Austrian Journal of Statistics
2023, Volume 52, 71–81.
http://www.ajs.or.at/
doi:10.17713/ajs.v52iSI.1757

Analytic Solutions of Equation for Random Evolution on a Complex Plane

Igor Samoilenko
Taras Shevchenko National University of Kyiv

Ganna Verovkina
Taras Shevchenko National University of Kyiv

Tetiana Samoilenko
National Technical University of Ukraine "Igor Sikorsky Kyiv Polytechnic Institute"

Abstract

We discuss a generalization of Goldstein-Kac model on a complex plane and apply probabilistic approach to construct solutions of the corresponding Cauchy problem for complex-analytic initial conditions. The method is based on reconstruction of complex-analytic functions by combination of power functions, for which corresponding solutions are the moments of evolution process.

As soon as in the hydrodynamic limit the equation for our model approximates a Schrödinger-type equation, the solutions constructed for pre-limit Cauchy problem may approximate solutions for corresponding Cauchy problem for a Schrödinger-type equation.

Keywords: Goldstein-Kac model, complex plane, Cauchy problem, complex-analytic initial conditions, Schrödinger-type equation.

1. Introduction

Numerous works are devoted to the description of random evolutions (random flights) that generalize original work Kac (1974) in the multidimensional case. They are mostly devoted to discussion of convergence of the model studied to the Wiener process, also description of corresponding equations and solving them for some well posed modes (Orsingher (1986), Orsingher (2002), Orsingher and De Gregorio (2007), Orsingher and Sommella (2004), Orsingher, Garra, and Zeifman (2020), Cinque and Orsingher (2023), Pinsky (1991), Pogorui and Rodríguez-Dagnino (2019), Pogorui, Swishchuk, and Rodríguez-Dagnino (2021), Kolesnik (2001), Kolesnik (2007), Kolesnik (2008), Kolesnik and Orsingher (2005), Samoilenko (2001a), Samoilenko (2002) and many others). The main problem is that the methods for solving equations proposed there can not be applied for any model as soon as they are, as a rule, strictly connected with the structure of the corresponding equation. In the article of Turbin and Samoilenko (2000) we proposed to change the approach, namely to solve a well posed Cauchy problem instead of a well posed equation. The idea is based on the use of real-analytic initial conditions and computation of the moments of Markov random evolution (see also Samoilenko (2001b)) which solve Cauchy problem for any evolutionary equation with pover functions as initial conditions. Thus, having corresponding solutions, we may approximate solution for any initial condition by real-analytic functions.

Here we apply the method, proposed in Turbin and Samoilenko (2000) to equations that appear in the case of Goldstein-Kac model on a complex plane, namely

$$\gamma_{r,z}^{\lambda,v}(t) = x + iy + v\int_0^t (-1)^{\xi_r^\lambda(s)} ds$$

$$+iv\int_0^t (-1)^{\xi_r^\lambda(s)} ds = z + (i+1)v\int_0^t (-1)^{\xi_r^\lambda(s)} ds,$$

where $x + iy$ is the starting point, $v > 0$ is the constant velocity of movement, $\xi_r^\lambda(s)$ is the Markov chain that takes values in $\{0, 1\}$ and has the infinitesimal matrix

$$Q_\lambda = \lambda \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}$$

and initial distribution $P\{\xi_r^\lambda(0) = 0\} = p, P\{\xi_r^\lambda(0) = 1\} = q, r = p - q$.

In Section 2 we show that the expectations of function of the evolution, namely the functionals defined in (5) satisfy Cauchy problem:

$$\frac{\partial^2 U}{\partial t^2} + 2\lambda\frac{\partial U}{\partial t} = 2iv^2\frac{\partial^2 U}{\partial z^2} \tag{1}$$

$$U(0, z) = f(z), \frac{\partial}{\partial t}U(t, z)|_{t=0} = rv(1 + i)f'(z).$$

In Section 3 this result is used to represent the solution to the Cauchy problem with complex-analytic initial conditions:

$$\begin{cases} f(z) = \sum_{k=0}^\infty f_k z^k, z \in C; \\ g(z) = \sum_{k=0}^\infty g_k z^k, z \in C. \end{cases} \tag{2}$$

Remark. *Solution of this Cauchy problem, like in the case of the one-dimensional telegraph equation, may be constructed by Riemann method in terms of Bessel functions (see, e.g. Pinsky (1991), Orsingher (1986), Orsingher (2002), Orsingher* et al. *(2020), Lachal, Leorato, and Orsingher (2006), Samoilenko (2001a)). But the probability approach proposed here is based on the explicit calculation of arbitrary moments of the random process and allows to avoid analytic difficulties and, moreover, to obtain new formulas for the Bessel functions. It can also be used in the study of models described by equations that are more complex than the one-dimensional telegraph equation.*

There is also another reason for the construction of a new representation of the solution of this Cauchy problem. Solution in terms of Bessel functions does not explicitly contain boundary-layer functions, which is explained by the fact that Riemann method is based on the analytic-geometric approach. The solutions constructed in the present paper explicitly contain the regular and boundary-layer components that may be usefull for calculation of approximate solutions.

Namely, a consequence of our representation of the solution of Cauchy problem is the following result. We set $\varepsilon = \frac{1}{2\lambda}$ and write equation (1) in the form

$$\varepsilon\frac{\partial^2}{\partial t^2}U(t, z) = \left(i\frac{v^2}{2\lambda}\frac{\partial^2}{\partial z^2} - \frac{\partial}{\partial t}\right)U(t, z). \tag{3}$$

In the hydrodynamic limit (see Kac (1974), Korolyuk and Turbin (1993)), when $v \to \infty$, $\lambda \to \infty$ so that $\frac{v^2}{\lambda} \to \sigma^2$ equation (3) has the form of a singularly perturbed differential equation with a small parameter for the highest derivative with respect to t. In the limit, we have a Schrödinger-type equation (see, e.g. free-particle wave equation):

$$\frac{\partial}{\partial t}U(t,z) = \frac{\sigma^2}{2} i \frac{\partial^2}{\partial z^2} U(t,z). \tag{4}$$

Solutions of Schrödinger-type equations may be found in the form of series for corresponding complex functions. Now we see that Schrödinger-type equation may be approximated by a singularly perturbed differential equation with a small parameter as the coefficient of the higher derivative with respect to t.

We show that the solution of Cauchy problem for the last equation also may be presented as a series. Moreover, in the theory of singularly perturbed evolutionary equations, solutions often contain a regular (with respect to ε) component and a boundary-layer component, which contains a function of the form $\exp(-t/\varepsilon)$ (in our case, $\exp(-2\lambda t) = \exp(-t/\varepsilon)$). The solutions constructed in the present paper explicitly contain the regular and boundary-layer components.

Thus, we will now construct solutions for complex-analytic initial conditions of Cauchy problem (1), (2). As soon as other functions may be approximated by complex-analytic functions, we may thus approximate solutions for Cauchy problems with other initial conditions, and, assuming $\varepsilon \to 0$ we shall see approximate solution of Cauchy problem for Schrödinger-type equation (4).

2. Cauchy problem for functionals of the evolution

Let us consider the functionals of the process $\gamma^{\lambda,v}_{r,z}(t)$ of the form

$$U_j(t,z) = E_j f(z + (i+1)v \int_0^t (-1)^{\xi^\lambda_r(s)} ds), \tag{5}$$

where $j \in \{0,1\}$ is the state of the process $\xi^\lambda_r(s)$ at the moment of time $s = 0$.

Theorem 1. *Let f be a continuously differentiable function with a compact support. Then the functions $U_j(t,z)$ defined in (5) satisfy the system of backward Kolmogorov equations:*

$$\begin{cases} \frac{\partial U_0}{\partial t} = (i+1)v\frac{\partial U_0}{\partial z} - \lambda U_0 + \lambda U_1 \\ \frac{\partial U_1}{\partial t} = -(i+1)v\frac{\partial U_0}{\partial z} - \lambda U_1 + \lambda U_0. \end{cases} \tag{6}$$

Proof. Consider the function $U_0(t,z)$:

$$U_0(t+\triangle t, z) = E_0 f(z + (1+i)v \int_0^{t+\triangle t} (-1)^{\xi^\lambda_1(s)} ds)$$

$$= Ef(z + (1+i)v[\int_0^{\triangle t} (-1)^{\xi^\lambda_1(s)} ds + \int_{\triangle t}^{t+\triangle t} (-1)^{\xi^\lambda_{-1}(s)} ds]).$$

The probability that the state of the system will change during the time interval $\triangle t$ is equal to $\lambda \triangle t$, therefore, as soon as $(-1)^{\xi^\lambda_{-1}} = (-1)^{1+\xi^\lambda_1}$:

$$U_0(t+\triangle t, z) = (1 - \lambda \triangle t) Ef(z + (i+1)v \int_{\triangle t}^{t+\triangle t} (-1)^{\xi^\lambda_1(s)} ds$$

$$+(i+1)v \triangle t) + \lambda \triangle t Ef(z + (i+1)v \int_{\triangle t}^{t+\triangle t} (-1)(-1)^{\xi^\lambda_1(s)} ds + O(\triangle t))$$

$$+o(\triangle t), \triangle t \to 0.$$

Due to the fact that the process is stationary $P\{\xi_1^\lambda(t+\triangle t)=k|\xi_1^\lambda(\triangle t)=j\}=P\{\xi_1^\lambda(t)=k|\xi_1^\lambda(0)=j\}$, thus we have:

$$U_0(t+\triangle t,z)-U_0(t,z)=Ef(z+(i+1)v\int_0^t(-1)^{\xi_1^\lambda(s)}ds$$

$$+(i+1)v\triangle t)-Ef(z+(i+1)v\int_0^t(-1)^{\xi_1^\lambda(s)}ds)$$

$$-\lambda\triangle tEf(z+(i+1)v\int_0^t(-1)^{\xi_1^\lambda(s)}ds+(i+1)v\triangle t)$$

$$+\lambda\triangle tEf(z+(i+1)v\int_0^t(-1)^{\xi_{-1}^\lambda(s)}ds+O(\triangle t))+o(\triangle t),\triangle t\to 0.$$

Dividing by $\triangle t$ and tending $\triangle t\to 0$ we have:

$$\frac{\partial U_0}{\partial t}=-\lambda U_0+\lambda U_1+(i+1)v\frac{\partial U_0}{\partial z}.$$

It is necessary to justify the passage to the limit:

$$\lim_{\triangle t\to 0}\frac{Ef(z+(i+1)v\triangle t+v\int_0^t(-1)^{\xi_1^\lambda(s)}ds)-Ef(z+v\int_0^t(-1)^{\xi_1^\lambda(s)}ds)}{\triangle t}$$

$$=E\lim_{\triangle t\to 0}\frac{f(z+(1+i)v\triangle t+v\int_0^t(-1)^{\xi_1^\lambda(s)}ds)-f(z+v\int_0^t(-1)^{\xi_1^\lambda(s)}ds)}{\triangle t}$$

$$=(i+1)vE_0f'_z=\frac{\partial U_0}{\partial z}(i+1)v.$$

The limit can be carried under the expectation sign due to Lebesgue's theorems on passage to the limit under the integral sign, pointwise convergence of the expression under the sign of mathematical expectations to $(i+1)vf'_z$ (because f is continuously differentiable) and boundedness of the integrand, which is different from 0 on a compact support.

By analogy, we have $\frac{\partial U_1}{\partial t}=-(i+1)v\frac{\partial U_1}{\partial z}-\lambda U_1+\lambda U_0$. This completes the proof of Theorem 1. □

According to the result in Kolesnik and Turbin (1991) (see also Pinsky (1991)), the functions satisfying system (6) satisfy equation:

$$det\begin{pmatrix}\frac{\partial}{\partial t}+\lambda-v(1+i)\frac{\partial}{\partial z} & -\lambda\\ -\lambda & \frac{\partial}{\partial t}+\lambda+v(1+i)\frac{\partial}{\partial z}\end{pmatrix}U(t,z)=0,$$

where $U(t,z)$ means $U_j(t,z),j=\overline{0,1}$.

Calculating the determinant, we have:

$$\frac{\partial^2 U}{\partial t^2}+2\lambda\frac{\partial U}{\partial t}=2iv^2\frac{\partial^2 U}{\partial z^2} \tag{7}$$

Let us formulate the Cauchy problem for equation (7), whose solutions are the functions $U_j(t,z)$. From the definition of functions we have: $U_j(t,z)=E_jf(z+(i+1)v\int_0^t(-1)^{\xi(s)}ds)$ and for $t=0$ we get $U_j(0,z)=E_jf(z)=f(z)$. For initial distribution $P\{\xi(0)=0\}=p,P\{\xi(0)=1\}=q$ we get $U(0,z)=pU_0(0,z)+qU_1(0,z)=f(z)$.

From the system of backward Kolmogorov equations we have:

$$\frac{\partial U_0(t,z)}{\partial t}|_{t=0}=-\lambda U_0(t,z)|_{t=0}+\lambda U_1(t,z)|_{t=0}$$

$$+(i+1)v\frac{\partial U_0(t,z)}{\partial z}|_{t=0}=-\lambda f(z)+\lambda f(z)+(i+1)vE_0f'_z|_{t=0}$$

$$= (i+1)vE_0 f_z'(z) = (i+1)vf_z'(z).$$

By analogy, $\frac{\partial U_1(t,z)}{\partial t}|_{t=0} = -(i+1)vf_z'(z)$. Taking into account the initial distribution, we have: $U_t'(t,z)|_{t=0} = p(U_0)_t' - q(U_1)_t' = (p-q)(i+1)vf'(z) = (i+1)vrf'(z)$.
Thus, the Cauchy problem for equation (7) has the form:

$$U(0,z) = f(z), U_t'(t,z)|_{t=0} = (i+1)vrf'(z). \tag{8}$$

We proved the following theorem.

Theorem 2. *The function $U(t,z) = pU_0(t,z) + qU_1(t,z)$ satisfies equation (7) with initial conditions (8).*

3. Solution of Cauchy problem

Let $c_r^{\lambda,v}(t,z;n) = E\left(\gamma_{r,z}^{\lambda,v}(t)\right)^n$, then the function $c_r^{\lambda,v}(t,z;n)$ solves equation (7) with initial conditions

$$f(z) = z^n; g(z) = rv(i+1)nz^{n-1}. \tag{9}$$

If $r = 0$, and thus in (9) $g(z) = 0$, then the solution $U(t,z)$ of the Cauchy problem

$$U(0,z) = f(z), U_t'(t,z)|_{t=0} = 0 \tag{10}$$

with complex analytic function $f(z) = \sum_{k=0}^{\infty} f_k z^k$ has the form

$$u_0(t,z) = \sum_{k=0}^{\infty} f_k c_0^{\lambda,v}(t,z;k).$$

If $r \neq 0$ then the function $\frac{1}{rvm(i+1)}\left[c_r^{\lambda,v}(t,z;m) - c_0^{\lambda,v}(t,z;m)\right]$ for $m \geq 1$ is a solution of equation (7) with initial conditions

$$f(z) = 0,\ g(z) = z^{m-1}$$

and then the function

$$u_r(t,z) = \sum_{m=1}^{\infty} \frac{g_{m-1}}{rvm(i+1)}\left[c_r^{\lambda,v}(t,z;m) - c_0^{\lambda,v}(t,z;m)\right]$$

solves (7) with initial conditions

$$f(z) = 0,\ g(z) = \sum_{m=1}^{\infty} g_{m-1} z^{m-1}.$$

Hence the function

$$u_0(t,z) + u_r(t,z) = f_0 c_0^{\lambda,v}(t,z;0) + \sum_{k=1}^{\infty} [f_k c_0^{\lambda,v}(t,z;k) + \frac{g_{k-1}}{rvk(i+1)}\left(c_r^{\lambda,v}(t,z;k) - c_0^{\lambda,v}(t,z;k)\right)]$$

is the solution of (7) with initial conditions $f(z) = \sum_{k=0}^{\infty} f_k z^k$, $g(z) = \sum_{k=0}^{\infty} g_k z^k$. Since the function $\frac{g_0}{2\lambda}\left(1 - e^{-2\lambda t}\right)$ solves (7) with initial conditions $f(z) = 0,\ g(z) = g_0$ we arrive at the next result.

Theorem 3. *Let $c_r^{\lambda,v}(t,z;n)$ be a solution of equation (7) with initial conditions (9). Then the solution $U(t,z)$ of the Cauchy problem (8) with complex analytic conditions (2) is given by*

$$U(t,z) = f_0 + \frac{g_0}{2\lambda}\left(1 - e^{-2\lambda t}\right) + \sum_{k=1}^{\infty} [f_k c_0^{\lambda,v}(t,z;k) + \frac{g_{k-1}}{rvk(i+1)}\left(c_r^{\lambda,v}(t,z;k) - c_0^{\lambda,v}(t,z;k)\right)]. \tag{11}$$

The problem of constructing the solution is thus reduced to calculating the moments $c_r^{\lambda,v}(t,z;k)$ of evolution defined by Kac model on a complex plane.

4. Moments of evolution on a complex plane

In this section we use the ideas of the article Turbin and Samoilenko (2000) to calculate moments of evolution on a complex plane. Note that the trivial substitution of new coefficients into the formulas of the work Turbin and Samoilenko (2000) does not give the desired result.

Lemma 1. *The following equality holds true:*

$$\lim_{\substack{\lambda\to\infty\\ v\to\infty\\ \frac{v^2}{\lambda}\to\sigma^2}} c_r^{\lambda,v}(t,z;n) = \sum_{j=0}^{\left[\frac{n}{2}\right]} \binom{n}{2j} z^{n-2j}\mu_{2j}\left(\sigma^2 t\right)^j (i+1)^j, \tag{12}$$

where $[p]$ *is the integer part of* p,

$$\mu_{2j} = \begin{cases} 1\cdot 3\cdot\ldots\cdot(2j-1) \ if\ j \text{ is an even number;} \\ 0\ if\ j \text{ is an odd number.}\end{cases}$$

Proof. By definition

$$c_{r,z}^{\lambda,v} = E\left(z + v(i+1)\int_0^t (-1)^{\xi_r^\lambda(s)}ds\right)^n = \sum_{j=0}^{n}\binom{n}{j} z^{n-j} E\left(v(i+1)\int_0^t (-1)^{\xi_r^\lambda(s)}ds\right)^j.$$

From the weak convergence of the process $v\int_0^t(-1)^{\xi_r^\lambda(s)}ds$ for $\lambda\to\infty$, $v\to\infty$, $\frac{v^2}{\lambda}\to\sigma^2$ to the Wiener process (see Kac (1974), Pinsky (1991)) $\sigma w(t)$, $Var\ w(t) = t$, it follows that

$$\lim_{\substack{\lambda\to\infty\\ v\to\infty\\ \frac{v^2}{\lambda}\to\sigma^2}} E\left[v(i+1)\int_0^t (-1)^{\xi_r^\lambda(s)}ds\right]^n = (i+1)^n\mu_n\left(\sigma^2 t\right)^n,$$

which gives (12). The proof of Lemma 1 is complete. □

The lemma allows one to seek a solution to equation (7) with initial conditions (9) in the form (see also Samoilenko (2001b), Kolesnik (2012)):

$$c_r^{\lambda,v}(t,z;n) = z^n + \frac{rvnz^{n-1}}{2\lambda}\left(1-e^{-2\lambda t}\right)(i+1) + \sum_{j=1}^{\left[\frac{n}{2}\right]}\binom{n}{2j} z^{n-2j}\mu_{2j}\left(\frac{v^2}{\lambda}t\right)^j (i+1)^j$$
$$+a_n(t,z) + b_n(t,z)\left(1-e^{-2\lambda t}\right), \tag{13}$$

where the functions $a_n(t,z)$ and $c_n(t,z)$ are polynomials in z (for a fixed t) and in t (for a fixed z) which degree is at most $n-2$. Since $c_r^{\lambda,v}(0,z;n) = z^n$, then the condition

$$a(0,z) = 0 \tag{14}$$

must be necessarily true. Differentiating (13) with respect to t, we find

$$\frac{\partial}{\partial t}c_r^{\lambda,v}(t,z;n) = rv(i+1)nz^{n-1}e^{-2\lambda t} + \sum_{j=1}^{\left[\frac{n}{2}\right]}\binom{n}{2j} z^{n-2j} j\mu_{2j}\left(\frac{v^2}{\lambda}\right)^j t^{j-1}(i+1)^j$$
$$+\frac{\partial}{\partial t}a(t,z) + \left(\frac{\partial}{\partial t}b(t,z)\right)\left(1-e^{-2\lambda t}\right) - 2\lambda b(t,z)e^{-2\lambda t}.$$

As soon as $\frac{\partial}{\partial t}c_r^{\lambda,v}(t,z;n)\Big|_{t=0} = rv(i+1)z^{n-1}$, we obtain one more condition on $a_n(t,z)$ and $c_n(t,z)$:

$$\binom{n}{2} z^{n-2}\mu_2\left(\frac{v^2}{\lambda}\right)^2 (i+1) + \frac{\partial}{\partial t}a_n(t,z)\Big|_{t=0} = 2\lambda b_n(0,z). \tag{15}$$

Theorem 4. *The functions* $a_n(t,z)$ *and* $b_n(t,z)$ *are given by:*
if n is even:

$$a_n(t,z) = \sum_{j=1}^{\frac{n}{2}-1}\sum_{k=1}^{\frac{n}{2}-j} l_j^{(k)} z^{n-2(k+j)} t^k (i+1)^{2(k+j)} + \sum_{j=0}^{\frac{n}{2}-2}\sum_{k=1}^{\frac{n}{2}-j-1} d_j^{(k)} z^{n-2(k+j)-1} t^k (i+1)^{2(k+j)+1},$$

$$b_n(t,z) = \sum_{j=0}^{\frac{n}{2}-1}\sum_{k=1}^{\frac{n}{2}-j} e_j^{(k)} z^{n-2(k+j)} t^{k-1} (i+1)^{2(k+j)} + \sum_{j=1}^{\frac{n}{2}-1}\sum_{k=1}^{\frac{n}{2}-j} f_j^{(k)} z^{n-2(k+j)+1}$$

$$\times t^{k-1}(i+1)^{2(k+j)-1} + \sum_{k=2}^{\frac{n}{2}} f_0^{(k)} z^{n-2k+1} t^{k-1} (i+1)^{2k-1};$$

if n is odd:

$$a_n(t,z) = \sum_{j=1}^{[\frac{n}{2}]-1}\sum_{k=1}^{[\frac{n}{2}]-j} l_j^{(k)} z^{n-2(k+j)} t^k (i+1)^{2(k+j)} + \sum_{j=0}^{[\frac{n}{2}]-2}\sum_{k=1}^{[\frac{n}{2}]-j-1} d_j^{(k)} z^{n-2(k+j)-1} t^k (i+1)^{2(k+j)+1},$$

$$b_n(t,z) = \sum_{j=0}^{[\frac{n}{2}]-1}\sum_{k=1}^{[\frac{n}{2}]-j} e_j^{(k)} z^{n-2(k+j)} t^{k-1} (i+1)^{2(k+j)} + \sum_{j=1}^{[\frac{n}{2}]}\sum_{k=1}^{[\frac{n}{2}]-j+1} f_j^{(k)} z^{n-2(k+j)+1}$$

$$\times t^{k-1}(i+1)^{2(k+j)-1} + \sum_{k=2}^{[\frac{n}{2}]+1} f_0^{(k)} z^{n-2k+1} t^{k-1} (i+1)^{2k-1}.$$

Here

$$e_j^{(1)} = \frac{l_j^{(1)}}{2\lambda};$$

$$e_j^{(k)} = \frac{1}{2\lambda(k-1)}\left(e_{j-1}^{(k+1)}\cdot k\cdot(k-1) - v^2 e_j^{(k-1)}\cdot(n-2(k+j-1))\right.$$
$$\left.\times(n-2(k+j-1)-1)\right),\ k\neq 1;$$

$$l_j^{(k)} = \frac{1}{2\lambda k}\left(2\lambda e_{j-1}^{(k+1)}\cdot k - l_{j-1}^{(k+1)}\cdot(k+1)\cdot k + e_{j-2}^{(n+1)}\cdot(k+1)\cdot k\right.$$
$$+v^2 l_j^{(k-1)}\cdot(n-2(k+j-1))\cdot(n-2(k+j-1)-1)$$
$$\left.-v^2 e_{j-1}^{(k)}\cdot(n-2(k+j-1))\cdot(n-2(k+j-1)-1)\right);$$

$$f_j^{(1)} = \frac{d_{j-1}^{(1)}}{2\lambda};$$

$$f_j^{(k)} = \frac{1}{2\lambda(k-1)}\left(f_{j-1}^{(k+1)}\cdot k\cdot(k-1) - v^2 f_j^{(k-1)}\cdot(n-2(k+j-1)+1)\right.$$
$$\left.\times(n-2(k+j-1))\right),\ k\neq 1;$$

$$d_j^{(k)} = \frac{1}{2\lambda k}\left(2\lambda f_j^{(k+1)}\cdot k - d_{j-1}^{(k+1)}\cdot(k+1)\cdot k\right.$$
$$+f_{j-1}^{(k+2)}\cdot(k+1)\cdot k + v^2 d_j^{(k-1)}\cdot(n-2(k+j-1)-1)\cdot(n-2(k+j-1))$$
$$\left.-v^2 f_j^{(k)}\cdot(n-2(k+j)+1)(n-2(k+j))\right).$$

In doing so, we assume $b_0^{(k)} = \binom{n}{2k}\mu_{2k}\left(\frac{v^2}{\lambda}\right)^k,\ k=0,\ldots,[\frac{n}{2}]$ *and ,* $f_0^{(1)} = -\frac{rvn}{2\lambda}$*, and in the case when k and j go beyond the specified limits of change we put* $b_j^{(k)} = d_j^{(k)} = e_j^{(k)} = f_j^{(k)} = 0$.

Proof. We put $\nu_n(t,z) = \sum_{k=1}^{\left[\frac{n}{2}\right]} \binom{n}{2k} z^{n-2k}\mu_{2k}\left(\frac{v^2}{\lambda}t\right)^k (i+1)^{2k} + a_n(t,z)$, then $c_r^{\lambda,v}(t,z;n) = z^n + \frac{rv(i+1)nz^{n-1}}{2\lambda}\left(1-e^{-2\lambda t}\right) + \nu_n(t,z) + b_n(t,z)\left(1-e^{-2\lambda t}\right)$.

Substitute this expression into equation (7), and then group and equate the free terms and the terms at $e^{-2\lambda t}$. We get:

$$\frac{\partial^2}{\partial t^2}b_n - 2\lambda\frac{\partial}{\partial t}b_n = (i+1)^2v^2\frac{\partial^2}{\partial z^2}b_n - \frac{n(n-1)(n-2)rv^3(i+1)^3z^{n-3}}{2\lambda} \tag{16}$$

and

$$\frac{\partial^2}{\partial t^2}\nu_n - \frac{\partial^2}{\partial t^2}b_n + 2\lambda\frac{\partial}{\partial t}\nu_n - 2\lambda\frac{\partial}{\partial t}b_n = v^2(i+1)^2(n-1)nz^{n-2}$$
$$+\frac{n(n-1)(n-2)rv^3(i+1)^3z^{n-3}}{2\lambda} + v^2(i+1)^2\frac{\partial^2}{\partial z^2}\nu_n - v^2(i+1)^2\frac{\partial^2}{\partial z^2}b_n. \tag{17}$$

Equality (15) may now be written as:

$$\left.\frac{\partial}{\partial t}\nu_n(t,z)\right|_{t=0} = 2\lambda b_n(0,t). \tag{18}$$

The polynomial $\nu_n(t,z)$ includes the sum $\sum_{k=1}^{\left[\frac{n}{2}\right]} \binom{n}{2k} z^{n-2k}\mu_{2k}\left(\frac{v^2}{\lambda}t\right)^k (i+1)^{2k}$, denote the coefficient of $z^{n-2k}t^k(i+1)^{2k}$ by $l_0^{(k)}$. Since in relations (16) and (17) connecting polynomials ν_n and b_n, differentiation with respect to z is performed twice, the degree of each next term in z will be 2 less than the previous one, so the coefficient of $z^{n-2(k+j)}t^k(i+1)^{2(k+j)}$ is denoted by $l_j^{(k)}$.

It can be seen from (18) that the polynomial $b_n(t,z)$ contains the term $\frac{l_0^{(1)}}{2\lambda}z^{n-2}$, denote $e_0^{(1)} = \frac{l_0^{(1)}}{2\lambda}$, and similarly to the previous one, the coefficient for $z^{n-2(k+j)}t^{k-1}(i+1)^{2(k+j)}$ via $e_j^{(k)}$. If in relation (16) we collect the coefficients at the powers of z and t coinciding after differentiation, we get:

$$e_{j-1}^{(k+1)}\cdot(k-1)\cdot k - 2\lambda(k-1)e_j^{(k)} = v^2e_j^{(k-1)}\cdot(n-2(k+j-1))\cdot(n-2(k+j-1)-1).$$

Since coefficients with smaller indices were found earlier, for example, $e_0^{(1)}$ is already known, while $e_{-1}^{(3)}$ is set equal to 0 (j goes beyond the specified limits of change), $e_0^{(2)}$ can be found, we have:

$$e_j^{(k)} = \frac{1}{2\lambda(k-1)}\left(e_{j-1}^{(k+1)}\cdot k\cdot(k-1) - v^2e_j^{(k-1)}\cdot(n-2(k+j-1))\cdot(n-2(k+j-1)-1)\right).$$

Since the degree of z is reduced by 2 when differentiating, the term with z^{k-3} does not yet occur, and therefore the presence of the expression $\frac{n(n-1)(n-2)rv^3(i+1)^3z^{n-3}}{2\lambda}$ in relations (16) and (17) is taken into account below.

Similarly, collecting the coefficients in (17), we obtain:

$$l_{j-1}^{(k+1)}\cdot(k+1)\cdot k - e_{j-2}^{(k+1)}\cdot(k+1)\cdot k + 2\lambda l_j^{(k)}k - 2\lambda e_{j-1}^{(k+1)}$$
$$= v^2(i+1)^2l_j^{(k-1)}\cdot(n-2(k+j-1))\cdot(n-2(k+j-1)-1) - v^2(i+1)^2e_{j-1}^{(k)}$$
$$\times(n-2(k+j-1))\cdot(n-2(k+j-1)-1).$$

Since we know coefficients of lower indices, e.g. $l_0^{(k)}$ and $e_0^{(k)}$, we have:

$$l_j^{(k)} = \frac{1}{2\lambda k}\left(2\lambda e_{j-1}^{(k+1)}\cdot k - l_{j-1}^{(k+1)}\cdot(k+1)\cdot k + e_{j-2}^{(k+1)}\cdot(k+1)\cdot k\right.$$

$$+v^2(i+1)^2 l_j^{(k-1)} \cdot (n-2(k+j-1)) \cdot (n-2(k+j-1)-1) - v^2(i+1)^2 e_{j-1}^{(k)} \times (n-2(k+j-1)) \cdot (n-2(k+j-1)-1)).$$

Then from (18) we find $e_j^{(1)} = \frac{l_j^{(1)}}{2\lambda}$ and repeat the same procedure. The limits of k and j can be found from the considerations that polynomials have degree at most $n-2$ and variables z and t cannot have degree less than 0.

However, relations (16) and (17) contain term $\frac{n(n-1)(n-2)rv^3(i+1)^3 z^{n-3}}{2\lambda}$ which will give its contribution to $\nu_n(t,z)$ and $b_n(t,z)$. Denote $f_0^{(1)} = -\frac{rvn}{2\lambda}$ - coefficient at $z^{n-1}\left(1-e^{-2\lambda t}\right)(i+1)$, and in general $f_j^{(k)}$ - coefficient at $z^{n-2(k+j)+1}t^{j-1}\left(1-e^{-2\lambda t}\right)(i+1)^{2(k+j)-1}$, and $d_j^{(k)}$ - at $z^{n-2(k+j)-1}t^k(i+1)^{2(k+j)+1}$. Applying the above procedure, we obtain the required relations. The proof of Theorem 4 is complete. □

In conclusion, we present equation (7) in the form of a singularly perturbed Cauchy problem:

$$\varepsilon^2 \frac{\partial^2}{\partial t^2} U(t,z) = \left(\frac{\sigma^2}{2} i \frac{\partial^2}{\partial z^2} - \frac{\partial}{\partial t}\right) U(t,z)$$

$$U(0,z) = f(z), \quad \frac{\partial}{\partial t} U(t,z)\bigg|_{t=0} = r\frac{\sigma}{\sqrt{2}\varepsilon}(i+1)f'(z), \ r \in R.$$

Using Theorem 4, we present solutions of this Cauchy problem for conditions of independent interest (in square brackets the regular part of the solution is distinguished):

$$f(z) = z \ : \ U(t,z) = \left[z + \frac{r}{\sqrt{2}}\sigma\varepsilon(i+1)\right] - \frac{r}{\sqrt{2}}\sigma\varepsilon(i+1)e^{-\frac{t}{\varepsilon^2}};$$

$$f(z) = z^2 \ : \ U(t,z) = \left[z^2 + \sigma^2 t(i+1)^2 + r\sigma\sqrt{2}\varepsilon z(i+1) - \sigma^2\varepsilon^2(i+1)^2\right] + \left(\sigma^2\varepsilon^2(i+1)^2 - r\sigma\sqrt{2}\varepsilon z(i+1)\right)e^{-\frac{t}{\varepsilon^2}};$$

$$f(z) = z^3 \ : \ U(t,z) = \left[z^3 + 3z\sigma^2 t - 3x\sigma^2\varepsilon^2(i+1)^2 + \frac{3r}{\sqrt{2}}\sigma\varepsilon z^2(i+1)^2 - \frac{3r}{\sqrt{2}}\sigma^3\varepsilon^3(i+1)^3\right] + (3z\sigma^2\varepsilon^2(i+1)^2 - \frac{3r}{\sqrt{2}}\sigma\varepsilon z^2(i+1) + \frac{3r}{\sqrt{2}}\sigma^3\varepsilon t(i+1)^3 + \frac{3r}{\sqrt{2}}\sigma^3\varepsilon^3(i+1)^3)e^{-\frac{t}{\varepsilon^2}}.$$

If we put $\varepsilon \to 0$ we easily see the solutions of Cauchy problem for Schrödinger-type equation (6).

Acknowledgements

The authors thank the anonymous reviewers for valuable comments that improved our work.

References

Cinque F, Orsingher E (2023). "Random Motions in R^3 with Orthogonal Directions." *Stochastic Processes and Their Applications*, **161**, 173–200. URL https://www.sciencedirect.com/science/article/abs/pii/S0304414923000716.

Kac M (1974). "A Stochastic Model Related to the Telegrapher's Equation." *Rocky Mountain Journal of Mathematics*, **4**(3), 497–509. URL https://projecteuclid.org/journals/rocky-mountain-journal-of-mathematics/volume-4/issue-3.

Kolesnik AD (2001). "Weak Convergence of a Planar Random Evolution to the Wiener Process." *Journal of Theoretical Probability*, **14**(2), 485–494. URL https://link.springer.com/article/10.1023/A:1011167815206.

Kolesnik AD (2007). "A Note on Planar Random Motion at Finite Speed." *Journal of Applied Probability*, **44**(3), 838–842. URL https://www.jstor.org/stable/27595886.

Kolesnik AD (2008). "Random Motions at Finite Speed in Higher Dimensions." *Journal of Statistical Physics*, **131**(6), 1039–1065. URL https://link.springer.com/article/10.1007/s10955-008-9532-0.

Kolesnik AD (2012). "Moment Analysis of the Telegraph Random Process." *Buletinul Academiei de Stiinte a Republicii Moldova. Matematica*, **68**(1), 90–107.

Kolesnik AD, Orsingher E (2005). "A Planar Random Motion with an Infinite Number of Directions Controlled by the Damped Wave Equation." *Journal of Applied Probability*, **42**(4), 1168–1182. URL https://doi.org/10.1239/jap/1134587824.

Kolesnik AD, Turbin AF (1991). "Infinitisimal Hyperbolic Operator for Markov Random Evolutions in R^n (in Russian)." *Reports of the National Academy of Sciences of Ukraine*, **1**, 11–14.

Korolyuk VS, Turbin AF (1993). *Mathematical Foundations of the State Lumping of Large Systems*. Springer Dordrecht. ISBN 978-0-7923-2413-3, URL https://link.springer.com/book/9780792324133.

Lachal A, Leorato S, Orsingher E (2006). "Minimal Cyclic Random Motion in R^n and Hyper-Bessel Functions." *Annales de l'institut Henri Poincare (B) Probability and Statistics*, **42**(6), 753–772. URL https://www.sciencedirect.com/science/article/abs/pii/S0246020306000173.

Orsingher E (1986). "A Planar Random Motion Governed by the Two-Dimentional Telegraph Equations." *Journal of Applied Probability*, **23**(2), 385–397. URL https://doi.org/10.2307/3214181.

Orsingher E (2002). "Bessel Functions of Third Order and the Distribution of Cyclic Planar Motions with Three Directions." *Stochastics and Stochastics Reports*, **74**(3-4), 617–631. URL https://www.tandfonline.com/doi/abs/10.1080/1045112021000060755.

Orsingher E, De Gregorio A (2007). "Random Flights in Higher Spaces." *Journal of Theoretical Probability*, **20**, 769–806. URL https://link.springer.com/article/10.1007/s10959-007-0093-y.

Orsingher E, Garra R, Zeifman AI (2020). "Cyclic Random Motions with Orthogonal Directions." *Markov Processes and Related Fields*, **26**(3), 381–402. URL math-mprf.org/journal/articles/id1578.

Orsingher E, Sommella A (2004). "Cyclic Random Motion in R^3 with Four Directions and Finite Velocity." *Stochastics and Stochastics Reports*, **76**(2), 113–133. URL https://www.tandfonline.com/doi/abs/10.1080/10451120420000218829?journalCode=gssr19.

Pinsky M (1991). *Lectures on Random Evolutions*. World Scientific. ISBN 978-981-02-0559-1, URL https://www.worldscientific.com/worldscibooks/10.1142/1328#t=aboutBook.

Pogorui A, Swishchuk A, Rodríguez-Dagnino RM (2021). *Random Motions in Markov and Semi-Markov Random Environments 2: High-Dimensional Random Motions and Financial Applications*. Wiley. ISBN 978-178-63-0706-4, URL https://onlinelibrary.wiley.com/doi/book/10.1002/9781119808152.

Pogorui AA, Rodríguez-Dagnino RM (2019). "Goldstein-Kac Telegraph Equations and Random Flights in Higher Dimensions." *Applied Mathematics and Computation*, **361**, 617–629. URL https://www.sciencedirect.com/science/article/abs/pii/S0096300319304473.

Samoilenko IV (2001a). "Markovian Random Evolution in R^n." *Random Operators and Stochastic Equations*, **9**(2), 139–160. URL https://www.degruyter.com/document/doi/10.1515/rose.2001.9.2.139/html.

Samoilenko IV (2001b). "Moments of Markov Random Evolutions." *Ukrainian Mathematical Journal*, **53**(7), 1197–1205. URL https://link.springer.com/article/10.1023/A:1013341602187.

Samoilenko IV (2002). "Fading Markov Random Evolution." *Ukrainian Mathematical Journal*, **54**(3), 448–459. URL https://link.springer.com/article/10.1023/A:1020517601268.

Turbin AF, Samoilenko IV (2000). "A Probability Method for the Solution of the Telegraph Equation with Real-Analytic Initial Conditions." *Ukrainian Mathematical Journal*, **52**(8), 1292–1299. URL https://link.springer.com/article/10.1023/A:1010313423230.

Affiliation:

Igor Samoilenko
Faculty of Computer Science and Cybernetics
Taras Shevchenko National University of Kyiv
Akademika Hlushkova Ave, 4d, Kyiv, Ukraine, 03680
E-mail: isamoil@i.ua
URL: http://csc.univ.kiev.ua/en/person/samoilenko

Ganna Verovkina
Faculty of Mechanics and Mathematics
Taras Shevchenko National University of Kyiv
Akademika Hlushkova Ave, 4e, Kyiv, Ukraine, 03680
E-mail: ganna.verov@gmail.com
URL: http://www.matfiz.univ.kiev.ua/department/10

Tetiana Samoilenko
Faculty of Physics and Mathematics
National Technical University of Ukraine
"Igor Sikorsky Kyiv Polytechnic Institute
Polytekhnichna str., 14, building 7, Kyiv-56, Ukraine, 03056
E-mail: tsamoil27@gmail.com

Austrian Journal of Statistics http://www.ajs.or.at/
published by the Austrian Society of Statistics http://www.osg.or.at/
Volume 52
2023