



# Revisiting the Memoryless Property – Testing for the Pareto Type I Distribution

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## Abstract

We propose new goodness-of-fit tests for the Pareto type I distribution. These tests are based on a multiplicative version of the memoryless property which characterises this distribution. We present the results of a Monte Carlo power study demonstrating that the proposed tests are powerful compared to existing tests. As a result of independent interest, we demonstrate that tests specifically developed for the Pareto type I distribution substantially outperform tests for exponentiality applied to log-transformed data (since Pareto type I distributed values can be transformed to exponentiality via a simple log-transformation). Specifically, the newly proposed tests based on the multiplicative memoryless property of the Pareto distribution substantially outperform a test based on the memoryless property of the exponential distribution. The practical use of tests is illustrated by testing the hypothesis that two sets of observed golfers' earnings (those of the PGA and LIV tours) are realised from Pareto distributions.

*Keywords:* memoryless property, goodness-of-fit testing, Pareto distribution.

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## 1. Introduction

The memoryless property of the exponential distribution is well-known and characterises this distribution among the continuous laws. This property implies that if, for example, the exponential distribution is used to model waiting times, then the probability of an event occurring within the next unit of time is independent of the amount of time that has passed. Let  $Y$  be a random variable denoting an exponential waiting time. The memoryless property of the exponential distribution can be expressed as

$$P(Y \geq s + t | Y > s) = P(Y > t), \quad t, s \geq 0. \quad (1)$$

For more details regarding this and some other characterisations of the exponential distribution, see [Tomy, Jose, and Veena \(2020\)](#) as well as [Galambos and Kotz \(2006\)](#). Other versions

of the memoryless property for distributions such as the uniform, Weibull, Pareto and extreme value are also discussed in Galambos and Kotz (2006). The memoryless property has been used to develop several goodness-of-fit tests for the exponential distribution, see Ahmad and Alwasel (1999) as well as Alwasel (2001). Below, we use a related characterisation to propose new goodness-of-fit tests for the Pareto distribution.

The Pareto distribution is named after the Italian economist Vilfredo Pareto, who first described this law in his 1897 book *Cours d'économie politique*, see Pareto (1897). Several generalisations of this distribution have since been proposed, see Arnold (2015), and the originally proposed distribution has subsequently become known as the Pareto type I distribution. In what follows, we refer to the Pareto type I distribution simply as the Pareto distribution. This distribution is closely related to the exponential distribution. If  $Y$  is exponential with mean  $1/\beta$ , then  $X = e^Y$  is a Pareto distributed random variable with shape parameter  $\beta > 0$ , denoted by  $X \sim P(\beta)$ . In this case, the distribution function of  $X$  is

$$F(x) = 1 - x^{-\beta}, \quad \beta > 0, \quad x > 1.$$

Note that the Pareto distribution is often specified to include a scale parameter  $\sigma$ ; we take this parameter to be equal to 1 throughout. Similar results to those presented in this paper can be obtained when the scale parameter is required to be estimated from observed data.

The Pareto distribution is widely used to model phenomena exhibiting heavy tails. Examples of the use of the Pareto distribution include the modelling of excess losses in insurance claims, see Rytgaard (1990), as well as the modelling of failure times of mechanical components, see Bourguignon and Saulo (2018). The Pareto is also a popular model found in survival analysis and reliability theory, see, Ndwandwe, Allison, and Visagie (2021) as well as Nofal and El Gebaly (2017). For a recent review of goodness-of-fit tests for the Pareto distribution, see Ndwandwe, Allison, Santana, and Visagie (2023) and for more general Pareto distributions see Chu, Dickin, and Nadarajah (2019).

The link between the exponential and Pareto laws means that the memoryless property of the exponential distribution has implications for the Pareto distribution as well. The Pareto distribution exhibits a multiplicative version of the memoryless property, which can be expressed as follows:  $X \sim P(\beta)$  for some  $\beta > 0$  if, and only if,

$$P(X \geq st | X > s) = P(X > t), \quad t, s \geq 1. \quad (2)$$

This property will be exploited to construct goodness-of-fit tests for the Pareto distribution. Naturally, due to the link between the Pareto and exponential distributions, tests for exponentially can also be applied to log-transformed data in order to test for the Pareto distribution. We also note that the property in (2) can be used to construct goodness-of-fit tests for the Pareto type II distribution. This can be done by using a suitable transformation; if a random variable  $Z$  follows a Pareto type II distribution with location parameter  $\mu$ , then the random variable  $X = Z - \mu + \sigma$  follows a Pareto type I distribution.

The remainder of the paper is structured as follows. In Section 2, we introduce three new classes of goodness-of-fit tests based on a characterization that uses the multiplicative memoryless property. Section 3 includes a Monte Carlo study comparing the performances of the newly proposed tests to those of existing tests for the Pareto distribution as well as tests for exponentiality applied to transformed data. The finite sample performances are considered against a wide range of fixed alternative distributions. Additionally, the performances of the tests against a range of mixture distributions are also compared in this section. In Section 4, we demonstrate the use of the tests in order to determine whether or not observed data sets, relating to the earning of golfers, are realised from a Pareto distribution. Finally, Section 5 presents some conclusions.

## 2. The test statistics based on the memoryless property

In this section we develop three classes of goodness-of-fit tests for the Pareto distribution based on the characterisation given in (2). Given a random sample  $X, X_1, X_2, \dots, X_n$ , from an unknown probability distribution  $F$  with density  $f$ , we wish to test the composite null hypothesis

$$H_0 : X \sim P(\beta), \quad (3)$$

for some  $\beta > 0$ , against general alternatives.

To estimate  $\beta$ , we use two different estimators, namely, the maximum likelihood estimator (MLE) and the method of moments estimator (MME). The MLE is

$$\tilde{\beta}_n := \tilde{\beta}(X_1, \dots, X_n) = \frac{n}{\sum_{j=1}^n \log(X_j)}, \quad (4)$$

while the MME is

$$\ddot{\beta}_n := \ddot{\beta}(X_1, \dots, X_n) = \frac{\bar{X}}{\bar{X} - 1},$$

with  $\bar{X} = \frac{1}{n} \sum_{j=1}^n X_j$ . Henceforth, we will use the notation  $\hat{\beta}_n$  to denote either of these two estimators. Note that the mean of the Pareto distribution only exists for  $\beta > 1$ , and thus any test using MME would be greatly affected if  $\beta < 1$ .

In order to propose the first two test statistics, we restrict our attention to the specific choice  $s = t$  in (2) to obtain that  $X \sim P(\beta)$  if, and only if,

$$P(X > t^2) = [P(X > t)]^2, \quad t \geq 1. \quad (5)$$

The first of the two test statistics is obtained by estimating the left hand side of (5) non-parametrically using the empirical survival function,  $S_n(t) = \frac{1}{n} \sum_{j=1}^n I(X_j > t)$ , and the right hand side parametrically by using the survival function of the Pareto distribution with estimated  $\hat{\beta}_n$ ,  $[S_{\hat{\beta}_n}(t)]^2 = t^{-2\hat{\beta}_n}$ :

$$\begin{aligned} MP_n^{(1)} &= \int_1^\infty \left( S_n(t^2) - [S_{\hat{\beta}_n}(t)]^2 \right)^2 dF_{\hat{\beta}_n}(t) \\ &= \int_1^\infty \left( \frac{1}{n} \sum_{j=1}^n I(X_j > t^2) - t^{-2\hat{\beta}_n} \right)^2 \hat{\beta}_n t^{-(\hat{\beta}_n+1)} dt. \end{aligned}$$

From some straightforward calculation we have that the test statistic simplifies to

$$MP_n^{(1)} = \frac{2}{3n} \sum_{j=1}^n X_j^{-3\hat{\beta}_n/2} - \frac{1}{n^2} \sum_{j=1}^n \sum_{k=1}^n \min(X_j, X_k)^{-\hat{\beta}_n/2} + \frac{8}{15}.$$

The double summation in the expression for  $MP_n^{(1)}$  can be reduced to a single summation based on the order statistics,  $X_{(1)} < X_{(2)} < \dots < X_{(n)}$ ,

$$MP_n^{(1)} = \frac{2}{3n} \sum_{j=1}^n X_j^{-3\hat{\beta}_n/2} - \frac{1}{n^2} \sum_{j=1}^n v_{j,n} X_{(j)}^{-\hat{\beta}_n/2} + \frac{8}{15},$$

where  $v_{j,n} := (n - j + 1)^2 - (n - j)^2$ .

Test second test statistic is obtained by estimating both sides of (5) nonparametrically:

$$MP_n^{(2)} = \int_1^\infty \left( S_n(t^2) - [S_n(t)]^2 \right)^2 dF_n(t),$$

where  $F_n(t) = 1 - S_n(t)$  is the empirical distribution function. Note that  $MP_n^{(2)}$  does not depend on the unknown parameter  $\beta$ , while this is not the case for  $MP_n^{(1)}$ .

The restriction in (5) does not incorporate full information regarding the memoryless property. As a result, we propose a third test statistic by noting that (2) implies that  $X$  has a Pareto distribution if, and only if,

$$P(X > st) = P(X > s)P(X > t), \quad s, t \geq 1. \quad (6)$$

Again, estimating the left hand side of (6) nonparametrically by the empirical survival function and the right hand side by the parametric survival function with estimated shape parameter, we have the test statistic

$$\begin{aligned} MP_n^{(3)} &= \int_1^\infty \int_1^\infty \left( \frac{1}{n} \sum_{i=1}^n I(X_j > st) - (st)^{-\hat{\beta}_n} \right)^2 dF_{\hat{\beta}_n}(s) dF_{\hat{\beta}_n}(t) \\ &= \int_1^\infty \int_1^\infty \left( \frac{1}{n} \sum_{i=1}^n I(X_j > st) - (st)^{-\hat{\beta}_n} \right)^2 (st)^{-(\hat{\beta}_n+1)} \hat{\beta}_n^2 ds dt. \end{aligned}$$

After some algebraic manipulation we have that

$$\begin{aligned} MP_n^{(3)} &= \frac{10}{9} - \frac{1}{n^2} \sum_{j=1}^n \sum_{k=1}^n \min(X_j, X_k)^{-\hat{\beta}_n} - \frac{\hat{\beta}_n}{n^2} \sum_{j=1}^n \sum_{k=1}^n \min(X_j, X_k)^{-\hat{\beta}_n} \log(\min(X_j, X_k)) \\ &\quad - \frac{\hat{\beta}_n}{n} \sum_{j=1}^n \left[ \frac{1 - X_j^{-2\hat{\beta}_n}}{2\hat{\beta}_n} - X_j^{-2\hat{\beta}_n} \log(X_j) \right]. \end{aligned}$$

Again this test statistic simplifies to a single sum involving order statistics

$$\begin{aligned} MP_n^{(3)} &= \frac{10}{9} - \frac{1}{n^2} \sum_{j=1}^n v_{j,n} X_{(j)}^{-\hat{\beta}_n} - \frac{\hat{\beta}_n}{n^2} \sum_{j=1}^n v_{j,n} X_{(j)}^{-\hat{\beta}_n} \log(X_{(j)}) \\ &\quad - \frac{\hat{\beta}_n}{n} \sum_{j=1}^n \left[ \frac{1 - X_j^{-2\hat{\beta}_n}}{2\hat{\beta}_n} - X_j^{-2\hat{\beta}_n} \log(X_{(j)}) \right]. \end{aligned}$$

All three proposed tests reject the hypothesis in (3) for large values.

The newly proposed test statistics are all in the form of an  $L2$  weighted distance measure. These type of statistics are typically consistent against fixed alternative distributions. Furthermore, the asymptotic null distribution corresponds to the weighted sum of  $\chi_1^2$  random variables (see, e.g., Cuparić, Milošević, and Obradović (2022), Meintanis, Milošević, and Jiménez-Gamero (2024) and Ngatchou-Wandji, Nombebe, Santana, and Allison (2024)).

### 3. Monte Carlo study

In this section, we use Monte Carlo simulation to compare the performances of our newly proposed tests to some of the existing tests found in the statistical literature. The classical goodness-of-fit tests included are the Kolmogorov-Smirnov, Cramér-von Mises and Anderson-Darling tests. Additionally, we include a test based on likelihood ratios as well as a test based on the Mellin transform. The calculable forms of the test statistics are specified below:

- The Kolmogorov-Smirnov test:

$$KS_n = \max \left\{ \max_{1 \leq j \leq n} \left[ \frac{j}{n} - F_{\hat{\beta}_n}(X_{(j)}) \right], \max_{1 \leq j \leq n} \left[ F_{\hat{\beta}_n}(X_{(j)}) - \frac{j-1}{n} \right] \right\}.$$

- The Cramér-von Mises test:

$$CM_n = \frac{1}{12n} + \sum_{j=1}^n \left[ F_{\hat{\beta}_n}(X_{(j)}) - \frac{2j-1}{2n} \right]^2.$$

- The Anderson-Darling test:

$$AD_n = -n - \frac{1}{n} \sum_{j=1}^n (2j-1) \left[ \log \left( F_{\hat{\beta}_n}(X_{(j)}) \right) + \log \left( 1 - F_{\hat{\beta}_n}(X_{(n+1-j)}) \right) \right].$$

- The test proposed by [Zhang \(2002\)](#), based on likelihood ratios:

$$ZA_n = - \sum_{j=1}^n \left\{ \frac{\log \left( F_{\hat{\beta}_n}(X_{(j)}) \right)}{n-j+\frac{1}{2}} + \frac{\log \left( 1 - F_{\hat{\beta}_n}(X_{(j)}) \right)}{j-\frac{1}{2}} \right\}.$$

- The test proposed by [Meintanis \(2009\)](#), based on the Mellin transform:

$$G_{n,a} = \frac{1}{n} \left[ (\hat{\beta}_n + 1)^2 \sum_{j,k=1}^n I_w^{(0)}(X_j X_k) + \sum_{j,k=1}^n I_w^{(2)}(X_j X_k) + 2(\hat{\beta}_n + 1) \sum_{j,k=1}^n I_w^{(1)}(X_j X_k) \right] \\ + \hat{\beta}_n \left[ n \hat{\beta}_n I_w^{(0)}(1) - 2(\hat{\beta}_n + 1) \sum_{j=1}^n I_w^{(0)}(X_j) - 2 \sum_{j=1}^n I_w^{(1)}(X_j) \right],$$

where

$$I_w^{(m)}(x) = \int_0^\infty (t-1)^m \frac{1}{x^t} w(t) dt, \quad m = 0, 1, 2.$$

Choosing  $w(t) = e^{-ax}$ , one has

$$I_a^{(0)}(x) = (a + \log x)^{-1},$$

$$I_a^{(1)}(x) = \frac{1 - a - \log x}{(a + \log x)^2},$$

and

$$I_a^{(2)}(x) = \frac{2 - 2a + a^2 + 2(a-1) \log x + \log^2 x}{(a + \log x)^3}.$$

To obtain the numerical results presented later in this section, the tuning parameter  $a$  has been assigned a value of 1.

- A test proposed by [Allison, Milošević, Obradović, and Smuts \(2022\)](#) based on a characterisation:

$$I_{n,m} = \int_1^\infty \Delta_{n,m}(x) dF_n(x),$$

where

$$\Delta_{n,m}(x) = \frac{1}{n} \sum_{j=1}^n I\{X_j^{\frac{1}{m}} \leq x\} - \frac{1}{n^m} \sum_{j_1, \dots, j_m=1}^n I\{\min(X_{j_1}, \dots, X_{j_m}) \leq x\}.$$

For the implementation of the test, [Allison et al. \(2022\)](#) recommend choosing  $m = 2$ .

All of the tests above reject the null hypothesis in (3) for large values of the test statistics. This will be discussed further in Section 3.

In addition to these tests for the Pareto distribution, we also compare the numerical powers achieved by the newly proposed tests against those of some tests for exponentiality (the interested reader is referred to [Allison, Santana, Smit, and Visagie \(2017\)](#) and [Henze and Meintanis \(2005\)](#) for an overview of tests for exponentiality). In these cases, the tests for exponentiality are applied to the log-transformed data (if  $X \sim P(\beta)$  then  $Y = \log X$  follows an exponential distribution with rate parameter  $\beta$ ).

The exponentiality tests we consider are again the three classical tests based on the empirical distribution function (denoted by  $\widetilde{KS}_n$ ,  $\widetilde{CV}_n$ , and  $\widetilde{AD}_n$ ). In addition, we consider a test based on likelihood ratios by Zhang (2002) ( $\widetilde{ZA}_n$ ), a test proposed by Angus (1982) which is based on the memoryless property of the exponential distribution ( $\widetilde{AG}_n$ ), a score based test proposed by Cox and Oakes (1984) ( $\widetilde{CO}_n$ ), and a Cramér-von Mises type test based on mean residuals life proposed by Baringhaus and Henze (2000) ( $\widetilde{BH}_n$ ). These tests have been found to be quite powerful in the literature; see, e.g., Torabi, Montazeri, and Grané (2018) and Jiménez-Gamero, Milošević, and Obradović (2020).

### 3.1. Simulation setting

We start by mentioning that the way we obtain the critical values differs depending on the estimation method.

When using the MLE in (4) to estimate  $\beta$ , the critical values of the tests considered do not depend on the value of  $\beta$ , i.e., if  $X \sim P(\beta)$ , then  $Y := X^\beta \sim P(1)$ . Consider the transformed sample  $Y_j = X_j^{\beta_n}$ ,  $j = 1, \dots, n$ , and note that

$$\widetilde{\beta}(Y_1, \dots, Y_n) = \frac{n}{\sum_{j=1}^n \log(Y_j)} = \frac{n}{\widetilde{\beta}(X_1, \dots, X_n) \sum_{j=1}^n \log(X_j)} = \frac{\widetilde{\beta}(X_1, \dots, X_n)}{\widetilde{\beta}(X_1, \dots, X_n)} = 1. \quad (7)$$

For more details regarding this transformation and the estimation of critical values using Monte Carlo simulation, see Section 3.1 of Ndwandwe *et al.* (2023). We simulate 100 000 Monte Carlo samples (from a  $P(1)$  distribution) to estimate these critical values. Next, we obtain 10 000 samples from an alternative distribution in order to obtain the empirical power against the distribution in question. Below, we calculate empirical powers against the distributions listed in Table 1. Note that the alternative distributions employed in the simulation study are right-shifted by one unit so as to ensure that they share the domain of the null distribution.

In the case of MME, fixed critical values cannot be obtained because  $\widetilde{\beta}(Y_1, \dots, Y_n) \neq 1$ . Instead we employ a parametric bootstrap approach to obtain the relevant empirical critical values. Due to the time-consuming nature of the bootstrap procedure, the warp-speed bootstrap technique proposed by Giacomini, Politis, and White (2013) is used to obtain empirical powers. To this end, we generate 50 000 Monte Carlo samples and for each Monte Carlo sample we obtain one parametric bootstrap sample. The estimated empirical power is then the proportion of these 50 000 Monte Carlo samples that lead to the rejection of the null hypothesis in (3), see Ndwandwe *et al.* (2023) for an algorithm for the implementation of the warp-speed bootstrap in the current context. Note that the two tests  $MP_n^{(2)}$  and  $I_{n,2}$  are free of the unknown parameter, and hence the estimated power results using MLE and MME are identical and therefore are displayed in only 1 column in the tables that follow. All calculations below are performed in R, see R Core Team (2020).

### 3.2. Powers against fixed alternatives

For each of the tests considered, Tables 2 and 3 report power estimates for samples of size  $n = 20$  and  $n = 30$ . These estimates are calculated against the alternative distributions given in Table 1 for various parameter values. The resulting numerical powers are calculated to be the percentage, rounded to the nearest integer, of the samples that lead to the rejection of the null hypothesis.

Consider first the tests specifically developed for the Pareto distribution, i.e., those which do not require the data to be log-transformed. The tables include the results obtained using both the MLE and the MME. For ease of comparison, we circle the highest two numerical powers associated with the MME and we indicate the two highest powers achieved using the MLE with a rectangle (including ties). In each case, the best performing test overall is

Table 1: Summary of various choices of the alternative distributions

Alternative	Density function	Notation
Gamma	$\frac{1}{\Gamma(\theta)}(x-1)^{\theta-1}e^{-(x-1)}$	$\Gamma(\theta)$
Weibull	$\theta(x-1)^{\theta-1}\exp(-(x-1)^\theta)$	$W(\theta)$
Log-normal	$\exp\left\{-\frac{1}{2}(\log(x-1)/\theta)^2\right\} / \left\{\theta(x-1)\sqrt{2\pi}\right\}$	$LN(\theta)$
Half-normal	$\frac{\sqrt{2}}{\theta\sqrt{\pi}}\exp\left(-\frac{(x-1)^2}{2\theta^2}\right)$	$HN(\theta)$
Linear failure rate	$(1+\theta(x-1))\exp(-(x-1)-\theta(x-1)^2/2)$	$LFR(\theta)$
Beta exponential	$\theta e^{-(x-1)}(1-e^{-(x-1)})^{\theta-1}$	$BE(\theta)$
Dhillon	$\frac{\theta+1}{x}\exp\left\{-\left(\log(x)\right)^{\theta+1}\right\}(\log(x))^\theta$	$D(\theta)$

indicated using shading as well. Additionally, these two power tables include results pertaining to the tests based on the transformed data. In the majority of the cases considered, the  $\widetilde{CO}_n$  test performs best among the tests for exponentiality. When comparing the powers of the tests developed for the Pareto distribution to those based on the transformed data, the powers of the former group are found to be substantially higher. As a result, we do not report the powers achieved by all of the tests based on log-transformed data in Tables 2 and 3, we only indicate the best performing test from this group together with the associated power; this is shown in the last column of each table. In cases where the tests based on the exponential distribution outperform or equal the performance of the others, these powers are indicated using shading. It should be noted that outright outperformance of this kind is only achieved against two alternative distributions; the  $LN(2.5)$  (the  $\widetilde{CO}_n$  performs best in this case) and the  $HN(0.5)$  (where the  $\widetilde{BH}_n$  test proves most powerful). In both of the mentioned cases, the outperformance is by a narrow margin. For completeness, we include the numerical powers associated with the exponentiality tests in the appendix. However, these results are not discussed further, except to point out that the newly proposed tests based on the multiplicative memoryless property substantially outperform the test based on the memoryless property of the exponential distribution against the considered alternatives.

Table 2: Numerical powers against fixed alternatives with  $n = 20$ 

Distribution	$KS_n$		$CV_n$		$AD_n$		$ZA_n$		$G_{n,1}$		$MP_n^{(1)}$		$MP_n^{(3)}$		$MP_n^{(2)}$	$I_{n,2}$	Exp. test
	MME	MLE	MME	MLE	MME	MLE	MME	MLE	MME	MLE	MME	MLE	MME	MLE			
$P(2)$	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5 (all)
$P(5)$	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5 (all)
$P(10)$	5	5	5	5	5	5	5	5	4	4	5	5	5	5	5	5	5 (all)
$\Gamma(0.8)$	15	9	16	10	15	10	11	11	19	10	18	12	21	14	10	8	13 ( $\widetilde{CO}_n$ )
$\Gamma(1)$	37	24	44	29	42	24	33	28	51	32	47	31	48	32	24	24	38 ( $\widetilde{CO}_n$ )
$\Gamma(1.2)$	64	45	74	56	72	51	64	55	80	60	75	57	75	57	44	50	69 ( $\widetilde{CO}_n$ )
$W(0.8)$	15	9	17	9	17	9	11	10	20	8	19	10	22	12	8	7	9 ( $\widetilde{CO}_n$ )
$W(1.2)$	65	50	75	62	73	57	65	60	80	65	76	63	76	62	48	55	73 ( $\widetilde{CO}_n$ )
$W(1.5)$	91	82	96	92	96	90	93	91	98	94	97	93	96	92	82	88	96 ( $\widetilde{CO}_n$ )
$LN(1)$	72	55	82	66	83	64	90	80	85	72	80	62	72	51	52	63	88 ( $\widetilde{CO}_n$ )
$LN(1.5)$	17	6	20	7	22	6	21	10	24	8	22	7	23	7	9	6	14 ( $\widetilde{CO}_n$ )
$LN(2.5)$	11	27	9	29	26	43	22	27	5	32	14	26	17	17	10	32	47 ( $\widetilde{CO}_n$ )
$HN(0.5)$	44	37	53	47	49	41	43	43	55	47	56	50	57	51	37	40	63 ( $\widetilde{BH}_n$ )
$HN(1)$	67	54	76	65	74	59	62	60	80	66	78	68	80	70	54	57	69 ( $\widetilde{BH}_n$ )
$HN(1.2)$	74	58	82	70	80	64	69	66	86	71	84	73	86	76	58	62	76 ( $\widetilde{BH}_n$ )
$LFR(0.2)$	45	31	52	38	49	33	40	35	59	41	55	41	57	43	31	33	47 ( $\widetilde{CO}_n$ )
$LFR(0.8)$	54	42	64	53	60	47	51	49	69	55	66	56	68	57	42	45	58 ( $\widetilde{CO}_n$ )
$LFR(1)$	56	45	65	55	62	50	53	52	70	57	68	58	69	60	45	48	61 ( $\widetilde{CO}_n$ )
$BE(0.8)$	17	11	19	12	18	11	13	12	23	12	22	13	25	16	11	9	13 ( $\widetilde{CO}_n$ )
$BE(1)$	39	25	46	31	43	25	34	28	52	33	48	32	50	34	24	25	39 ( $\widetilde{CO}_n$ )
$BE(1.5)$	84	67	91	80	91	75	86	80	94	83	91	80	90	77	66	75	89 ( $\widetilde{CO}_n$ )
$D(0.4)$	50	29	60	34	60	30	57	39	64	38	59	33	57	30	29	33	52 ( $\widetilde{CO}_n$ )
$D(0.6)$	73	51	82	61	83	56	80	65	86	67	82	59	79	54	49	57	80 ( $\widetilde{CO}_n$ )
$D(0.8)$	89	70	95	81	94	79	93	84	96	87	94	81	92	76	68	78	93 ( $\widetilde{CO}_n$ )



Table 3: Numerical powers against fixed alternatives with  $n = 30$

Distribution	$KS_n$		$CV_n$		$AD_n$		$ZA_n$		$G_{n,1}$		$MP_n^{(1)}$		$MP_n^{(3)}$		$MP_n^{(2)}$	$I_{n,2}$	Exp. test
	MME	MLE	MME	MLE	MME	MLE	MME	MLE	MME	MLE	MME	MLE	MME	MLE			
$P(2)$	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5 (all)
$P(5)$	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5 (all)
$P(10)$	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5	5 (all)
$\Gamma(0.8)$	19	12	21	14	21	13	14	14	25	14	25	16	29	19	12	10	14 ( $\widetilde{CO}_n$ )
$\Gamma(1)$	53	36	61	44	59	39	47	42	67	49	64	47	65	48	32	38	53 ( $\widetilde{CO}_n$ )
$\Gamma(1.2)$	82	64	89	76	89	73	82	76	93	80	90	78	90	77	60	71	85 ( $\widetilde{CO}_n$ )
$W(0.8)$	20	10	21	11	21	11	13	12	23	10	25	13	29	16	9	8	11 ( $\widetilde{CO}_n$ )
$W(1.2)$	83	69	90	81	90	78	83	81	93	85	91	83	90	81	66	77	89 ( $\widetilde{CO}_n$ )
$W(1.5)$	99	95	100	99	100	99	99	99	100	99	100	99	100	99	95	98	100 ( $\widetilde{CO}_n$ )
$LN(1)$	88	75	94	85	95	87	98	95	95	90	92	82	87	71	72	86	98 ( $\widetilde{CO}_n$ )
$LN(1.5)$	21	8	26	8	28	7	28	14	30	10	28	8	28	8	9	8	20 ( $\widetilde{CO}_n$ )
$LN(2.5)$	13	38	10	42	31	58	30	39	4	44	16	39	19	26	23	48	61 ( $\widetilde{CO}_n$ )
$HN(0.5)$	61	54	71	66	68	61	61	63	73	68	74	69	75	70	52	59	80 ( $\widetilde{BH}_n$ )
$HN(1)$	85	73	91	84	90	81	82	83	93	86	92	87	93	88	72	77	84 ( $\widetilde{BH}_n$ )
$HN(1.2)$	90	78	95	88	94	86	87	88	96	90	96	91	96	92	78	82	90 ( $\widetilde{BH}_n$ )
$LFR(0.2)$	63	47	71	57	69	52	56	54	77	61	74	60	75	62	42	47	62 ( $\widetilde{CO}_n$ )
$LFR(0.8)$	73	60	82	73	80	69	70	71	85	76	84	76	85	78	58	65	76 ( $\widetilde{CO}_n$ )
$LFR(1)$	74	63	83	75	81	71	72	73	86	78	85	78	86	79	60	67	79 ( $\widetilde{CO}_n$ )
$BE(0.8)$	24	15	27	17	26	15	16	17	31	17	30	19	34	23	13	11	17 ( $\widetilde{CO}_n$ )
$BE(1)$	52	36	60	44	59	39	46	42	67	48	63	46	64	48	32	37	52 ( $\widetilde{CO}_n$ )
$BE(1.5)$	95	86	98	94	99	93	97	95	99	96	98	95	98	93	84	92	98 ( $\widetilde{CO}_n$ )
$D(0.4)$	66	41	75	51	76	48	73	56	78	57	74	49	71	43	39	49	70 ( $\widetilde{CO}_n$ )
$D(0.6)$	89	70	94	81	94	80	93	84	96	86	94	80	92	73	66	80	94 ( $\widetilde{CO}_n$ )
$D(0.8)$	97	89	99	95	99	95	99	96	99	97	99	95	98	92	87	95	99 ( $\widetilde{CO}_n$ )

Tables 2 and 3 indicate that all of the tests considered achieve the specified size of 5% against each of the Pareto distributions included in the Monte Carlo study. Unsurprisingly, the powers reported in Table 3 are generally higher than those reported in Table 2, indicating that an increase in sample size results in an increase in power against the alternatives considered.

Consider the result obtained using the MLE. The tables indicate that, using this estimation technique,  $MP_n^{(1)}$  and  $G_{n,1}$  perform best, closely followed by  $MP_n^{(3)}$ . The  $ZA_n$  and  $CV_n$  tests also generally perform well in this setting. When turning our attention to the results obtained using the MME, we see that  $MP_n^{(3)}$  and  $G_{n,1}$  exhibit the highest powers, followed by  $MP_n^{(1)}$  and  $AD_n$ .

We now compare the results obtained using the MLE to those based on the MME. Although the power achieved by a given test using the MLE is higher than that achieved using the MME in a few cases, the reverse is generally true. In the vast majority of cases, the results obtained using the MME is superior to those associated with the MLE. In some cases this outperformance is substantial. Taking all of the observations made above into consideration, our recommendation when testing for the Pareto distribution is to use either the  $MP_n^{(3)}$  or  $G_{n,1}$  tests together with the MME.

### 3.3. Powers against mixture distributions

In addition to the powers obtained against fixed alternatives, we include results relating to a family of mixture distributions. Consider a mixture distribution with mixing proportion  $p$ , contaminated by some distribution  $G$ . When simulating from this distribution, we generate a realisation from  $G$  with probability  $p \in [0, 1]$  and we generate a Pareto distributed random variate with probability  $1 - p$ . The shape parameter of the Pareto distribution used is chosen so that the means of the two distributions under consideration are equal. The contaminating distribution considered is the shifted exponential distribution mean 3. The calculated numerical powers can be found in Table 4. Again, to ease our comparison, we encircle the highest numerical power linked to the MME and mark the highest power attained through the MLE with a rectangle, including instances of ties. Additionally, we shade the highest power overall. Note that, in Table 4, the results associated with sample sizes  $n = 20$  (top) and  $n = 30$  (bottom) are included in the same table.

Table 4: Numerical powers against exponential mixture distributions

Proportion	$n$	$KS_n$		$CV_n$		$AD_n$		$ZA_n$		$G_{n,1}$		$MP_n^{(1)}$		$MP_n^{(3)}$		$MP_n^{(2)}$	$I_{n,2}$
		MME	MLE	MME	MLE	MME	MLE	MME	MLE	MME	MLE	MME	MLE	MME	MLE		
0.1	20	6	6	6	5	5	5	6	6	6	5	6	6	6	6	5	5
	30	5	5	6	5	5	5	6	6	6	5	6	6	6	6	6	5
0.3	20	10	8	10	9	9	7	9	9	11	8	11	9	12	10	8	6
	30	11	10	13	11	11	9	11	11	13	11	13	11	14	13	9	8
0.5	20	16	13	19	15	16	12	15	15	20	15	20	17	21	18	12	11
	30	22	18	26	22	23	18	20	20	26	22	27	23	28	25	15	15
0.7	20	27	22	32	27	29	23	25	24	34	27	34	29	35	31	22	22
	30	39	32	46	40	42	35	35	36	47	41	47	42	49	45	30	32
0.9	20	42	35	50	43	46	37	40	39	53	44	52	45	54	48	36	38
	30	59	50	68	62	65	56	57	58	70	64	70	65	71	66	49	54

The powers shown in Table 4 indicate that the best performing test using the MLE is  $MP_n^{(3)}$ , very closely followed by  $MP_n^{(1)}$ . Turning our attention to the results pertaining to the MME, we see that  $MP_n^{(3)}$  is not outperformed by any test against any of the mixtures considered. However, we again note that the  $G_{n,1}$  test achieves powers close to those of  $MP_n^{(3)}$ .

The results above serve to support our findings obtained against fixed alternatives. That is, we recommend the use of the MME in order to perform parameter estimation, and we recommend using either the  $MP_n^{(2)}$  or  $G_{n,1}$  to test the hypothesis of the Pareto distribution.

Table 5: Season earnings of LIV golf players earning in excess of \$3.5 million

\$36 071 517	\$16 993 416	\$15 124 499	\$13422785	\$12 765 714	\$9 792 500	\$8 755 785
\$8 297 000	\$8169167	\$8 033 500	\$7 638 000	\$6 755 314	\$5 741 000	\$5 718 500
\$5 109 000	\$4 992 618	\$4 843 367	\$4 614 500	\$4 596 000	\$4 535 000	\$4 459 964
\$4 434 314	\$4 382 417	\$3 877 583	\$3 700 000	\$3 693 666	\$3 599 100	\$3 584 333

Table 6: Season earnings of PGA players earning in excess of \$3.5 million

\$14 046 909	\$10 107 897	\$9 405 081	\$9 369 605	\$8 654 566	\$7 427 299	\$7 073 986
\$7 012 672	\$6 829 575	\$6 520 597	\$6 117 886	\$5 776 298	\$5 567 974	\$5 289 842
\$5 248 220	\$5 076 060	\$5 018 443	\$4 940 600	\$4 868 461	\$4 837 271	\$4 722 433
\$4 310 047	\$3 940 513	\$3 876 590	\$3 757 425	\$3 718 990	\$3 623 137	\$3 616 679

## 4. Practical application

We now use the tests developed for the Pareto distribution (i.e., we do not include the tests based on log-transformed data) to test whether or not two observed data sets are realised from the Pareto distribution. The data sets considered are the 2022 earnings of male golf players on the PGA (Professional Golfers' Association of America) and LIV Golf tours, respectively. LIV Golf is a professional golf tour started in Saudi Arabia and financed by their sovereign wealth oil fund. LIV represents the number 54 in Roman numerals and refers to both the number of holes played at each LIV tour event, as well as the number of shots played on a standard par 72 course if every hole played results in a birdie. The inaugural season started on 9 June 2022. To make the LIV Golf tour more enticing, the organisers offer prize earnings that are much higher than those offered by the more established PGA tour.

Table 5 shows the earnings of the 28 LIV Golf tour players who earned more than \$3.5 million in the 2022 season. The average number of events played by these 26 players on the LIV Golf tour is only 7.85, whilst the average earnings is \$7 989 306 per player. Table 6 shows the earnings of the 28 PGA tour players who earned more than \$3.5 million in the 2022 season. The average number of events played by these 28 players on the PGA tour is 23.18, where this average decreases to 17.89 if one only counts the tournaments in which a player made the cut (i.e., played on the final two days of the tournament). The average earnings is \$6 098 395 per player and the average player earnings per tournament (based on 17.89 events) is \$340 882.90. From these summary statistics, as well as the entries in Tables 5 and 6, it is clear that the earnings for the top earners are much higher for those playing in the LIV tour compared to those playing in the PGA tour.

Figure 1 shows side-by-side violin plots with overlaid box plots for the earnings data given in Tables 5 and 6. Both density estimates indicate that the distributions of the earnings are positively skewed. Furthermore, both plots show evidence of the presence of outliers.

We test the hypotheses that the earnings of the top players of the PGA tour and the LIV Golf tour, respectively, are realised from a Pareto distribution. Tables 7 and 8 show the values of the test statistics as well as the corresponding  $p$ -values for the PGA and LIV Golf earnings, respectively. In each case we use both the MLE and the MME for estimation.

A striking feature of Table 7 is the substantial difference between the  $p$ -values obtained using the two estimation methods. The results indicate that the null hypothesis of the Pareto distribution is rejected by  $KS_n$ ,  $MP_n^{(1)}$ ,  $MP_n^{(3)}$  and  $G_{n,1}$  at a 5% significance level when using the MME to estimate  $\beta$ , while  $CV_n$  fails to reject the null hypothesis in this case by the smallest of margins. If we set the nominal significance level to 10%, then all of the tests considered (which require the parameter  $\beta$  to be estimated), with the exception of  $ZA_n$ , reject the null hypothesis. On the other hand, none of the tests reject the hypothesis of the Pareto distribution at the 10% level when using the MLE. In the Monte Carlo power study, we

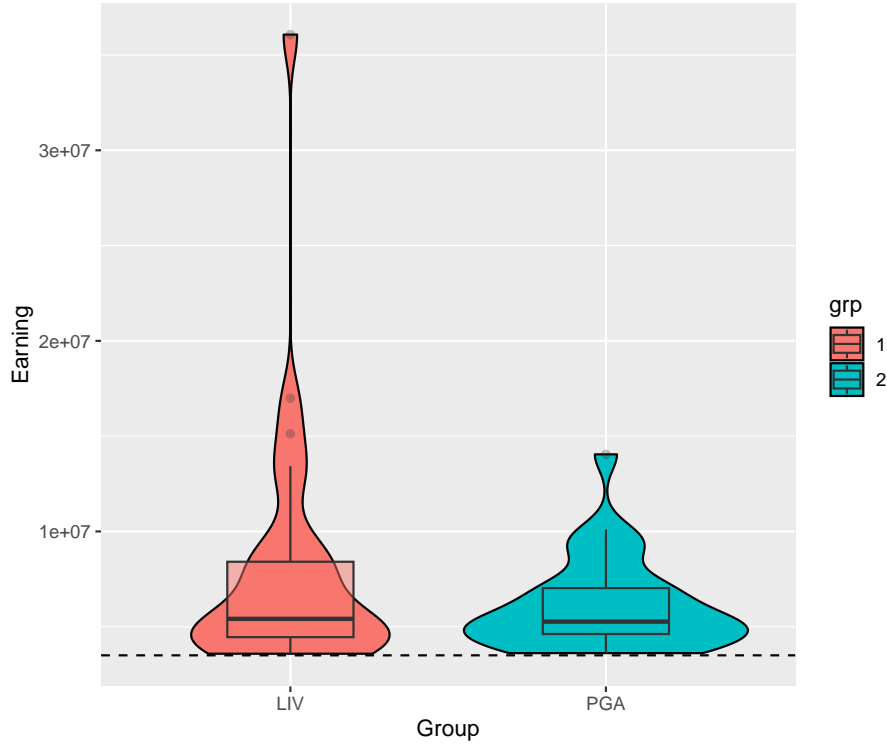


Figure 1: Violin and box plots for PGA and LIV golf earnings above \$3.5 millions

Table 7: Summary results for the PGA results and ranking

Test	MME		MLE	
	Statistic	$p$ -value	Statistic	$p$ -value
$KS_n$	0.255	0.0243	0.206	0.1284
$CV_n$	0.356	0.0505	0.177	0.2668
$AD_n$	1.655	0.0741	0.891	0.3313
$ZA_n$	3.484	0.1302	3.440	0.2273
$G_{n,1}$	0.225	0.0412	0.045	0.4480
$I_{n,2}$	0.008	0.7630	0.008	0.7630
$MP_n^{(1)}$	0.009	0.0489	0.005	0.2849
$MP_n^{(2)}$	0.004	0.5499	0.004	0.5499
$MP_n^{(3)}$	0.009	0.0448	0.004	0.3135

Table 8: Summary results for the LIV Golf results and ranking

Test	MME		MLE	
	Statistic	$p$ -value	Statistic	$p$ -value
$KS_n$	0.151	0.4865	0.119	0.7714
$CV_n$	0.110	0.5154	0.047	0.8961
$AD_n$	0.603	0.5926	0.284	0.9473
$ZA_n$	3.337	0.7631	3.309	0.9584
$G_{n,1}$	0.069	0.3791	0.004	0.8813
$I_{n,2}$	0.039	0.1522	0.039	0.1577
$MP_n^{(1)}$	0.003	0.4958	0.001	0.9212
$MP_n^{(2)}$	0.006	0.2377	0.006	0.2377
$MP_n^{(3)}$	0.003	0.4457	0.002	0.9174

recommended using  $MP_n^{(3)}$  or  $G_{n,1}$  together with the MME. As a result, the rejection of these tests using the MME casts serious doubt as to the hypothesis that the earnings associated with the PGA tour are realised from a Pareto distribution.

We now turn our attention to the LIV Golf results shown in Table 8. In this case, all tests considered fail to reject the null hypothesis for any reasonable level of significance, regardless of how  $\beta$  is estimated. We conclude that there is little doubt as to the compatibility of these data with the Pareto assumption. One possible reason for the discrepancy in the results of Tables 7 and 8 is the higher prize money associated with LIV Golf compared to the PGA tour. The heavy tail of the Pareto seems to provide a reasonable model for these extreme earnings.

## 5. Conclusions

We propose three new goodness-of-fit tests for the Pareto distribution based on a multiplicative version of the memoryless property which characterises this distribution. The resulting tests are computationally inexpensive as both test statistics require only the calculation of single summations.

A Monte Carlo study investigates the empirical powers associated with the new tests and compares these results to those of existing tests in the literature. We consider powers against fixed alternatives as well as mixture distributions. In both cases, the newly proposed tests are found to be competitive, often outperforming the existing tests against the alternative distributions considered.

A result of independent interest is to compare the powers associated with tests developed specifically for the Pareto distribution to that of tests developed for the exponential distribution when applied to log-transformed data. The simulation results indicate that the latter underperform relative to the former by some margin.

A practical example relating to the earnings of golfers is included. The seasons' earnings of golfers earning in excess of \$3.5 million in both the PGA and LIV Golf are considered separately. In each case, we test the hypothesis that the observed earnings are realised from a Pareto distribution. Interestingly, we find that the Pareto distribution provides an accurate model for the earnings from LIV Golf, while there is evidence suggesting that the earnings of the PGA are not realised from this distribution.

## Appendix

Table 9 and 10 show the empirical powers associated with the exponentiality tests based on the log-transformed data. For ease of comparison, the two highest powers (including ties) against each alternative distribution are indicated in bold.

Table 9: Numerical powers of tests based on log-transformed data with  $n = 20$

Distribution	$\widetilde{KS}_n$	$\widetilde{CV}_n$	$\widetilde{AD}_n$	$\widetilde{ZA}_n$	$\widetilde{AG}_n$	$\widetilde{CO}_n$	$\widetilde{BH}_n$
P(2)	<b>5</b>	<b>5</b>	<b>5</b>	<b>5</b>	<b>5</b>	<b>5</b>	<b>5</b>
P(5)	<b>5</b>	<b>5</b>	<b>5</b>	<b>5</b>	<b>5</b>	<b>5</b>	<b>5</b>
P(10)	<b>5</b>	<b>5</b>	<b>5</b>	<b>5</b>	<b>5</b>	<b>5</b>	<b>5</b>
$\Gamma(0.8)$	10	<b>11</b>	10	<b>11</b>	10	<b>13</b>	<b>11</b>
$\Gamma(1)$	25	31	26	28	25	<b>39</b>	<b>33</b>
$\Gamma(1.2)$	47	58	53	56	46	<b>69</b>	<b>62</b>
W(0.8)	8	<b>9</b>	<b>9</b>	<b>9</b>	8	<b>9</b>	8
W(1.2)	51	62	57	60	49	<b>73</b>	<b>66</b>
W(1.5)	82	92	90	91	83	<b>96</b>	<b>94</b>
LN(1)	55	66	65	<b>80</b>	52	<b>88</b>	66
LN(1.5)	7	7	6	<b>10</b>	9	<b>14</b>	7
LN(2.5)	26	29	<b>43</b>	26	10	<b>47</b>	25
HN(0.5)	47	59	54	54	48	<b>62</b>	<b>63</b>
HN(1)	53	65	60	60	53	<b>68</b>	<b>69</b>
HN(1.5)	62	71	69	68	61	<b>74</b>	<b>76</b>
LFR(0.2)	33	41	35	37	32	<b>47</b>	<b>44</b>
LFR(0.8)	42	53	47	48	43	<b>58</b>	<b>57</b>
LFR(1)	44	56	50	51	45	<b>61</b>	<b>60</b>
BE(0.8)	11	12	11	12	11	<b>13</b>	<b>13</b>
BE(1)	24	31	26	28	24	<b>39</b>	<b>33</b>
BE(1.5)	67	80	77	80	67	<b>89</b>	<b>83</b>
D(0.4)	28	34	30	<b>38</b>	28	<b>52</b>	35
D(0.6)	50	61	57	<b>65</b>	48	<b>80</b>	63
D(0.8)	71	82	79	<b>84</b>	69	<b>93</b>	<b>84</b>

Table 10: Numerical powers of tests based on log-transformed data with  $n = 30$ 

Distribution	$\widetilde{KS}_n$	$\widetilde{CV}_n$	$\widetilde{AD}_n$	$\widetilde{ZA}_n$	$\widetilde{AG}_n$	$\widetilde{CO}_n$	$\widetilde{BH}_n$
$P(2)$	<b>5</b>	<b>5</b>	<b>5</b>	<b>5</b>	<b>5</b>	<b>5</b>	<b>5</b>
$P(5)$	<b>5</b>	<b>5</b>	<b>5</b>	<b>5</b>	<b>5</b>	<b>5</b>	<b>5</b>
$P(10)$	<b>5</b>	<b>5</b>	<b>5</b>	<b>5</b>	<b>5</b>	<b>5</b>	<b>5</b>
$\Gamma(0.8)$	13	<b>14</b>	13	<b>14</b>	12	<b>14</b>	<b>15</b>
$\Gamma(1)$	36	45	40	44	33	<b>53</b>	<b>49</b>
$\Gamma(1.2)$	64	77	73	77	61	<b>85</b>	<b>81</b>
$W(0.8)$	10	<b>11</b>	<b>11</b>	<b>12</b>	10	<b>11</b>	<b>11</b>
$W(1.2)$	69	82	79	81	66	<b>89</b>	<b>86</b>
$W(1.5)$	96	99	99	99	96	<b>100</b>	<b>100</b>
$LN(1)$	76	85	87	<b>96</b>	72	<b>98</b>	86
$LN(1.5)$	8	9	8	<b>15</b>	10	<b>20</b>	9
$LN(2.5)$	39	43	<b>58</b>	39	24	<b>61</b>	36
$HN(0.5)$	67	78	75	77	65	<b>80</b>	<b>83</b>
$HN(1)$	72	<b>84</b>	81	83	72	<b>84</b>	<b>88</b>
$HN(5.5)$	77	89	86	89	76	<b>90</b>	<b>95</b>
$LFR(0.2)$	46	56	51	56	43	<b>62</b>	<b>61</b>
$LFR(0.8)$	60	72	68	71	58	<b>76</b>	<b>77</b>
$LFR(1)$	64	77	72	74	61	<b>79</b>	<b>81</b>
$BE(0.8)$	14	<b>17</b>	15	<b>17</b>	14	<b>17</b>	<b>17</b>
$BE(1)$	35	44	39	42	32	<b>52</b>	<b>49</b>
$BE(1.5)$	85	94	93	95	84	<b>98</b>	<b>96</b>
$D(0.4)$	42	50	48	<b>56</b>	40	<b>70</b>	52
$D(0.6)$	71	81	80	<b>85</b>	67	<b>94</b>	84
$D(0.8)$	89	95	95	<b>96</b>	86	<b>99</b>	<b>96</b>

## References

- Ahmad IA, Alwasel IA (1999). “A Goodness-of-Fit Test for Exponentiality Based on the Memoryless Property.” *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, **61**(3), 681–689. doi:10.1111/1467-9868.00200.
- Allison JS, Milošević B, Obradović M, Smuts M (2022). “Distribution-Free Goodness-of-Fit Tests for the Pareto Distribution Based on a Characterization.” *Computational Statistics*, **37**(1), 403–418. doi:10.1007/s00180-021-01126-y.
- Allison JS, Santana L, Smit N, Visagie I (2017). “An ‘Apples to Apples’ Comparison of Various Tests for Exponentiality.” *Computational Statistics*, **32**(4), 1241–1283. doi:10.1007/s00180-017-0733-3.
- Alwasel I (2001). “On Goodness of Fit Testing of Exponentiality Using the Memoryless Property.” *Journal of Nonparametric Statistics*, **13**(4), 569–581. doi:10.1080/10485250108832865.
- Angus JE (1982). “Goodness-of-Fit Tests for Exponentiality Based on a Loss-Of-Memory Type Functional Equation.” *Journal of Statistical Planning and Inference*, **6**(3), 241–251. doi:10.1016/0378-3758(82)90029-5.
- Arnold BC (2015). *Pareto Distributions*. CRC Press, New York. doi:10.1201/b18141.
- Baringhaus L, Henze N (2000). “Tests of Fit for Exponentiality Based on a Characterization via the Mean Residual Life Function.” *Statistical Papers*, **41**(2), 225–236. doi:10.1007/bf02926105.
- Bourguignon M, Saulo H (2018). “On a New Approach to Estimate the Shape Parameter of the Inverse Gaussian Distribution.” *South African Statistical Journal*, **52**(1), 15–27. doi:10.37920/sasj.2018.52.1.2.
- Chu J, Dickin O, Nadarajah S (2019). “A Review of Goodness of Fit Tests for Pareto Distributions.” *Journal of Computational and Applied Mathematics*, **361**, 13–41. doi:10.1016/j.cam.2019.04.018.
- Cox DR, Oakes D (1984). *Analysis of Survival Data*, volume 21. CRC press. doi:10.2307/2289259.
- Cuparić M, Milošević B, Obradović M (2022). “Asymptotic Distribution of Certain Degenerate V-and U-Statistics with Estimated Parameters.” *Mathematical Communications*, **27**(1), 77–100. doi:https://hrcak.srce.hr/275694.
- Galambos J, Kotz S (2006). *Characterizations of Probability Distributions.: A Unified Approach with an Emphasis on Exponential and Related Models*, volume 675. Springer. doi:10.1007/s00362-007-0103-2.
- Giacomini R, Politis DN, White H (2013). “A Warp-Speed Method for Conducting Monte Carlo Experiments Involving Bootstrap Estimators.” *Econometric Theory*, **29**, 567–589. doi:10.1017/s0266466612000655.
- Henze N, Meintanis SG (2005). “Recent and Classical Tests for Exponentiality: A Partial Review with Comparisons.” *Metrika*, **61**, 29–45. doi:10.1007/s001840400322.
- Jiménez-Gamero MD, Milošević B, Obradović M (2020). “Exponentiality Tests Based on Basu Characterization.” *Statistics*, **54**(4), 714–736. doi:10.1080/02331888.2020.1774768.
- Meintanis SG (2009). “A Unified Approach of Testing for Discrete and Continuous Pareto Laws.” *Statistical Papers*, **50**(3), 569–580.



- Meintanis SG, Milošević B, Jiménez-Gamero MD (2024). “Goodness-of-Fit Tests Based on the Min-Characteristic Function.” *Computational Statistics & Data Analysis*, p. 107988. doi:10.1016/j.csda.2024.107988.
- Ndwandwe L, Allison JS, Santana L, Visagie IJH (2023). “Testing for the Pareto Type I Distribution: A Comparative Study.” *Metron*, **81**, 215–256. doi:10.1007/s40300-023-00252-5.
- Ndwandwe LM, Allison JS, Visagie IJH (2021). “A New Fixed Point Characterisation Based Test for the Pareto Distribution in the Presence of Random Censoring.” *Proceedings of the South African Statistical Association*, **62**, 17–23. doi:hdl.handle.net/10520/ejc-sasj\_proc\_v2021\_n1\_a3.
- Ngatchou-Wandji J, Nombebe T, Santana L, Allison J (2024). “On Classes of Consistent Tests for the Type I Pareto Distribution Based on a Characterization Involving Order Statistics.” *Statistics*, pp. 1–31. doi:10.1080/02331888.2024.2347342.
- Nofal ZM, El Gebaly YM (2017). “New Characterizations of the Pareto Distribution.” *Pakistan Journal of Statistics and Operation Research*, **13**, 63–74. doi:10.18187/pjsor.v13i1.1742.
- Pareto V (1897). *Cours D’economie Politique, Vol. II*. Lausanne: F. Rouge.
- R Core Team (2020). *R: A Language and Environment for Statistical Computing*. R Foundation for Statistical Computing, Vienna, Austria. URL <http://www.R-project.org/>.
- Rytgaard M (1990). “Estimation in the Pareto Distribution.” *ASTIN Bulletin: The Journal of the IAA*, **20**(2), 201–216. doi:10.2143/ast.20.2.2005443.
- Tomy L, Jose M, Veena G (2020). “A Review on Recent Generalizations of Exponential Distribution.” *Biometrics & Biostatistics International Journal*, **9**, 152–156. doi:10.15406/bbij.2020.09.00313.
- Torabi H, Montazeri NH, Grané A (2018). “A Wide Review on Exponentiality Tests and Two Competitive Proposals with Application on Reliability.” *Journal of Statistical Computation and Simulation*, **88**(1), 108–139. doi:10.1080/00949655.2017.1379522.
- Zhang J (2002). “Powerful Goodness-Of-Fit Tests Based on the Likelihood Ratio.” *Journal of the Royal Statistical Society: Series B (Statistical Methodology)*, **64**(2), 281–294. doi:10.1111/1467-9868.00337.

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