Austrian Journal of Statistics
2025, Volume 54, 18–43.
http://www.ajs.or.at/
doi:10.17713/ajs.v54i2.1902

On the Flexible Alpha Skew-logistic Distribution: Properties and Applications

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Abstract

This article introduces a novel class of skew-logistic distribution, providing flexibility to fit data with up to three modes. Various important properties of this new distribution are thoroughly examined, including its moment generating function, moments, entropy, and characterizations. Considering the location and scale parameters, a model extension and its parameter estimation technique are included. The study also includes a simulation study using the Metropolis–Hastings algorithm to observe the behavior of the estimated parameters. Furthermore, the adaptability and utility of the new model are assessed using three real-life datasets. Finally, a likelihood ratio test is applied to distinguish the introduced model from some other simpler models.

Keywords: trimodlity, multimodality, skew logistic, skewness, AIC.

1. Introduction

Gaussian distribution is considered as one of the most useful continuous probability distribution. This distribution, however, is specifically used to describe only the situation involving symmetric behavior and when asymmetry arises in the data then it isn't the appropriate one to describe the situation. To handle these issues, Azzalini (2013) proposed a new class of continuous probability distribution namely skew normal (SN) distribution. The pdf (probability density function) of SN is given by

$$f(x;\lambda) = 2\phi(x)\Phi(\lambda x); \quad x \in R,\, \lambda \in R, \tag{1}$$

where, $\phi(\cdot)$ and $\Phi(\cdot)$ are the pdf and cumulative distribution function (cdf) of the standard normal distribution, respectively. Considering the flexibility and adverse usefulness of SN model, subsequent research in the literature was carried out extensively into the skew normal distribution. Nadarajah (2009), Azzalini (2013), Wang, Boyer, and Genton (2004), and many others contributed remarkably towards those studies. Thus a continuous research study was going on considering the possibilities and research gaps found in skew normal model (Azzalini 2013). Later on, replacing the pdf and cdf of the standard normal distribution in SN model by pdf and cdf of some other popular symmetric distribution, some new classes of asymmetric distributions were introduced along with some of their important properties. Some perfect examples towards this study are skew-logistic (SL) distribution (Nadarajah 2009), skew-Laplace (SLa) distribution (Aryal and Nadarajah 2005), skew generalized t distribution (Theodossiou 1998), skew-cauchy distribution (Arnold and Beaver 2000), skew uniform distribution (Nadarajah and Aryal 2004), and so on.

The pdf of skew-logistic distribution was proposed by Nadarajah (2009) and is given

$$f(x;\lambda) = 2\left[\frac{\exp(-x)}{(1+\exp(-x))^2}\right]\left[\frac{1}{1+\exp(-\lambda x)}\right], \quad x \in R,\ \lambda \in R. \tag{2}$$

Nadarajah (2009) discussed some important properties of the SL distribution and reported the usefulness of the asymmetric version of logistic distribution. He also stated that the SL distribution has several interesting applications in various fields such as geology, medicine, and several ones. Consequently, various research studies were conducted on the SL distribution which reflects on some notable works of Chakraborty, Hazarika, and Ali (2012), Gupta and Kundu (2010), Sastry and Bhati (2016), Mirzadeh and Iranmanesh (2019), and so on.

While numerous research studies have been undertaken on various models of unimodal skew distributions, there exists a research gap in exploring new skew distributions capable of accommodating data exhibiting both unimodal, bimodal or trimodal characteristics. Alpha skew-normal distribution was proposed by Elal-Olivero (2010) which allows to fit data with both uni- and bi-modality. Considering the same idea, alpha SL distribution (Hazarika and Chakraborty 2014) and alpha SLa distribution (Harandi and Alamatsaz 2013) were also introduced to fit the uni-bimodal data. A new distribution namely Generalized alpha skew-normal distribution were also introduced by Sharafi, Sajjadnia, and Behboodian (2017) which is more flexible to fit bimodal data than that of alpha skew-normal distribution (Elal-Olivero 2010). Two parameter skew-normal distribution (Elal-Olivero, Olivares-Pacheco, Venegas, Bolfarine, and Gomez 2020) was another skew distribution introduced with different mechanism to fit data with uni- and bi-modal character. For modelling uni- and bi-modality data, some other class of probability distributions were presented under Balakrishnan mechanism (Arnold, Beaver, Azzalini, Balakrishnan, Bhaumik, Dey, Cuadras, and Sarabia 2002), including Balakrishnan alpha skew-normal distribution (Hazarika, Shah, and Chakraborty 2020), Balakrishnan alpha skew-logistic distribution (Shah, Chakraborty, and Hazarika 2020a), Log-Balakrishnan alpha skew-normal distribution (Shah, Chakraborty, Hazarika, and Ali 2020b), bimodal skew-symmetric normal distribution (Contreras-Reyes 2020), Balakrishnan alpha Skew-Generalized t distribution (Pathak, Shah, Hazarika, Chakraborty, and Das 2023b), Generalized Balakrishnan alpha skew normal distribution (Shah, Hazarika, Chakraborty, and Hamedani 2024), and so on.

On the other hand, alpha beta skew-normal distribution (Shafiei, Doostparast, and Jamalizadeh 2016) was another skewed model which fits data with trimodal behavior. Some new distributions including alpha beta SL distribution (Esmaeili, Lak, Alizadeh *et al.* 2020), alpha beta generalized t distribution (Lak, Alizadeh, Monfared, and Esmaeili 2019), generalized alpha beta skew-normal distribution (Shah, Hazarika, Chakraborty, and Ali 2023a), Balakrishnan alpha beta skew-normal distribution (Shah, Hazarika, Chakraborty, and Ali 2021), Balakrishnan alpha beta SLa distribution (Shah, Hazarika, Chakraborty, and Alizadeh 2023b), and many more were also introduced the same idea for fitting trimodal data. Moreover, Martínez-Flórez, Tovar-Falón, and Elal-Olivero (2022) proposed some new classes of continuous probability distributions for fitting data with uni-bimodal as well as trimodal character. Trimodal SL distribution (Pathak, Hazarika, Chakraborty, Das, and Hamedani 2023a) and the flexible alpha skew-normal distribution (Das, Pathak, Hazarika, Chakraborty, and Hamedani 2023) were two new skew distributions which also fits data up to three modes. The pdf of flexible alpha skew-normal distribution (Das *et al.* 2023) is given as

$$f(x;\alpha,\lambda) = \frac{2}{C}\left(1+\alpha\left(\frac{(x^2-1)^2+2}{4}\right)\right)\phi(x)\Phi(\lambda x), \quad x\in\mathbb{R},\, \lambda\in\mathbb{R}, \alpha\geq 0. \tag{3}$$

where, $C = 1+\alpha$.

Furthermore, various new families of distributions designed to address multimodal data are presented in the literature. Examples includes the multimodal skew-normal distribution (Chakraborty, Hazarika, and Ali 2015), multimodal alpha skew-normal distribution Hazarika, Shah, Chakraborty, Alizadeh, and Hamedani (2023), multimodal Balakrishnan alpha skew-normal distribution (Shah, Hazarika, Pathak, Chakraborty, and Ali 2023c), multimodal alpha SLa distribution (Chakraborty, Hazarika, and Ali 2014), and many others. These studies represented notable contributions in exploring the multimodal approach to modeling data.

Though many research have been contacted on uni-bimodal symmetric or asymmetric probability distribution, a little studies are reported regarding the probability distribution which allows to fit data with more than two modes. In this article, a new class of skew distribution is addressed borrowing the idea of flexible skew normal distribution (Das *et al.* 2023) which is flexible enough to fit data up to three modes. Some important mathematical properties of the new distribution are discussed along with graphical visualization. Consequently, the adaptability and usefulness of the new distribution is also checked using real life datasets.

The rest of the article is organized as follows: Section 2 includes the new family of SL distribution along with some pictorial visualizations of the pdfs as well as special cases. Some important mathematical properties of the distribution are also included in Section 2. Section 3 provides the characterizations of the FASL distribution via two truncated moments and conditional distribution. Section 4 addresses a location scale extension and parameter estimation of the new distribution, while Section 5 provides the results of simulation study. Applications of the new distribution using three real-life data sets are considered in Section 6. The summary of the hypothesis testing results is given in Section 7. Finally, Section 8 concludes the article.

2. Flexible alpha skew-logistic distribution

This section includes a new extension of SL distribution discussing some of its significant properties and graphical presentations.

Definition 1. Let X be a random variable, then X is said to follow a flexible alpha SL distribution, denoted as $X \sim FASL(\alpha,\lambda)$, if its pdf is given by

$$f(x;\alpha,\lambda) = \frac{2\left(1+\alpha\left(\frac{(x^2-1)^2+2}{4}\right)\right)\exp(-x)}{C_1\Big(1+\exp(-x)\Big)^2\Big(1+\exp(-\lambda x)\Big)} \tag{4}$$

where $x\in\mathbb{R}$, $\lambda\in\mathbb{R}$ $\alpha\geq 0$ and $C_1 = 1+\frac{\alpha}{60}\left[45-10\pi^2+7\pi^4\right]$. The detailed derivation of C_1 is provided in Appendix A.

Using the Taylor series expansion for $(1+x)^{-1}$, single series representation of the pdf can be written as

$$f(x) = \begin{cases} \dfrac{2}{C_1\Big(1+\exp(-x)\Big)^2}\left(1+\alpha\left(\dfrac{(x^2-1)^2+2}{4}\right)\right) \\ \times\sum_{j=0}^{\infty}(-1)^j \exp\Big(-1-\lambda j\Big)x, & \text{if} \quad x \geq 0, \\ \dfrac{2}{C_1\Big(1+\exp(-x)\Big)^2}\left(1+\alpha\left(\dfrac{(x^2-1)^2+2}{4}\right)\right) \\ \times\sum_{j=0}^{\infty}(-1)^j \exp\Big(-(1-\lambda-\lambda j)\,x\Big), & \text{if} \quad x < 0. \end{cases} \tag{5}$$

Again by expanding the terms involved in (5), a double series representation of the pdf can be written as

$$f(x) = \begin{cases} \dfrac{2}{C_1}\left(1+\alpha\left(\dfrac{(x^2-1)^2+2}{4}\right)\right)\sum_{j,k=0}^{\infty}(-1)^{j+k}(k+1)\exp\left(-C_2 x\right), & \text{if} \quad x \geq 0, \\ \dfrac{2}{C_1}\left(1+\alpha\left(\dfrac{(x^2-1)^2+2}{4}\right)\right)\sum_{j,k=0}^{\infty}(-1)^{j+k}(k+1)\exp\left(Dx\right), & \text{if} \quad x < 0, \end{cases} \tag{6}$$

where $C_2 = 1+\lambda j+k$ and $D = C_2+\lambda$.

Some properties of flexible alpha SL distribution are:

1. If $\lambda = 0$, then the pdf of flexible alpha logistic distribution reduces to a new symmetric pdf and is denoted as $X \sim FAL(\alpha)$. The pdf of X is given as

$$f(x;\alpha,\lambda) = \frac{2\left(1+\alpha\left(\dfrac{(x^2-1)^2+2}{4}\right)\right)\exp(-x)}{C(\alpha)\Big(1+\exp(x)\Big)^2}, \tag{7}$$

   where $C(\alpha) = 2+\dfrac{\alpha}{30}(45-10\pi^2+7\pi^4)$. The detailed derivation of $C(\alpha)$ is provided in Appendix B.

2. If $\alpha = 0$, then $X \sim SL(\lambda)$.

3. If $\lambda = 0$ and $\alpha = 0$, then $X \sim L(0,1)$.

4. If $X \sim FASL(\alpha,\lambda)$, then $-X \sim FASL(\alpha,-\lambda)$.

The pdf of the FASL distribution is plotted in Figure 1 for various parameters. It can be seen that for both positive values of α and λ, the pdf can fit data up to three modes (see Fig 1 (a)). Additionally, if λ is assumed to be negative, the pdf likewise exhibits trimodality (see Fig. 1 (d)), although the graphs' peaks point in the opposite direction from Fig. 1 (a). Furthermore, if $\alpha = 0$ and $(\alpha,\lambda) = (0,0)$, the new distribution simplifies to $SL(\lambda)$ and $L(0,1)$, respectively (see Fig 1 (c)). Eventually, for $\lambda = 0$, the new distribution switches to a symmetric distribution with trimodal behavior known as the $FAL(\alpha)$ distribution (see Fig 1 (b)).

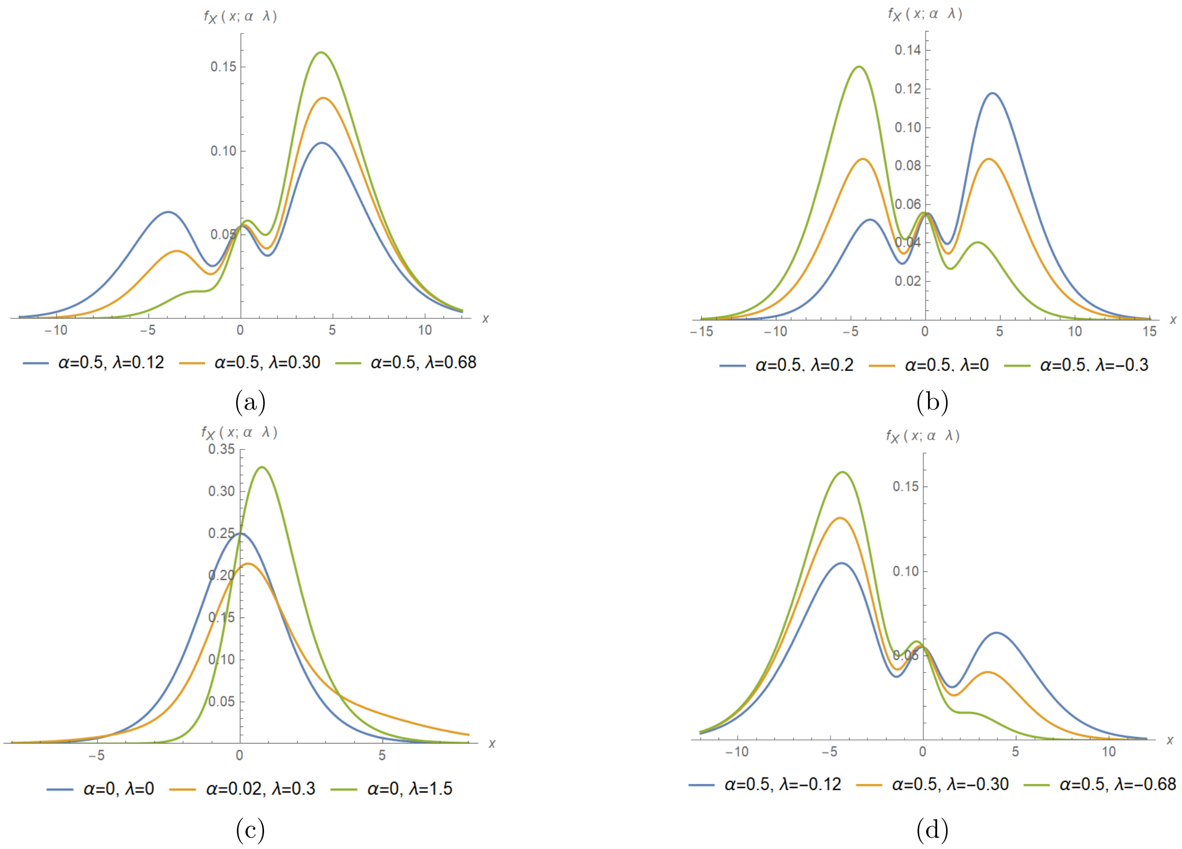


Figure 1: Pdf of $FASL(\alpha,\lambda)$ with different choices of α and λ

Lemma 2.1. *Let $X \sim FASL(\alpha,\lambda)$, then the cdf of X is*

$$F(x)=\begin{cases}\frac{2}{C_1}\sum_{j,k=0}^{\infty}(-1)^{j+k}(k+1)\Bigg[\frac{1}{C_2}\Big[1-\exp(-C_2x)\Big] \\ +\frac{\alpha}{4C_2^5}\Big[24-\exp(-C_2x)\Big(24+C_2x\Big(24+12C_2x+4(C_2x)^2 \\ +(C_2x)^3\Big)\Big)\Big]-\frac{2\alpha}{4C_2^3}\Big[2-\exp(-C_2x)\Big(2+C_2x+(C_2x)^2\Big)\Big] \\ +\frac{3\alpha}{4C_2}\Big[\Big(1-\exp(-C_2x)\Big)\Big]\Bigg], & \text{if} \quad x\le 0, \\ \frac{2}{C_1}\sum_{j,k=0}^{\infty}(-1)^{j+k}(k+1)\Bigg[\frac{\exp(Dx)}{D}+\frac{\alpha}{4D^5} \\ \Big[\exp(Dx)\Big(24+Dx\Big((Dx)^3 \quad -4(Dx)^2+12Dx-24\Big)\Big)\Big] \\ -\frac{2\alpha}{4D^3}\Big[\exp(Dx)\Big((Dx)^2-(Dx)+2\Big)\Big]+\frac{3}{D}\exp(Dx)\Bigg], & \text{if} \quad x>0.\end{cases} \tag{8}$$

Proof. **Case I:** If $x \geq 0$ then cdf $F(x)$ can be written as

$$\begin{aligned}
F(x) &= P(X \leq x) \\
&= \int_0^x \frac{2}{C_1}\left(1+\alpha\left(\frac{(x^2-1)^2+2}{4}\right)\right)\sum_{j,k=0}^{\infty}(-1)^{j+k}(k+1)\exp\left(-C_2x\right)dx \\
&= \frac{2}{C_1}\sum_{j,k=0}^{\infty}(-1)^{j+k}(k+1)\int_0^x\left(1+\alpha\left(\frac{(x^2-1)^2+2}{4}\right)\right)\exp(-C_2x)dx \\
&= \frac{2}{C_1}\sum_{j,k=0}^{\infty}(-1)^{j+k}(k+1)\Bigg[\int_0^x \exp(-C_2x)dx \\
&\quad + \frac{\alpha}{4}\left(\int_0^x x^4\exp(-C_2x)dx - 2\int_0^x x^2\exp(-C_2x)dx + 3\int_0^x \exp(-C_2x)dx\right)\Bigg] \\
&= \frac{2}{C_1}\sum_{j,k=0}^{\infty}(-1)^{j+k}(k+1)\Big[I_1+\frac{\alpha}{4}I_2\Big].
\end{aligned}$$

Here I_1 and I_2 can be calculated using the functions involved in the cdf of trimodal skew-normal distribution (Pathak *et al.* 2023a) as

$$I_1 = \frac{1}{C_2}\Big[1-\exp(-C_2x)\Big],$$

and

$$\begin{aligned}
I_2 = &\frac{1}{C_2^5}\left[24-\exp(-C_2x)\Big(24+C_2x\Big(24+12C_2x+4(C_2x)^2+(C_2x)^3\Big)\Big)\right] \\
&- \frac{2}{C_2^3}\left[2-\exp(-C_2x)\Big(2+C_2x+(C_2x)^2\Big)\right] + \frac{3}{C_2}\left[\Big(1-\exp(-C_2x)\Big)\right].
\end{aligned}$$

Hence the final results of cdf for $X \geq 0$ is obtained as

$$\begin{aligned}
F(x) = &\frac{2}{C_1}\sum_{j,k=0}^{\infty}(-1)^{j+k}(k+1)\Bigg[\frac{1}{C_2}\Big[1-\exp(-C_2x)\Big] \\
&+ \frac{\alpha}{4C_2^5}\Big[24-\exp(-C_2x)\Big(24+C_2x\Big(24+12C_2x+4(C_2x)^2+(C_2x)^3\Big)\Big)\Big] \\
&- \frac{2\alpha}{4C_2^3}\Big[2-\exp(-C_2x)\Big(2+C_2x+(C_2x)^2\Big)\Big] + \frac{3\alpha}{4C_2}\Big[\Big(1-\exp(-C_2x)\Big)\Big]\Bigg].
\end{aligned}$$

Case II: if $x < 0$, the cdf $F(x)$ can be written as

$$\begin{aligned}
F(x) &= P(X \leq x) \\
&= \int_{-\infty}^x \frac{2}{C_1}\left(1+\alpha\left(\frac{(x^2-1)^2+2}{4}\right)\right)\sum_{j,k=0}^{\infty}(-1)^{j+k}(k+1)\exp\left(Dx\right)dx
\end{aligned}$$

The calculation procedure of this case is similar to the Case I and hence the result for Case II is obtained as

$$\begin{aligned}
F(x) = &\frac{2}{C_1}\sum_{j,k=0}^{\infty}(-1)^{j+k}(k+1)\Bigg[\frac{\exp(Dx)}{D} + \frac{\alpha}{4D^5}\Big[\exp(Dx)\Big(24+Dx\Big((Dx)^3 \\
&- 4(Dx)^2+12Dx-24\Big)\Big)\Big] - \frac{2\alpha}{4D^3}\Big[\exp(Dx)\Big((Dx)^2-(Dx)+2\Big)\Big] \\
&+ \frac{3}{D}\exp(Dx)\Bigg].
\end{aligned}$$

□

Lemma 2.2. *Let $X \sim FASL(\alpha, \lambda)$, then the moment generating function (mgf) of X is*

$$M(t) = \frac{2}{C_1} \sum_{j,k=0}^{\infty} (-1)^{j+k}(k+1) \Bigg[\left(\frac{1}{C_2 - t} + \frac{1}{D+t} \right) + \frac{\alpha}{4} \Bigg(3 \left(\frac{1}{(C_2 - t)} + \frac{1}{(D+t)} \right) - 4 \left(\frac{1}{(C_2 - t)^3} + \frac{1}{(D+t)^3} \right) + 24 \left(\frac{1}{(C_2 - t)^5} + \frac{1}{(D+t)^5} \right) \Bigg) \Bigg] \tag{9}$$

Proof. Using the double series expansion of (6), the mgf of FASL distribution can be written as

$$M(t) = E(e^{tx}) = \frac{2}{C_1} \Big[I_3 + I_4 \Big], \tag{10}$$

where,

$$\begin{aligned} I_3 &= \int_0^{\infty} \left(1 + \alpha \left(\frac{(x^2-1)^2+2}{4} \right) \right) \sum_{j,k=0}^{\infty} (-1)^{j+k}(k+1) \exp\Big((t - C_2)\, x \Big) dx \\ &= \sum_{j,k=0}^{\infty} (-1)^{j+k}(k+1) \Bigg[\int_0^{\infty} \exp\Big((t - C_2)\, x \Big) dx \\ &\qquad + \frac{\alpha}{4} \int_0^{\infty} \Big((x^2-1)^2 + 2 \Big) \exp\Big((t - C_2)\, x \Big) dx \Bigg] \\ &= \sum_{j,k=0}^{\infty} (-1)^{j+k}(k+1) \Big[I_3^1 + I_3^2 \Big]. \end{aligned} \tag{11}$$

The integrals I_3^1 and I_3^2 can be calculated by using Equations 3.351.1 and 3.326.2 in Gradshteyn and Ryzhik (2000) and putting in (11), I_3 can be obtained as

$$I_3 = \sum_{j,k=0}^{\infty} (-1)^{j+k}(k+1) \left[\frac{1}{(C_2 - t)} + \frac{\alpha}{4} \left(\frac{3}{(C_2 - t)} - \frac{4}{(C_2 - t)^3} + \frac{24}{(C_2 - t)^5} \right) \right]. \tag{12}$$

Analogously,

$$I_4 = \int_{-\infty}^{0} \left(1 + \alpha \left(\frac{(x^2-1)^2+2}{4} \right) \right) \sum_{j,k=0}^{\infty} (-1)^{j+k}(k+1) \exp\Big((t + D)\, x \Big) dx. \tag{13}$$

Proceeding similarly to the calculation involved in I_3, the results of the integration in I_4 can also be computed, and the final results can be written as

$$I_4 = \sum_{j,k=0}^{\infty} (-1)^{j+k}(k+1) \left[\frac{1}{(D+t)} + \frac{\alpha}{4} \left(\frac{3}{(D+t)} - \frac{4}{(D+t)^3} + \frac{24}{(D+t)^5} \right) \right] \tag{14}$$

Substituting the results of (12) and (14) in (10), the mgf of the FASL distribution can be obtained as

$$M(t) = \frac{2}{C_1} \sum_{j,k=0}^{\infty} (-1)^{j+k}(k+1) \Bigg[\left(\frac{1}{C_2 - t} + \frac{1}{D+t} \right) + \frac{\alpha}{4} \Bigg(3 \left(\frac{1}{(C_2 - t)} + \frac{1}{(D+t)} \right) - 4 \left(\frac{1}{(C_2 - t)^3} + \frac{1}{(D+t)^3} \right) + 24 \left(\frac{1}{(C_2 - t)^5} + \frac{1}{(D+t)^5} \right) \Bigg) \Bigg].$$

□

Corollary 2.2.1. *Replacing t by it in (9), the characteristic function of a $FASL(\lambda,\alpha)$ random variable can be calculated as*

$$K(t)=\frac{2}{C_1}\sum_{j,k=0}^{\infty}(-1)^{j+k}(k+1)\Bigg[\Bigg(\frac{1}{C_2-it}+\frac{1}{D+it}\Bigg)$$
$$+\frac{\alpha}{4}\Bigg(3\Bigg(\frac{1}{(C_2-it)}+\frac{1}{(D+it)}\Bigg)-4\Bigg(\frac{1}{(C_2-it)^3}+\frac{1}{(D+it)^3}\Bigg)$$
$$+24\Bigg(\frac{1}{(C_2-it)^5}+\frac{1}{(D+it)^5}\Bigg)\Bigg)\Bigg]. \qquad (15)$$

Lemma 2.3. *The nth order moment of a $FASL(\alpha,\lambda)$ random variable can be obtained as*

$$E(X^n)=\frac{1}{C_1}\Bigg[2n!\left(1-2^{1-n}\right)\zeta(n)+\frac{\alpha}{2}\Big[2(n+4)!\left(1-2^{1-(n+4)}\right)\zeta(n+4)$$
$$-2(2n+4)!\left(1-2^{1-(n+2)}\right)\zeta(n+2)+3(2n)!\left(1-2^{1-(n)}\right)\zeta(n)\Big]\Bigg]. \qquad (16)$$

if n is even, and

$$E(X^n)=\frac{1}{C_1}\Bigg[2n!\Bigg\{\Big(1-2^{1-n}\Big)\zeta(n)+\frac{1}{2^n\lambda^{n+1}}\sum_{j=0}^{\infty}(-1)^j(j+1)\xi(j,n+1)\Bigg\}+\frac{\alpha}{2}\Bigg\{2(n+4)!$$
$$\Bigg(\Big(1-2^{1-(n+4)}\Big)\zeta(n+4)+\frac{1}{2^{n+4}\lambda^{n+5}}\sum_{j=0}^{\infty}(-1)^j(j+1)\xi(j,n+5)\Bigg)-4(n+2)!$$
$$\Bigg(\Big(1-2^{1-(n+2)}\Big)\zeta(n+2)+\frac{1}{2^{n+2}\lambda^{n+3}}\sum_{j=0}^{\infty}(-1)^j(j+1)\xi(j,n+3)\Bigg)+6n!$$
$$\Bigg(\Big(1-2^{1-(n)}\Big)\zeta(n)+\frac{1}{2^{n}\lambda^{n+1}}\sum_{j=0}^{\infty}(-1)^j(j+1)\xi(j,n+1)\Bigg)\Bigg\}\Bigg] \qquad (17)$$

if n is odd, where $\zeta(a)=\sum_{j=0}^{\infty}j^{-a}$ is the Riemann's zeta function (Contreras-Reyes 2021), $\zeta(a,q)=\sum_{j=0}^{\infty}(q+j)^{-a}$, and $\xi(j,k)=\zeta\left(k,\frac{1+2\lambda+j}{2\lambda}\right)-\zeta\left(k,\frac{1+\lambda+j}{2\lambda}\right)$.

Proof.

$$E(X^n)=\int_{-\infty}^{\infty}x^n f(x)dx$$
$$=\int_{-\infty}^{\infty}x^n\frac{2\left(1+\alpha\left(\frac{(x^2-1)^2+2}{4}\right)\right)\exp(-x)}{C_1\Big(1+\exp(-x)\Big)^2\Big(1+\exp(-\lambda x)\Big)}dx$$
$$=\frac{1}{C_1}\Bigg[\int_{-\infty}^{\infty}\frac{x^n 2\exp(-x)}{\Big(1+\exp(-x)\Big)^2\Big(1+\exp(-\lambda x)\Big)}dx$$
$$+\frac{\alpha}{2}\int_{-\infty}^{\infty}\frac{x^n\left((x^2-1)^2+2\right)\exp(-x)}{\Big(1+\exp(-x)\Big)^2\Big(1+\exp(-\lambda x)\Big)}dx\Bigg]$$
$$=\frac{1}{C_1}\Big[I_4+I_5\Big],$$

where now, I_4 is the rth order moment of SL distribution given by Nadarajah (2009), while I_5 can be calculated using the rth order moment of tri-modal SL distribution given by Pathak *et al.* (2023a). Hence, the expression for the even and odd order moments of $FASL(\alpha, \lambda)$ random variable is

$$E(X^n) = \frac{1}{C_1}\bigg[2n!\left(1 - 2^{1-n}\right)\zeta(n) + \frac{\alpha}{2}\Big[2(n+4)!\left(1 - 2^{1-(n+4)}\right)\zeta(n+4) - 2(2n+4)!\left(1 - 2^{1-(n+2)}\right)\zeta(n+2) + 3(2n)!\left(1 - 2^{1-(n)}\right)\zeta(n)\Big]\bigg]$$

if n is even, and

$$E(X^n) = \frac{1}{C_1}\Bigg[2n!\left\{\left(1 - 2^{1-n}\right)\zeta(n) + \frac{1}{2^n\lambda^{n+1}}\sum_{j=0}^{\infty}(-1)^j(j+1)\xi(j, n+1)\right\} + \frac{\alpha}{2}\Bigg\{2(n+4)! \left(\left(1 - 2^{1-(n+4)}\right)\zeta(n+4) + \frac{1}{2^{n+4}\lambda^{n+5}}\sum_{j=0}^{\infty}(-1)^j(j+1)\xi(j, n+5)\right) - 4(n+2)! \left(\left(1 - 2^{1-(n+2)}\right)\zeta(n+2) + \frac{1}{2^{n+2}\lambda^{n+3}}\sum_{j=0}^{\infty}(-1)^j(j+1)\xi(j, n+3)\right) + 6n! \left(\left(1 - 2^{1-n}\right)\zeta(n) + \frac{1}{2^n\lambda^{n+1}}\sum_{j=0}^{\infty}(-1)^j(j+1)\xi(j, n+1)\right)\Bigg\}\Bigg]$$

if n is odd. □

Corollary 2.3.1. *From equations (16) and (17), the first four order moments of a $FASL(\alpha, \lambda)$ random variable can be obtained as*

$$E(X) = \frac{1}{C_1}\left[\frac{1}{\lambda^2}\sum_{j=0}^{\infty}(-1)^j(j+1)\xi(j,2) + \frac{\alpha}{2}\left[\frac{15\beta^5}{2}\left\{-\frac{5}{4}\psi'''(1) + \frac{1}{\lambda^6}\sum_{j=0}^{\infty}(-1)^j(j+1)\xi(j,6)\right\} - 3\beta^3\left\{-3\psi''(1) + \frac{1}{\lambda^4}\sum_{j=0}^{\infty}(-1)^j(j+1)\xi(j,4)\right\} + 3\frac{\beta}{\lambda^2}\sum_{j=0}^{\infty}(-1)^j(j+1)\xi(j,2)\right]\right],$$

$$E(X^2) = \frac{\pi^2}{3C_1}\left[1 + \frac{\alpha}{2}\left(\frac{115\pi^4 - 98\pi^2 + 105}{35}\right)\right],$$

$$E(X^3) = \frac{1}{C_1}\left[-\frac{9\psi''(1)}{2} + \frac{3}{2\lambda^4}\sum_{j=0}^{\infty}(-1)^j(j+1)\xi(j,4) + \frac{\alpha}{2}\left[\frac{63\beta^5}{4}\left\{-\frac{7}{8}\psi'''(1) + \frac{5}{\lambda^8}\sum_{j=0}^{\infty}(-1)^j(j+1)\xi(j,8)\right\} - 15\beta^5\left\{-\frac{15}{4}\psi''(1) + \frac{1}{\lambda^6}\sum_{j=0}^{\infty}(-1)^j(j+1)\xi(j,6)\right\} + \frac{9}{2}\beta^3\left\{-3\psi''(1) + \frac{1}{\lambda^4}\sum_{j=0}^{\infty}(-1)^j(j+1)\xi(j,4)\right\}\right]\right],$$

$$E(X^4) = \frac{\pi^4}{3C_1}\left[\frac{7}{5} + \frac{889\pi^4 - 310\pi^2 + 147}{35}\right].$$

The expression of mean and variance as well as skewness and kurtosis coefficients of the $FASL(\alpha, \lambda)$ distribution contains very complex mathematical expressions. Therefore, numerical values for the mean ($E(X)$) and variance ($V(X) = E(X^2) - E(X)^2$) of $FASL(\alpha, \lambda)$ distribution for particular values of the parameters are listed in Table 1. Similarly, numerical values of the skewness ($\beta_1 = \mu_3^2/\mu_2^3$) and kurtosis ($\beta_2 = \mu_4/\mu_2^2$) coefficients are presented in Table 2.

The following Lemma provides the Rényi entropy (see e.g. Contreras-Reyes 2022) of a $FASL(\alpha, \lambda)$ random variable. The entropy of a random variable is a measure of the uncertainty or randomness associated with that variable.

Table 1: Mean and Variance of a $FASL(\alpha, \lambda)$ random variable for different parameter values

$\lambda \rightarrow$	-2		-1		2		1	
$\alpha \downarrow$	$E(X)$	$Var(X)$	$E(X)$	$Var(X)$	$E(X)$	$Var(X)$	$E(X)$	$Var(X)$
0.5	-4.513	6.989	-4.382	8.157	4.513	6.989	4.382	8.157
1.5	-4.906	6.173	-4.787	7.326	4.906	6.173	4.787	7.326
2.0	-4.961	6.031	-4.844	7.182	4.961	6.031	4.844	7.182
2.5	-4.996	5.942	-4.879	7.089	4.996	5.942	4.879	7.089
3.0	-5.019	5.879	-4.903	7.026	5.019	5.879	4.903	7.026

Table 2: Skewness (β_1) and kurtosis (β_2) of a $FASL(\alpha, \lambda)$ random variable for different parameter values

$\lambda \rightarrow$	-2		-1		2		1	
$\alpha \downarrow$	β_1	β_2	β_1	β_2	β_1	β_2	β_1	β_2
0.5	1.830	0.225	1.805	0.321	1.830	0.225	1.805	0.321
1.5	1.679	0.159	1.657	0.249	1.679	0.159	1.657	0.249
2.0	1.659	0.152	1.638	0.241	1.659	0.152	1.638	0.241
2.5	1.648	0.146	1.627	0.235	1.648	0.146	1.627	0.235
3.0	1.640	0.151	1.619	0.239	1.640	0.151	1.619	0.239

Definition 2. For a random variable X, the Rényi entropy of order γ (Rényi 1961) is defined as

$$R_\gamma(X) = \frac{1}{1-\gamma} \log \int f(x)^\gamma dx, \tag{18}$$

where $\gamma > 1$ and $\gamma \neq 1$.

Shannon entropy is a special case of Rényi one an is obtained in the limit $\lim_{\gamma \to 1} R_\gamma(X)$ (Contreras-Reyes 2022).

Lemma 2.4. *The Rényi entropy of a $FASL(\alpha, \lambda)$ random variable is*

$$R_\gamma(X) = \frac{1}{1-\gamma} \Bigg[\log C_6 + \gamma \log 2 - \gamma \log C_1 + \log \sum_{j,k=0}^{\infty} \binom{-\gamma}{j} \binom{-2\gamma}{k} \Big\{ (\gamma + j\lambda + k)^{-4p+q-1} \\ \Gamma(4p - q + 1) + (-1)^{4p-q} (\gamma + j\lambda + k + \lambda)^{-4p+q-1} \Gamma(4p - q + 1) \Big\} \Bigg] \tag{19}$$

where $C_6 = (-1)^q 2^{m-p} \left(\frac{\alpha}{4}\right)^m \sum_{m=0}^{\gamma} \sum_{p=0}^{m} \sum_{q=0}^{2p} \binom{\gamma}{m} \binom{m}{p} \binom{2p}{q}$.

Proof. By definition,

$$\int_{-\infty}^{\infty} f(x)^\gamma dx = \frac{2^\gamma}{C_1^\gamma} \int_{-\infty}^{\infty} \left\{ 1 + \alpha \left(\frac{(x^2-1)^2 + 2}{4} \right) \right\}^\gamma \frac{\exp(-x)^\gamma}{\Big(1 + \exp(-x)\Big)^{2\gamma} \Big(1 + \exp(-\lambda x)\Big)^\gamma} dx, \tag{20}$$

Now, using the binomial expansion, the polynomial involved in the calculation of entropy can be expressed as

$$\left\{1+\alpha\left(\frac{(x^2-1)^2+1}{4}\right)\right\}^{\gamma} = \frac{\alpha^m}{4^m}\sum_{m=0}^{\gamma}\binom{\gamma}{m}\left((x^2-1)^2+2\right)^m$$
$$= 2^{m-p}\left(\frac{\alpha}{4}\right)^m\sum_{m=0}^{\gamma}\sum_{p=0}^{m}\binom{\gamma}{m}\binom{m}{p}(x^2-1)^{2p}$$
$$= (-1)^q 2^{m-p}\left(\frac{\alpha}{4}\right)^m\sum_{m=0}^{\gamma}\sum_{p=0}^{m}\sum_{q=0}^{2p}\binom{\gamma}{m}\binom{m}{p}\binom{2p}{q}x^{4p-q}$$

Hence,

$$\left\{1+\alpha\left(\frac{(x^2-1)^2+1}{4}\right)\right\}^{\gamma} = C_6 x^{4p-q}$$

where

$$C_6 = (-1)^q 2^{m-p}\left(\frac{\alpha}{4}\right)^m\sum_{m=0}^{\gamma}\sum_{p=0}^{m}\sum_{q=0}^{2p}\binom{\gamma}{m}\binom{m}{p}\binom{2p}{q}$$

Now, from (20) and using the Taylor series expansion, we get

$$\int_{-\infty}^{\infty} f(x)^{\gamma}dx = C_6\left(\frac{2}{C_1}\right)^{\gamma}\int_{-\infty}^{\infty} x^{4p-q}\frac{\exp(-x)^{\gamma}}{\Big(1+\exp(-x)\Big)^{2\gamma}\Big(1+\exp(-\lambda x)\Big)^{\gamma}}dx$$
$$= C_6\left(\frac{2}{C_1}\right)^{\gamma}\sum_{j,k=0}^{\infty}\binom{-\gamma}{j}\binom{-2\gamma}{k}\Bigg[\int_0^{\infty} x^{4p-q}e^{-x(\gamma+\lambda j+k)}$$
$$+\int_{-\infty}^{0} x^{4p-q}e^{x(\gamma+\lambda j+k+\lambda)}\Bigg]$$
$$= C_6\left(\frac{2}{C_1}\right)^{\gamma}\sum_{j,k=0}^{\infty}\binom{-\gamma}{j}\binom{-2\gamma}{k}\Big[(\gamma+j\lambda+k)^{-4p+q-1}\Gamma(4p-q+1)$$
$$+(-1)^{4p-q}(\gamma+j\lambda+k+\lambda)^{-4p+q-1}\Gamma(4p-q+1)\Big]$$

Now, the final expression of Rényi entropy is obtained as

$$R_{\gamma}(X) = \frac{1}{1-\gamma}\Bigg[\log C_6+\gamma\log 2-\gamma\log C_1+\log\sum_{j,k=0}^{\infty}\binom{-\gamma}{j}\binom{-2\gamma}{k}\Big\{(\gamma+j\lambda+k)^{-4p+q-1}$$
$$\Gamma(4p-q+1)+(-1)^{4p-q}(\gamma+j\lambda+k+\lambda)^{-4p+q-1}\Gamma(4p-q+1)\Big\}\Bigg].$$

□

Lemma 2.5. *If $X \sim FASL(\alpha,\lambda)$, then $Y = |X| \sim 2Z$, with $Z \sim FAL(\alpha)$.*

Proof. The pdf $f(y)$ of $Y = |X|$ can be written as

$$f(y) = f(x;\alpha,\lambda)+f(-x;\alpha,\lambda)$$
$$= \left[\frac{2\left(1+\alpha\left(\frac{(x^2-1)^2+2}{4}\right)\right)\exp(-x)}{C_1\Big(1+\exp(-x)\Big)^2\Big(1+\exp(-\lambda x)\Big)}\right]+\left[\frac{2\left(1+\alpha\left(\frac{(x^2-1)^2+2}{4}\right)\right)\exp(x)}{C_1\Big(1+\exp(x)\Big)^2\Big(1+\exp(\lambda x)\Big)}\right]$$
$$= \frac{2}{C_1}\left[1+\alpha\left(\frac{(x^2-1)^2+2}{4}\right)\right]\frac{\exp(x)}{(1+\exp(x))^2}$$
$$= \frac{4}{C(\alpha)}\left[1+\alpha\left(\frac{(x^2-1)^2+2}{4}\right)\right]\frac{\exp(x)}{(1+\exp(x))^2} = 2z.$$

□

Lemma 2.6. *Using the graphical method of Behboodian (1970), the* $FASL(\alpha,\lambda)$ *pdf has at most three modes.*

Proof. Let $X \sim FASL(\alpha,\lambda)$. From (4) and (7), we get

$$f(x;\alpha,\lambda) = \frac{C(\alpha)}{C_1} f(x,\alpha)G(\lambda x), \tag{21}$$

where $G(\cdot)$ is the cdf of the standard logistic distribution. Then,

$$f'(x;\alpha,\lambda) = \frac{C(\alpha)}{C_1}\left[f'(x;\alpha)G(\lambda x) + \lambda f(x;\alpha)g(\lambda x)\right] \tag{22}$$

where $g(\cdot)$ is the pdf of the standard logistic distribution. Now to show that (21) has at most three modes, it should be proved that (22) has one or five answers. For this, the graphical method has been applied here. Therefore,

$$f'(x;\alpha,\lambda) = F_1(x) - F_2(x), \tag{23}$$

where

$$F_1(x) = \frac{C(\alpha)}{C_1} f'(x;\alpha)G(\lambda x), \tag{24}$$

and

$$F_2(x) = -\frac{C(\alpha)}{C_1}\lambda f(x;\alpha)g(\lambda x). \tag{25}$$

Now, setting, $f'(x;\alpha,\lambda) = 0$, it has been obtained that

$$F_1(x) = F_2(x). \tag{26}$$

□

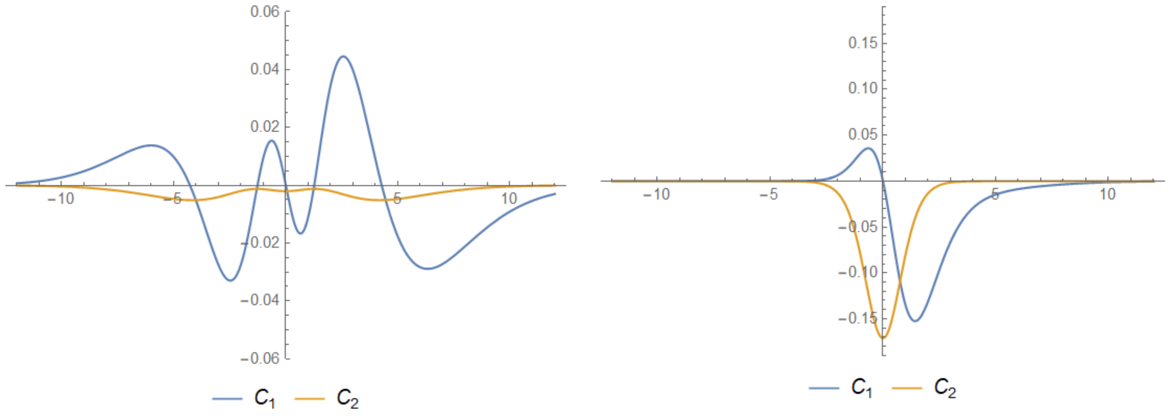
From the Figure 2 it can be seen that, since pdf (4) vanishes out of $(-11, 11)$, the curves C1 ($y = F_1(x)$) and C2 ($y = F_2(x)$) have been drawn for $(\alpha,\lambda) = (1.3, 0.12)$ and $(\alpha,\lambda) = (0.01, 1.5)$, respectively; with $x \in (-11, 11)$. It can be also seen that these two curves have at least one and at most five intersection points, where the values of x of these points are the roots of (26). Given that $\lim_{x\to\pm\infty} f(x;\alpha,\lambda) = 0$, if (26) possesses one answer then it should be the mode of (21) and, if (26) possesses five answers, then (21) should have three modes. Hence, the $FASL(\alpha,\lambda)$ pdf has between one and three modes.

3. Characterizations results

This section considers the characterizations of the FASL distribution via two truncated moments and the conditional distribution.

3.1. Characterization based on two truncated moments

This subsection deals with the characterizations of FASL distribution based on a relationship between two truncated moments. For the characterization based on truncated moments, a Theorem of Glänzel (1987) in the next Lemma. Clearly, the result holds when the H is not a closed interval. This characterization is stable in the sense of weak convergence, see Glanzel (1990).



(a) (b)

Figure 2: (a) Plots of $C1$ and $C2$ for $(\alpha, \lambda) = (1.3, 0.12)$; (b) Plots of $C1$ and $C2$ for $(\alpha, \lambda) = (0.01, 1.5)$

Lemma 3.1. *Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a given probability space and let $H = [d, e]$ be an interval for some $d < e$ ($d = -\infty$ and $e = \infty$ are allowed). Let $X : \Omega \to H$ be a continuous random variable with the distribution function F and let k and h be two real functions defined on H such that*

$$E\left[k\left(X\right) \mid X \geq x\right] = E\left[h\left(X\right) \mid X \geq x\right] \eta\left(x\right), \quad x \in H,$$

is defined with some real function η. Assume that $k, h \in C^1(H)$, $\eta \in C^2(H)$ and F is twice continuously differentiable and strictly monotone function on the set H. Finally, assume that $\eta h = k$ has no real solution in the interior of H. Then F is uniquely determined by the functions k, h and η, particularly

$$F\left(x\right) = \int_a^x C \left| \frac{\eta'\left(u\right)}{\eta\left(u\right) h\left(u\right) - k\left(u\right)} \right| \exp\left(-s\left(u\right)\right)\, du,$$

where the function s is a solution of the differential equation $s' = \dfrac{\eta' h}{\eta h - k}$ and C is the normalization constant, such that $\int_H dF = 1$.

Proposition 1. *Let $X : \Omega \to \mathbb{R}$ be a continuous random variable, and let*

$$h(x) = \frac{1 + e^{-\lambda x}}{1 + \alpha\left(\dfrac{(x^2-1)^2 + 2}{4}\right)}$$

and $k(x) = h(x)(1 + e^{-x})^{-1}$, for $x \in \mathbb{R}$. Then, the density of X is given in (4) if and only if the function η defined in Lemma 3.1 is

$$\eta(x) = \frac{1}{2}\left\{1 + \left(1 + e^{-x}\right)^{-1}\right\}, \quad x \in \mathbb{R}.$$

Proof. If X has pdf given in (4), then

$$\left(1 - F\left(x\right)\right) E\left[h\left(X\right) \mid X \geq x\right] = \frac{2}{C_1}\left\{1 - \left(1 + e^{-x}\right)^{-1}\right\}, \quad x \in \mathbb{R},$$

and

$$\left(1 - F\left(x\right)\right) E\left[k\left(X\right) \mid X \geq x\right] = \frac{1}{C_1}\left\{1 - \left(1 + e^{-x}\right)^{-2}\right\}, \quad x \in \mathbb{R}.$$

Hence

$$\eta(x) = \frac{\frac{1}{C_1}\left\{1-(1+e^{-x})^{-2}\right\}}{\frac{2}{C_1}\left\{1-(1+e^{-})^{-1}\right\}} = \frac{1}{2}\left\{1+(1+e^{-x})^{-1}\right\}.$$

Finally

$$\eta\left(x\right)h\left(x\right)-k\left(x\right) = \frac{1}{2}h\left(x\right)\left\{1-\left(1+e^{-x}\right)^{-1}\right\} > 0, \quad \text{for } x \in \mathbb{R}.$$

Conversely, if η has the above form, then

$$s'\left(x\right) = \frac{\eta'\left(x\right)h\left(x\right)}{\eta\left(x\right)h\left(x\right)-k\left(x\right)} = \frac{e^{-x}\left(1+e^{-x}\right)^{-2}}{1-\left(1+e^{-x}\right)^{-1}}.$$

Hence

$$s\left(x\right) = -\log\left\{1-\left(1+e^{-x}\right)^{-1}\right\}, \quad x \in \mathbb{R}.$$

Based on Lemma 3.1, X has pdf given in (4). □

Corollary 3.1.1. *If $X : \Omega \to \mathbb{R}$ is a continuous random variable and $h(x)$ is as in Proposition 1. Then, X has pdf given in (4) if and only if there exist functions k and η defined in Lemma 3.1 satisfying the following first order differential equation*

$$\frac{\eta'\left(x\right)h\left(x\right)}{\eta\left(x\right)h\left(x\right)-k\left(x\right)} = \frac{e^{-x}\left(1+e^{-x}\right)^{-2}}{1-\left(1+e^{-x}\right)^{-1}}.$$

Corollary 3.1.2. *The general solution of the above differential equation is*

$$\eta(x) = \left[1-\left(1+e^{-x}\right)^{-1}\right]\left[-\int e^{-x}\left(1+e^{-x}\right)^{-2}\left(h\left(x\right)\right)^{-1}k\left(x\right)+D\right],$$

where D is a constant. A set of functions satisfying this differential equation is presented in Proposition 1 with $D = 1/2$.

Clearly, there are other set (h, k, ξ) satisfying the conditions of Lemma 3.1 of which one is given in the following Proposition.

Proposition 2. *Let the random variable $X : \Omega \to \mathbb{R}$ be continuous, and let*

$$h(x) = \frac{(1+e^{-x})^2(1+e^{-\lambda x})}{1+\alpha\left(\frac{(x^2-1)^2+2}{4}\right)}$$

and $k(x) = h(x)e^{-x}$, for $x \in \mathbb{R}$. Then, the density of X is given in (4) if and only if the function η defined in Lemma 3.1 is

$$\eta(x) = \frac{e^{-x}}{2}, \quad x \in \mathbb{R}.$$

3.2. Characterization based on conditional distribution

Characterization results of FASL distribution based on conditional distribution is discussed in this subsection.

Lemma 3.2. *If $W \sim FAL(\alpha)$ and $Z \sim Logistic(0, 1)$ are independent. Then $W|\{\lambda W > Z\} \sim FASL(\alpha, \lambda)$.*

Proof. Let $X = W|\{\lambda W > Z\}$. Then

$$\begin{aligned} P(X \le x) &= P(W \le x|\lambda W > Z) \\ &= \frac{P(W \le x, \lambda W > Z)}{P(\lambda W > Z)}. \end{aligned}$$

Now, using the pdf given in (7), we have

$$P(W \le x, \lambda W > Z) = \int_{-\infty}^{x} \frac{1+\alpha\left(\dfrac{(x^2-1)^2+2}{4}\right)}{C(\alpha)} g(x)G(\lambda x)dx$$

and

$$P(\lambda W > Z) = \int_{-\infty}^{\infty} \frac{1+\alpha\left(\dfrac{(x^2-1)^2+2}{4}\right)}{C(\alpha)} g(x)G(\lambda x)dx = \frac{C_1}{C(\alpha)}.$$

where $g(\cdot)$ and $G(\cdot)$ are the pdf and cdf of standard logistic distribution, respectively. Therefore,

$$\begin{aligned} P(X \le x) &= \frac{\int_{-\infty}^{x} \frac{1+\alpha\left(\dfrac{(x^2-1)^2+2}{4}\right)}{C(\alpha)} g(x)G(\lambda x)dx}{\dfrac{1}{C(\alpha)}C_1} \\ &= \int_{-\infty}^{x} \frac{1+\alpha\left(\dfrac{(x^2-1)^2+2}{4}\right)}{C_1} g(x)G(\lambda x)dx \end{aligned}$$

and then the density function of $X = W|\{\lambda W > Z\}$ is

$$f_X(x) = \frac{1+\alpha\left(\dfrac{(x^2-1)^2+2}{4}\right)}{C_1} g(x)G(\lambda x)$$

Finally,

$$W|\{\lambda W > Z\} \sim FASL(\alpha, \lambda).$$

□

4. Maximum likelihood estimation

Considering the location parameter μ and scale parameter σ, a location scale generalized FASL distribution is proposed in this section using the transformation $Y = \mu + \sigma x$. Then the pdf of the location scale generalized FASL distribution is given by

$$f(y;\mu,\sigma,\alpha,\lambda) = \frac{2\left\{1+\alpha\left(\dfrac{\left(\left(\frac{y-\mu}{\sigma}\right)^2-1\right)^2+2}{4}\right)\right\}\exp\left(-\frac{y-\mu}{\sigma}\right)}{\sigma C_1\left\{\exp\left(-\frac{y-\mu}{\sigma}\right)+1\right\}^2\left\{\exp\left(-\frac{\lambda(y-\mu)}{\sigma}\right)+1\right\}} \tag{27}$$

where $y \in \mathbb{R}$, $\mu \in \mathbb{R}$, $\lambda \in \mathbb{R}$, $\sigma > 0$, and $\alpha \geq 0$. If Y has the pdf expressed in (27), then it is denoted as $Y \sim EFASL(\mu, \sigma, \alpha, \lambda)$.

Next, the estimation method of parameter vector $\theta = (\mu, \sigma, \alpha, \lambda)$ for $EFASL(\mu, \sigma, \alpha, \lambda)$ pdf is discussed. Let $y_1, y_2, y_3, \ldots, y_n$ be a set of n independently and identically distributed random variables drawn from the $EFASL(\mu, \sigma, \alpha, \lambda)$ random variable, then the log-likelihood equations of the set of three parameters is obtained as

$$\begin{aligned} l(\theta) &= n\log 2 + \sum_{i=1}^{n} \log\left\{1 + \alpha\left(\frac{\left(\left(\left(\frac{y_i-\mu}{\sigma}\right)^2 - 1\right)^2 + 2\right)}{4}\right)\right\} - \frac{1}{\sigma}\sum_{i=1}^{n}(y_i - \mu) \\ &\quad - n\log\sigma - n\log\left(1 + \frac{\alpha}{60}\left(45 - 10\pi^2 + 7\pi^4\right)\right) - 2\sum_{i=1}^{n}\log\left(\left\{\exp\left(-\frac{y_i-\mu}{\beta}\right) + 1\right\}\right) \\ &\quad - \sum_{i=1}^{n}\log\left(\left\{\exp\left(-\frac{\lambda\left(y_i-\mu\right)}{\beta}\right) + 1\right\}\right) \qquad (28) \end{aligned}$$

Differentiating the equation (28) with respect to the set of parameters, one can have the likelihood equations given as

$$\begin{aligned} \frac{\partial l(\theta)}{\partial \mu} &= \frac{n}{\sigma} - \frac{2}{\sigma}\sum_{i=1}^{n}\frac{1}{A_1} - \frac{\lambda}{\sigma}\sum_{i=1}^{n}\frac{1}{A_2} - \sum_{i=1}^{n}\frac{\alpha(y_i-\mu)(y_i-\mu+\sigma)(y_i-\mu-\sigma)}{\sigma^4 A_4} \\ \frac{\partial l(\theta)}{\partial \sigma} &= -\frac{n}{\sigma} - \frac{1}{\sigma^2}\sum_{i=1}^{n}(y_i-\mu) - \frac{2}{\sigma^2}\sum_{i=1}^{n}\frac{\exp\left(\frac{\mu}{\sigma}\right)(y_i-\mu)}{\exp\left(\frac{\mu}{\sigma}\right) + \exp\left(\frac{y_i}{\sigma}\right)} - \frac{\lambda}{\sigma^2}\sum_{i=1}^{n}\frac{y_i-\mu}{A_1} \\ &\quad + \sum_{i=1}^{n}\frac{\alpha\left(\frac{y_i-\mu}{\sigma}\right)^2 A_1}{A_4} \\ \frac{\partial l(\theta)}{\partial \alpha} &= -\frac{n\left(45 - 10\pi^2 + 7\pi^4\right)}{60\left(1 + \frac{1}{60}\left(45 - 10\pi^2 + 7\pi^4\right)\right)} + \frac{1}{4A_4}\sum_{i=1}^{n} 2 + \left(1 - \left(\frac{y_i-\mu}{\sigma}\right)^2\right)^2 \\ \frac{\partial l(\theta)}{\partial \lambda} &= -\frac{y_i-\mu}{\sigma}\frac{1}{A_2}, \end{aligned}$$

where

$$\begin{aligned} A_1 &= 1 + \exp\left(\frac{y_i-\mu}{\sigma}\right), \\ A_2 &= 1 + \exp\left(\lambda\frac{y_i-\mu}{\sigma}\right) \\ A_3 &= \frac{(\mu+\sigma-y_i)(y_i+\sigma-\mu)}{\sigma^3}, \\ A_4 &= 1 + \frac{\alpha}{2}\left[2\left(1 - \left(\frac{y_i-\mu}{\sigma}\right)^2\right)^2\right]. \end{aligned}$$

Now, by solving simultaneously the above equations, one can obtain the estimates. However, the direct solution of the above normal equations is not explicit and thus numerical procedure could be implemented using the `GenSA` package of `R` software.

5. Simulation study

A simulation study is conducted to evaluate the efficacy of maximum likelihood estimates for the parameters of new distribution. The Metropolis–Hastings (MH) algorithm (see e.g.

Contreras-Reyes, Quintero, and Wiff 2018) is employed to generate a set of random numbers which involves the replication of the process 10,000 times, incorporating three distinct sample sizes ($n = 100$, 300, and 500). The algorithm(s) for generating the random samples are given in Appendix C. Subsequently, the maximum likelihood estimates for each generated sample are estimated using the `GenSA` package in the `R` software. Finally, the estimated statistics are presented in terms of biases and mean square errors (MSEs) of the estimates:

$$Bias(\hat{\theta}) = E(\hat{\theta}) - \theta$$
$$MSE(\hat{\theta}) = V(\hat{\theta}) + Bias(\hat{\theta})^2,$$

where $\hat{\theta} = (\hat{\mu}, \hat{\sigma}, \hat{\alpha}, \hat{\lambda})$.

From Tables 3 and 4, it is observed that the Maximum Likelihood Estimates (MLEs) effectively estimate the model parameters. Moreover, the results also shows that when sample size increases, the bias and mean-square error of the MLEs decrease, indicating the asymptotic consistency of the MLEs of $FASL(\mu, \sigma, \alpha, \lambda)$ pdf.

Table 3: Results of simulation

			$\mu = 0, \quad \sigma = 1$							
			μ		σ		λ		α	
α	λ	n	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
	-2	100	-0.0983	0.1026	0.0879	0.0981	-0.1098	0.0864	0.0900	0.1204
		300	0.0756	0.0845	0.0654	0.0885	-0.0564	0.0847	-0.0794	0.0685
		500	0.0354	0.0369	-0.0441	0.0542	0.0487	0.0689	0.0722	0.0698
	-1	100	0.0854	0.0956	0.0954	0.0599	0.0810	0.0656	0.0753	0.0545
		300	-0.0663	0.0583	-0.0813	0.0652	-0.8210	0.0521	0.0517	0.0506
		500	-0.0507	0.0643	-0.0211	0.0451	-0.0421	0.0440	-0.0207	0.0444
0.5	0	100	0.1025	0.0745	-0.0540	0.0500	0.0787	0.0821	0.0858	0.0722
		300	-0.0677	0.0663	-0.0439	0.0496	-0.0553	0.0753	0.0921	0.0653
		500	0.0531	0.0541	0.0223	0.0339	-0.0449	0.0702	0.0340	0.0555
	1	100	0.0959	0.0753	0.0797	0.0653	0.0818	0.1394	0.0954	0.0686
		300	0.0756	0.0721	-0.0810	0.0430	0.0653	0.0631	-0.0653	0.0589
		500	-0.0453	0.0652	0.0352	0.0392	0.0398	0.0703	-0.0400	0.0440
	2	100	0.0954	0.2006	0.0438	0.0641	0.0959	0.0593	0.0821	0.2646
		300	0.0777	0.0975	-0.0721	0.0606	0.0721	0.0606	0.0653	0.0451
		500	-0.0309	0.0439	0.0550	0.0493	0.0421	0.0430	-0.0446	0.0440

6. Applications

This section addressed the applicability and adaptability of the proposed distribution in comparison to several competing models. The SL distribution (Nadarajah 2009), the alpha SL distribution (Hazarika and Chakraborty 2014), and the alpha beta SL distribution (Esmaeili *et al.* 2020) are considered for this comparative analysis. The Akaike Information Criteria (AIC) and Bayesian Information Criteria (BIC) are employed for model comparisons. The datasets under consideration for this comparison are given in the Appendix D.

6.1. Environmental performance index dataset

In this application, the Environmental Performance Index (EPI) dataset with $n = 163$ observations is used (Chakraborty *et al.* 2014). The results of the MLEs, log-likelihood, AIC and BIC of the models are reported in Table 5. From Table 5, it can be observed that the new distribution is more appropriate and better fitted among the other competitors in terms

Table 4: Results of simulation

			$\mu = 1, \quad \sigma = 2$							
			μ		σ		λ		α	
α	λ	n	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
	-2	100	0.1304	0.0694	0.0943	0.0843	-0.1100	0.6100	0.0950	0.1394
		300	-0.0920	0.0643	0.0840	0.0807	-0.0793	0.0941	-0.0648	0.0883
		500	0.0536	0.0549	-0.0700	0.0251	0.0344	0.0651	-0.0621	0.0721
	-1	100	0.0880	0.0597	0.1026	0.0888	0.0917	0.0886	0.0834	0.2041
		300	-0.0653	0.0732	-0.0821	0.1003	0.0867	0.0703	-0.0756	0.0953
		500	-0.0600	0.0439	-0.0703	0.0541	0.0203	0.0590	-0.0441	0.0403
1	0	100	0.0956	0.0867	-0.1315	0.0683	0.0921	0.0633	-0.0739	0.0663
		300	-0.0853	0.0825	0.1017	0.0519	-0.0833	0.0601	-0.0421	0.0589
		500	-0.0800	0.0630	0.0313	0.0512	-0.0710	0.0429	-0.0371	0.0423
	1	100	0.1406	0.0793	0.1030	0.0993	0.0965	0.1932	0.1344	0.0968
		300	0.0632	0.0703	-0.0727	0.0643	-0.0824	0.0654	-0.0674	0.0431
		500	-0.0508	0.0491	0.0636	0.0537	0.0437	0.0543	0.0537	0.0501
	2	100	0.1036	0.0934	0.0943	0.0886	0.0859	0.0722	0.0959	0.1333
		300	0.0941	0.0813	-0.0725	0.0721	-0.0721	0.0932	-0.0821	0.0806
		500	0.0453	0.0518	-0.0658	0.0692	-0.0539	0.0434	-0.0655	0.0721

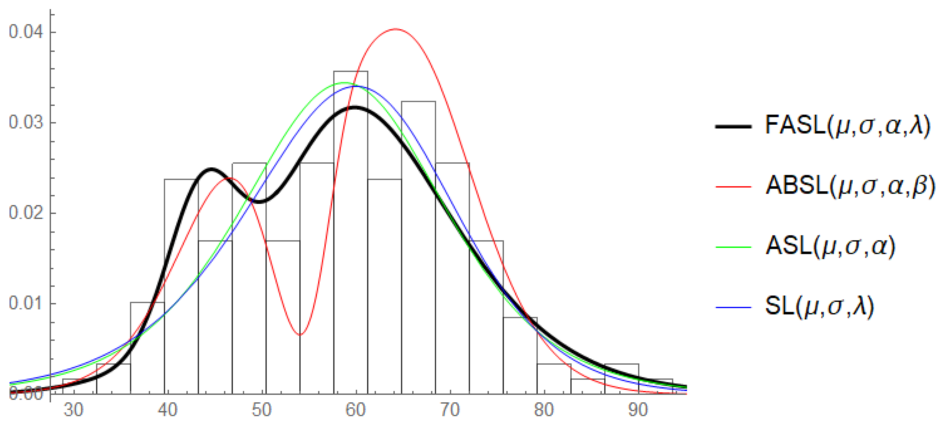


Figure 3: Observed and expected densities for EPI dataset

of log-likelihood, AIC and BIC. The adaptability of the $FASL(\mu, \sigma, \alpha, \lambda)$ pdf can also be observed in Figure 3.

6.2. Female height dataset

Female height dataset with $n = 67$ observations is considered for this application (Cruz-Medina 2001). The results of the MLEs, log-likelihood, AIC and BIC of the models are reported in Table 6. From Table 6, it can be observed that the new distribution is more appropriate and better fitted among the other competitors in terms of log-likelihood, AIC and BIC. The adaptability of the $FASL(\mu, \sigma, \alpha, \lambda)$ can also be observed in Figure 4.

6.3. Breaking strength of carbon fibres dataset

For this application, a dataset related to $n = 66$ breaking strength of carbon fibres of 50 mm length (GPa) is applied (Enogwe, Nwankwo, and Oti 2022). The results of the MLEs, log-likelihood, AIC and BIC of the models are reported in Table 7. From Table 7, it can be observed that the new distribution is more appropriate and better fitted among the other competitors in terms of log-likelihood and AIC. Nevertheless, when considering the BIC,

Table 5: MLEs, log-likelihood, AIC and BIC of models fitted to 163 EPI observations

Distributions	μ	σ	λ	α	β	$logL$	AIC	BIC
$SL(\mu,\sigma,\lambda)$	63.482	7.765	-0.508	–	–	-644.652	1295.304	1304.585
$ASL(\mu,\sigma,\alpha)$	59.185	7.242	—	0.034	—	-645.149	1296.298	1305.578
$ABSL(\mu,\sigma,\alpha,\beta)$	57.669	2.055	—	-0.456	-0.053	-642.589	1293.178	1305.553
$FASL(\mu,\sigma,\alpha,\lambda)$	41.797	4.400	1.131	0.201	—	-638.343	1284.686	1297.061

Table 6: MLEs, log-likelihood, AIC and BIC of models fitted to female height dataset

Distributions	μ	σ	λ	α	β	$logL$	AIC	BIC
$SL(\mu,\sigma,\lambda)$	67.513	1.919	-0.702	–	–	-168.738	343.476	350.101
$ASL(\mu,\sigma,\alpha)$	66.488	1.684	—	0.099	—	-168.913	343.826	350.440
$ABSL(\mu,\sigma,\alpha,\beta)$	65.356	1.664	—	-0.187	0.010	-168.565	345.130	353.079
$FASL(\mu,\sigma,\alpha,\lambda)$	71.036	1.156	-2.542	0.427	—	-166.343	340.686	349.390

both $SLG(\mu,\sigma\lambda)$ and $FASL(\mu,\sigma,\alpha,\lambda)$ pdfs demonstrate a similar fit performance. However, the $FASL(\mu,\sigma,\alpha,\lambda)$ pdf outperforms by accurately capturing all the modes present in the dataset. The adaptability of the $FASL(\mu,\sigma,\alpha,\lambda)$ pdf can also be observed in Figure 5.

7. Hypothesis testing

To discriminate between $FASL(\mu,\sigma,\alpha,\lambda)$ pdfs and some other nested models, Likelihood Ratio Test (LRT) is performed in this section. The test statistics as well as the null hypothesis of the test are as follows:

(i) to discriminate SL from FASL distribution, the null hypothesis $H_0 : \alpha = 0$ have to test against the alternative hypothesis $H_1 : \alpha \neq 0$. The test statistic is

$$-2\log(LR) = -2[\log L(\hat{\mu}_1,\hat{\sigma}_1,\hat{\lambda}_1,\alpha=0|x) - \log L(\hat{\mu}_2,\hat{\sigma}_2,\hat{\alpha}_2,\hat{\lambda}_2)] \sim \chi^2_1,$$

where $(\hat{\mu}_1,\hat{\sigma}_1,\hat{\mu}_2)$ and $(\hat{\sigma}_2,\hat{\lambda}_2,\hat{\alpha}_2)$ are the MLEs of $SL(\mu,\sigma,\lambda)$ and $FASL(\mu,\sigma,\alpha,\lambda)$ pdfs, respectively; and $r = 1$ (difference between the numbers of parameters).

(ii) To discriminate ASL from FASL distribution, the null hypothesis $H_0 : \lambda = 0$ have to

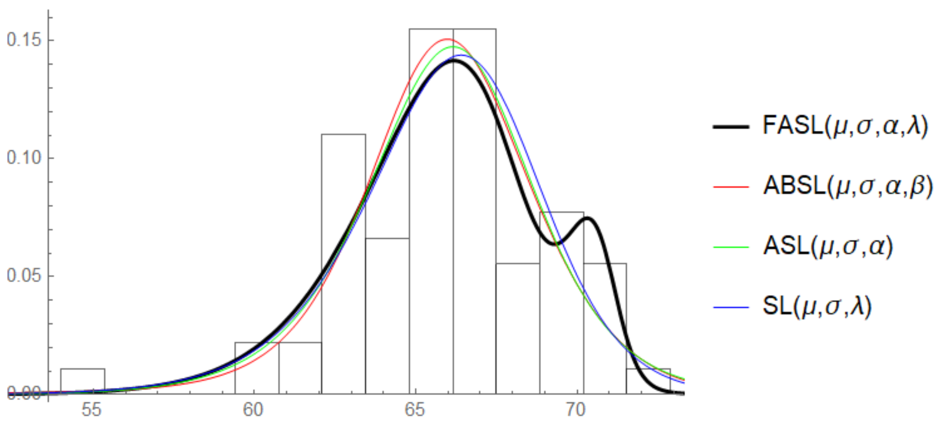


Figure 4: Observed and expected densities of female height dataset

Table 7: MLEs, log-likelihood, AIC and BIC of models fitted to breaking strength of carbon fibre dataset

Distributions	μ	σ	λ	α	β	$logL$	AIC	BIC
$SL(\mu,\sigma,\lambda)$	2.999	0.509	-0.315	–	–	-85.213	176.426	183.010
$ASL(\mu,\sigma,\alpha)$	2.445	0.274	—	-1.255	—	-85.468	176.936	183.505
$ABSL(\mu,\sigma,\alpha,\beta)$	2.287	0.487	—	-0.482	0.010	-85.679	179.358	188.117
$FASL(\mu,\sigma,\alpha,\lambda)$	1.736	0.266	0.629	0.337	—	-83.310	174.620	183.380

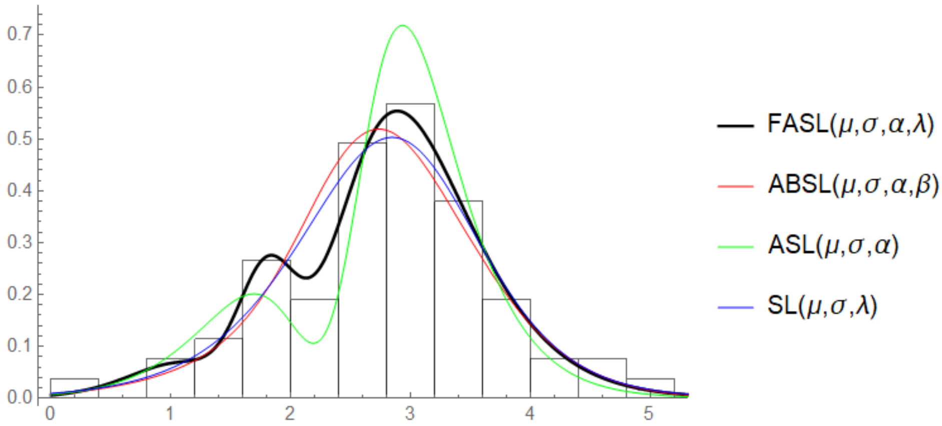


Figure 5: Observed and expected densities of breaking strength of carbon fibre dataset

test against the alternative hypothesis $H_1 : \lambda \neq 0$. The test statistic is

$$-2\log(LR) = -\,2[\log L(\hat{\mu}_1, \hat{\sigma}_1, \hat{\alpha}_1, \lambda = 0|x) - \log L(\hat{\mu}_2, \hat{\sigma}_2, \hat{\alpha}_2, \hat{\lambda}_2)] \sim \chi_1^2,$$

where $(\hat{\mu}_1, \hat{\sigma}_1, \hat{\alpha}_1)$ and $(\hat{\mu}_2, \hat{\sigma}_2, \hat{\lambda}_2, \hat{\alpha}_2)$ are the MLEs' of $ASL(\mu,\sigma,\alpha)$ and $FASL(\mu,\sigma,\alpha,\lambda)$ pdfs, respectively; and $r = 1$ (difference between the numbers of parameters).

Table 8: LRT for different hypotheses for the data set I, II and III

Hypothesis	LRT statistic			d.f.	Critical Values at 5 %
	Dataset I	Dataset II	Dataset III		
$H_0 : \alpha = 0$ Vs $H_1 : \alpha \neq 0$	12.618	13.612	3.810	1	3.841
$H_0 : \lambda = 0$ Vs $H_1 : \lambda \neq 0$	4.790	5.141	4.316	1	3.841

From Table 8, it can be noted that the value of the LRT statistics is higher than the tabulated critical value at 95% of confidence level for dataset I and II. However, for the dataset III, it can be seen that both SL and FASL distributions demonstrate a similar fit performance. However, the FASL distribution outperforms by accurately capturing all the modes present in the dataset (refer to Figure 5). Consequently, it can be stated that the collected data originates from the FASL distribution.

8. Conclusion

This paper introduces a new class of asymmetric probability distribution which is known as flexible alpha SL distribution. This new family of continuous probability distribution is flexible enough to fit data up to three modes. Some important mathematical properties of the proposed distribution are also discussed in this study. In addition, using a graphical method it has been proved that proposed pdf may have between one and three modes. Some characterization results of the proposed distribution are also computed for different approaches. A simulation study is also conducted using the MH algorithm and it is found the estimation of MLEs for the FASL distribution are asymptotically consistent for moderate and large sample sizes. Furthermore, to observe the accountability as well as flexibility, the novel family of distribution was checked via three real-life datasets and comparison was done with some other competitors in terms of AIC and BIC. Finally, LRT was applied to distinguish between the new model with some other simpler models.

A. Derivation of C_1

Considering $g(x)=\dfrac{exp(-x)}{(1-exp(x))^2}$ and $G(\lambda x)=\dfrac{1}{1+exp(-\lambda x)}$, the coefficient C_1 is calculated as follows:

$$\begin{aligned}
C_1 &= \int_{-\infty}^{\infty} 2\left[1+\alpha\left(\frac{(x^2-1)^2+2}{4}\right)\right] g(x)G(\lambda x)dx \\
&= 2\Bigg[\int_{-\infty}^{\infty} g(x)G(\lambda x)dx + \frac{\alpha}{4}\Bigg\{\int_{-\infty}^{\infty} x^4 g(x)G(\lambda x)dx - 2\int_{-\infty}^{\infty} x^2 g(x)G(\lambda x)dx \\
&\qquad + 3\int_{-\infty}^{\infty} g(x)G(\lambda x)dx\Bigg\}\Bigg] \\
&= 2\left[\frac{1}{2}+\frac{\alpha}{4}\left\{\frac{7\pi^4}{30}-\frac{2\pi^2}{6}+\frac{3}{2}\right\}\right] \\
&= 1+\frac{\alpha}{60}\left[45-10\pi^2+7\pi^4\right].
\end{aligned}$$

B. Derivation of $C(\alpha)$

Considering $g(x)=\dfrac{exp(-x)}{(1-exp(x))^2}$ the coefficient $C(\alpha)$ is calculated as follows:

$$\begin{aligned}
C(\alpha) &= \int_{-\infty}^{\infty} 2\left[1+\alpha\left(\frac{(x^2-1)^2+2}{4}\right)\right] g(x)dx \\
&= 2\Bigg[\int_{-\infty}^{\infty} g(x)dx + \frac{\alpha}{4}\Bigg\{\int_{-\infty}^{\infty} x^4 g(x)dx - 2\int_{-\infty}^{\infty} x^2 g(x)dx + 3\int_{-\infty}^{\infty} g(x)dx\Bigg\}\Bigg] \\
&= 2\left[1+\frac{\alpha}{4}\left\{\frac{7\pi^4}{15}-\frac{2\pi^2}{3}+3\right\}\right] \\
&= 2+\frac{\alpha}{30}\left\{45-10\pi^2+7\pi^4\right\}.
\end{aligned}$$

C. Algorithm(s) for generating random samples in Section 5

- Consider FASL as target density denoted as f.
- Consider Logistic distribution as proposal density denoted as g.
- Initialize y_0 (an arbitrary value).

For each iteration i from 1 to N,

- Generate x^* from g.
- Compute acceptance ratio $r = \frac{f(x^*)}{f(y_t)} \times \frac{g(y_t)}{g(x^*)}$.
- Generate a uniform random number u.
- If $u < min(1, r)$, then accept x^* ($y_{t+1} = x^*$); otherwise $y_{t+1} = y_t$.
- Store the accepted sample.
- Discard the first $N - n$ sample values (Burn-in values) and use the rest sample value as filtered sample.

D. Datasets used in Section 6

a) Environmental Performance Index (EPI) dataset:

93.5, 89.1, 86.4, 86.0, 81.1, 80.6, 78.2, 78.1, 78.1, 76.8, 76.3, 74.7, 74.5, 74.2, 73.4, 73.3, 73.2, 73.1, 73.0, 72.5, 72.5, 71.6, 71.4, 71.4, 70.6, 69.9, 69.8, 69.6, 69.4, 69.3, 69.3, 69.2, 69.1, 69.1, 68.7, 68.4, 68.3, 68.2, 68.2, 68.0, 67.8, 67.4, 67.3, 67.1, 67.0, 66.4, 66.4, 65.9, 65.9, 65.7, 65.7, 65.6, 65.4, 65.0, 65.0, 64.6, 63.8, 63.7, 63.6, 63.5, 63.5, 63.4, 63.1, 62.9, 62.5, 62.4, 62.2, 62.0, 61.2, 61.0, 60.9, 60.8, 60.6, 60.6, 60.5, 60.4, 60.4, 60.0, 59.7, 59.6, 59.3, 59.2, 59.1, 59.1, 59.0, 58.8, 58.2, 58.1, 58.0, 57.9, 57.3, 57.3, 57.1, 57.0, 56.4, 56.3, 56.1, 55.9, 55.3, 54.6, 54.4, 54.3, 54.2, 54.0, 54.0, 51.6, 51.4, 51.4, 51.3, 51.3, 51.3, 51.2, 51.1, 51.1, 50.8, 50.3, 50.1, 49.9, 49.8, 49.2, 49.0, 48.9, 48.3, 48.3, 48.0, 47.9, 47.8, 47.3, 47.1, 47.0, 45.9, 44.7, 44.6, 44.6, 44.6, 44.4, 44.3, 44.3, 44.0, 43.9, 43.1, 42.8, 42.3, 42.3, 42.0, 41.9, 41.8, 41.7, 41.3, 41.0, 40.8, 40.7, 40.2, 39.6, 39.5, 39.4, 38.4, 37.6, 36.4, 36.3, 33.7, 33.3, 32.1

b) Female Height dataset:

55.00, 60.00, 60.25, 61.00, 61.75, 62.25, 62.25, 62.63, 62.75, 63.00, 63.25, 63.25, 63.25, 63.25, 63.38, 64.00, 64.25, 64.25, 64.50, 64.75, 64.75, 65.00, 65.00, 65.13, 65.13, 65.17, 65.25, 65.25, 65.25, 65.25, 65.25, 65.50, 65.75, 66.00, 66.00, 66.25, 66.25, 66.25, 66.50, 66.75, 66.75, 66.75, 66.75, 67.00, 67.13, 67.25, 67.38, 67.50, 67.50, 67.75, 67.75, 67.75, 68.13, 68.75, 69.00, 69.00, 69.25, 69.50, 69.88, 70.00, 70.00, 70.25, 70.38, 71.00, 71.00, 71.25, 71.75

c) Breaking strength of carbon fibres dataset:

0.39, 0.85, 1.08, 1.25, 1.47, 1.57, 1.61, 1.61, 1.69, 1.80, 1.84, 1.87, 1.89, 2.03, 2.03, 2.05, 2.12, 2.35, 2.41, 2.43, 2.48, 2.50, 2.53, 2.55, 2.55, 2.56, 2.59, 2.67, 2.73, 2.74, 2.79, 2.81, 2.82, 2.85, 2.87, 2.88, 2.93, 2.95, 2.96, 2.97, 3.09, 3.11, 3.11, 3.15, 3.15, 3.19, 3.22, 3.22, 3.27, 3.28, 3.31, 3.31, 3.33, 3.39, 3.39, 3.56, 3.60, 3.65, 3.68, 3.70, 3.75, 4.20, 4.38, 4.42, 4.70, 4.90

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Austrian Journal of Statistics http://www.ajs.or.at/
published by the Austrian Society of Statistics http://www.osg.or.at/
Volume 54 *Submitted:* 2024-03-06
2025 *Accepted:* 2024-07-15