad, Ahmad, and Ahmed (2017) and Rayleigh Distribution (RD) Rayleigh (1880).

Data set 1: The first data represents a COVID-19 mortality rate data belongs to Mexico of 108 days, that is recorded from 4 March to 20 July 2020. This data formed of rough mortality

rate. The data has been previously used by Almongy, Almetwally, Aljohani, Alghamdi, and Hafez (2021). The data are as follows:
8.826 6.105 10.383 7.267 13.220 6.015 10.855 6.122 10.685 10.035 5.242 7.630 14.604 7.903 6.327 9.391 14.962 4.730 3.215 16.498 11.665 9.284 12.878 6.656 3.440 5.854 8.813 10.043 7.260 5.985 4.424 4.344 5.143 9.935 7.840 9.550 6.968 6.370 3.537 3.286 10.158 8.108 6.697 7.151 6.560 2.988 3.336 6.814 8.325 7.854 8.551 3.228 3.499 3.751 7.486 6.625 6.140 4.909 4.661 1.867 2.838 5.392 12.042 8.696 6.412 3.395 1.815 3.327 5.406 6.182 4.949 4.089 3.359 2.070 3.298 5.317 5.442 4.557 4.292 2.500 6.535 4.648 4.697 5.459 4.120 3.922 3.219 1.402 2.438 3.257 3.632 3.233 3.027 2.352 1.205 2.077 3.778 3.218 2.926 2.601 2.065 1.041 1.800 3.029 2.058 2.326 2.506 1.923.

Data set 2: The second data represents a COVID-19 data belonging to the Netherlands of 30 days, which recorded from 31 March to 30 April 2020. This data formed of rough mortality rate.The data has been previously used by Almongy *et al.* (2021). The data are as follows:
14.918 10.656 12.274 10.289 10.832 7.099 5.928 13.211 7.968 7.584 5.555 6.027 4.097 3.611 4.960 7.498 6.940 5.307 5.048 2.857 2.254 5.431 4.462 3.883 3.461 3.647 1.974 1.273 1.416 4.235.

The results shown in Tables 4 and 5 reveals that MTIRD is having a smallest value of AIC, BIC and AICC as compared to other competing models, Power Rayleigh Distribution (PRD) Bhat and Ahmad (2020), Transmuted Rayleigh Distribution (TRD) Merovci (2013), Exponentiated Rayleigh Distribution (ERD) Surles and Padgett (2001), Weibull Rayleigh Distribution (WRD) Ahmad *et al.* (2017) and Rayleigh Distribution (RD) Rayleigh (1880) and thus outperforms base model of Rayleigh distribution as well as other mentioned competing models. The results are further supported by Figs. 3 and 4.

Table 2: MLEs of MTIRD and competitive models with corresponding SE (given in parenthesis) for Data set 1

Model	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\eta}$
MTIRD	2.71827	5.26545	-	-
	(6.62562)	(0.28212)		
PRD	0.94840	4.185772	-	-
	(0.06891)	(0.64307)		
TRD	-	5.41934	-	0.52717
		(0.76236)		(0.39096)
ERD	0.98411	0.02268	-	-
	(0.12530)	(0.00290)		
WRD	1.00100	3.08438	2.29510	-
	(0.07157)	(144.5199)	(215.07606)	
RD	-	4.67154	-	-
		(0.22476)		

Table 3: MLEs of MTIRD and competitive models with corresponding SE (given in parenthesis) for Data set 2

Model	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\eta}$
MTIRD	2.71828	5.62707	-	-
	(13.13437)	(0.57136)		
PRD	0.93999	4.38275	-	-
	(0.13145)	(1.31766)		
TRD	-	5.57738	-	0.41225
		(1.13100)		(0.57859)
ERD	0.94725	0.01932	-	-
	(0.22631)	(0.00471)		
WRD	1.00100	3.20531	2.43308	-
	(0.13772)	(299.08934)	(454.06364)	
RD	-	4.99853	-	-
		(0.45630)		

Table 4: Comparison of MTIRD and competitive models for Data set 1

Model	$-2ll$	AIC	BIC	AICC	K-S	p-value
MTIRD	536.2300	540.2300	545.5943	540.3443	0.0591	0.8444
PRD	537.9138	541.9138	547.2781	542.0281	0.0735	0.6028
TRD	536.2621	540.2621	545.6263	540.3763	0.0690	0.6821
ERD	538.4436	542.4436	547.8078	542.5579	0.0899	0.3465
WRD	538.4816	544.4816	552.5280	544.5959	0.0937	0.2985
RD	538.4605	540.4605	543.1426	540.4982	0.0933	0.303

Table 5: Comparison of MTIRD and competitive models for Data set 2

Model	$-2ll$	AIC	BIC	AICC	K-S	p-value
MTIRD	153.63200	157.6320	160.4344	158.0764	0.0826	0.9759
PRD	154.0685	158.0685	160.8709	158.5129	0.0999	0.8962
TRD	153.7250	157.7250	160.5274	158.1694	0.0893	0.9528
ERD	154.2168	158.2168	161.0192	158.6613	0.1104	0.8192
WRD	154.2772	160.2772	164.4808	160.7216	0.1170	0.763
RD	154.2705	156.2705	157.6717	156.4134	0.1167	0.7655

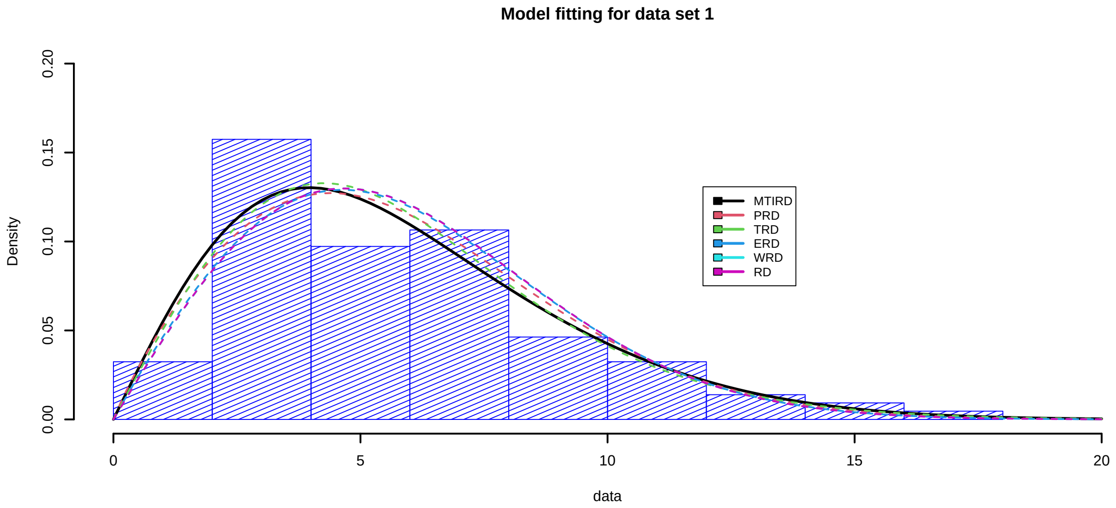

Figure 3: Fitted density plots for data set 1

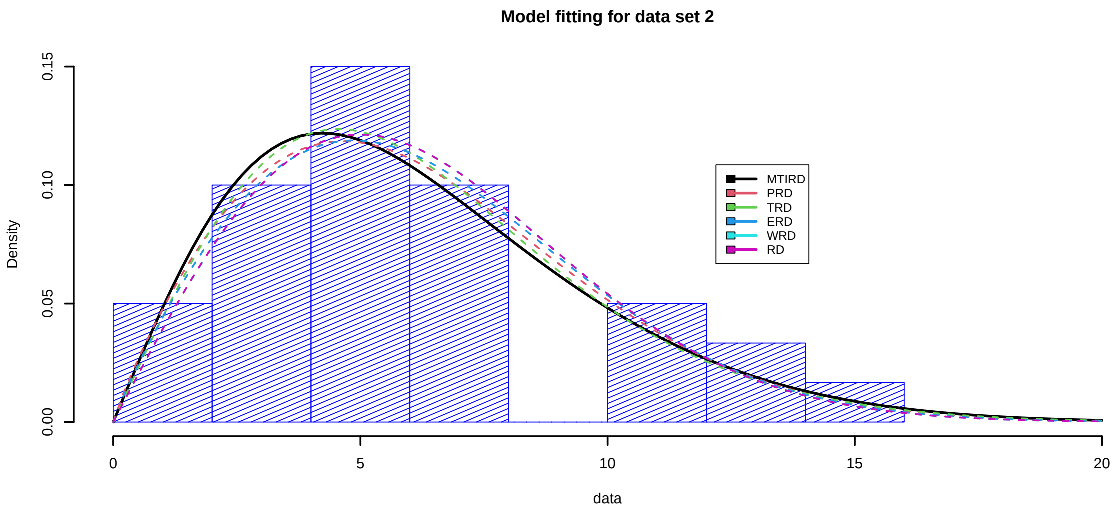

Figure 4: Fitted density plots for data set 2

9. Conclusion

In this manuscript, we propose a new model called the MTI Rayleigh distribution (MTIRD) which extends the Rayleigh distribution in the analysis of data with real support. An obvious reason for generalizing a standard distribution is because the generalized form provides larger flexibility in modeling real data. For the proposed model, important statistical properties have been derived. In addition to this, some reliability analysis have also been discussed. The new distribution is more flexible and its hazard rate function exhibits complex shapes. A simulation study is carried out to evaluate the performance of maximum likelihood estimate. The outcomes prove that the maximum likelihood estimate is consistent and precise. The estimation of parameters is approached by the method of maximum likelihood estimation. Also, two real data sets of COVID-19 mortality rates were considered, and they showed that MTIRD provides the best fit for these kinds of data compared with other competitive distributions. We prospect that the proposed model will draw wider applications in statistics.

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