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Bounds for the Tail Distributions of Suprema of Sub-Gaussian Type Random Fields

Olha Hopkalo
Taras Shevchenko
National University
of Kyiv

Lyudmyla Sakhno
Taras Shevchenko
National University
of Kyiv

Olga Vasylyk
National Technical
University of Ukraine
"Igor Sikorsky Kyiv
Polytechnic Institute"

Abstract

The paper presents bounds for the distributions of suprema for particular classes of φ-sub-Gaussian random fields. Results stated depend on representations of bounds for increments of the fields in different metrics. Several examples of applications are provided to illustrate the results, in particular, to random fields related to stochastic partial differential equations and partial differential equations with random initial conditions.

Keywords: sub-Gaussian random fields, distribution of supremum, heat equation, Airy equation, random initial conditions, stochastic heat equation.

1. Introduction

In this paper, we study sample paths properties of random fields belonging to the spaces of φ-sub-Gaussian random variables, which generalize Gaussian and sub-Gaussian ones and represent, on the other side, subclass of exponential type Orlicz spaces of random variables. Our aim is to investigate tail distributions of suprema of φ-sub-Gaussian random fields under different conditions imposed on their increments.

Recall that a random variable ξ is sub-Gaussian if its moment generating function is majorized by that of a Gaussian centered random variable $\eta \sim N(0, \sigma^2)$, that is,

$$\mathsf{E} \exp(\lambda\xi) \leq \mathsf{E} \exp(\lambda\eta) = \exp(\sigma^2\lambda^2/2).$$

The generalization of this notion to the classes of φ-sub-Gaussian random variables is introduced as follows (see, Buldygin and Kozachenko (2000) (Ch.2), Giuliano, Kozachenko, and Nikitina (2003), Kozachenko and Ostrovskij (1986), Vasylyk, Kozachenko, and Yamnenko (2008)).

Definition 1.1. Giuliano *et al.* (2003); Kozachenko and Ostrovskij (1986) A continuous even convex function φ is called *an Orlicz N-function* if $\varphi(0) = 0$, $\varphi(x) > 0$ as $x \neq 0$ and $\lim_{x\to 0} \frac{\varphi(x)}{x} = 0$, $\lim_{x\to\infty} \frac{\varphi(x)}{x} = \infty$. We say that an N-function satisfies *Condition Q* if $\liminf_{x\to 0} \frac{\varphi(x)}{x^2} = c > 0$, where the case $c = \infty$ is possible.

Definition 1.2. Giuliano *et al.* (2003); Kozachenko and Ostrovskij (1986) Let φ be an N-function satisfying condition Q and $\{\Omega, L, \mathsf{P}\}$ be a standard probability space. The random variable ζ is φ-sub-Gaussian, or belongs to the space $\mathrm{Sub}_\varphi(\Omega)$, if $\mathsf{E}\zeta = 0$, $\mathsf{E}\exp\{\lambda\zeta\}$ exists for all $\lambda \in \mathbb{R}$ and there exists a constant $a > 0$ such that the following inequality holds for all $\lambda \in \mathbb{R}$

$$\mathsf{E}\exp\{\lambda\zeta\} \leq \exp\{\varphi(\lambda a)\}.$$

The stochastic process (or random field) $X(t)$, $t \in T$, is called φ-sub-Gaussian if the random variables $\{X(t), t \in T\}$ are φ-sub-Gaussian.

The space $\mathrm{Sub}_\varphi(\Omega)$ is a Banach space with respect to the norm (see Giuliano *et al.* (2003); Kozachenko and Ostrovskij (1986)):

$$\tau_\varphi(\zeta) = \inf\{a > 0 : \mathsf{E}\exp\{\lambda\zeta\} \leq \exp\{\varphi(a\lambda)\}.$$

Definition 1.3. Giuliano *et al.* (2003); Kozachenko and Ostrovskij (1986) The function φ^* defined by $\varphi^*(x) = \sup_{y\in\mathbb{R}}(xy - \varphi(y))$ is called *the Young-Fenchel transform (or convex conjugate)* of the function φ.

The function φ^* is known also as the Legendre or Legendre-Fenchel transform. For φ-sub-Gaussian random variable ζ one can write the estimate for its tail probability in the form

$$P\{|\zeta| > u\} \leq 2\exp\left\{-\varphi^*\left(\frac{u}{\tau_\varphi(\zeta)}\right)\right\}, \quad u > 0. \tag{1.1}$$

Moreover, a random variable ζ is a φ-sub-Gaussian if and only if $\mathsf{E}\zeta = 0$ and there exist constants $C > 0$, $D > 0$ such that

$$P\{|\zeta| > u\} \leq C\exp\left\{-\varphi^*\left(\frac{u}{D}\right)\right\} \tag{1.2}$$

(see Buldygin and Kozachenko (2000), Corollary 4.1, p. 68).

Therefore, the property of φ-sub-Gaussianity can be characterized in a double way: by introducing a bound on the exponential moment of a random variable (Definition 1.2), or by the tail behavior (1.1) or (1.2), which is even more important from the practical point of view.

The class of φ-sub-Gaussian random variables includes centered compactly supported distributions, reflected Weibull distributions, centered bounded distributions, Gaussian, Poisson distributions. In the case of $\varphi = \frac{x^2}{2}$, the notion of φ-sub-Gaussianity reduces to the classical sub-Gaussianity. The main theory for the spaces of φ-sub-Gaussian random variables and stochastic processes was presented in Buldygin and Kozachenko (2000); Giuliano *et al.* (2003); Kozachenko and Ostrovskij (1986) followed by numerous further studies. Various classes of φ-sub-Gaussian processes and fields were studied, in particular, in Beghin, Kozachenko, Orsingher, and Sakhno (2007); Hopkalo and Sakhno (2021); Kozachenko and Olenko (2016); Kozachenko, Orsingher, Sakhno, and Vasylyk (2018, 2020).

Estimates for distribution of supremum $P\{\sup_{t\in T}|X(t)| \geq u\}$ of φ-sub-Gaussian stochastic process X were derived in various forms in the monograph Buldygin and Kozachenko (2000) basing on entropy methods.

Recall that the entropy approach in studing sample paths of a stochastic process $X(t)$, $t \in T$, requires to evaluate entropy characteristics of the set T with respect to a particular metrics generated by the process X. The origins of this approach are due to Dudley, who stated conditions for boundedness of Gaussian processes in the form of convergence of metric entropy integrals (which we call now Dudley entropy integrals). We address for corresponding references, e.g., to Adler and Taylor (2007) and Buldygin and Kozachenko (2000), where in the latter one the entropy approach was extended to different classes of processes, more general than Gaussian ones.

We will base our study on the following theorem proved in Kozachenko and Olenko (2016) (see also Buldygin and Kozachenko (2000)).

Theorem 1.1 (Buldygin and Kozachenko (2000); Kozachenko and Olenko (2016))**.** *Let $X(t)$, $t \in T$, be a φ-sub-Gaussian process and ρ_X be the pseudometrics generated by X, that is, $\rho_X(t,s) = \tau_\varphi(X(t) - X(s))$, $t, s \in T$. Assume that*

(i) the pseudometric space (T, ρ_X) is separable, the process X is separable on (T, ρ_X);

(ii) $\varepsilon_0 := \sup_{t \in T} \tau_\varphi(X(t)) < \infty$;

(iii) for a non-negative, monotone increasing function $r(x)$, $x \geq 1$ such that $r(e^x), x \geq 0$, is convex, it holds for $0 < \varepsilon \leq \varepsilon_0$

$$I_r(\varepsilon) := \int_0^\varepsilon r(N(v))\, dv < \infty, \tag{1.3}$$

where $N(v)$, $v > 0$, is the massiveness of the pseudometric space (T, ρ_X), that is, $N(v)$ denotes the smallest number of elements in a v-covering of T by closed balls of a radius at most v.

Then for all $\lambda > 0$, $0 < \theta < 1$ and $u > 0$ it holds

$$\mathsf{E} \exp\left\{\lambda \sup_{t \in T} |X(t)|\right\} \leq 2 \exp\left\{\varphi\left(\frac{\lambda \varepsilon_0}{1 - \theta}\right)\right\} A(\theta \varepsilon_0) \tag{1.4}$$

and

$$P\{\sup_{t \in T} |X(t)| \geq u\} \leq 2 \exp\left\{ -\varphi^*\left(\frac{u(1-\theta)}{\varepsilon_0}\right)\right\} A(\theta \varepsilon_0), \tag{1.5}$$

where

$$A(\theta \varepsilon_0) = r^{(-1)}\left(\frac{I_r(\theta \varepsilon_0)}{\theta \varepsilon_0}\right). \tag{1.6}$$

In the above theorem and in what follows we denote by $f^{(-1)}$ the inverse function for a function f.

The integrals of the form (1.3) with $r(x)$, $x \geq 1$, being some non-negative nondecreasing function are called entropy integrals. Entropy characteristics of the parameter set T with respect to the pseudometrics ρ_X generated by the process X and the rate of growth of the metric massiveness $N(v) = N_{\rho_X}(v)$, $v > 0$, or metric entropy $H(v) := \ln(N(v)$ play an important role in treating sample paths properties of the underlying process X (see Buldygin and Kozachenko (2000) for details).

Consider now a metric space (T, d), with an arbitrary metrics d and suppose that this metric space is separable. Suppose further that we have evaluated the metric massiveness N_d of T with respect to the metrics d and have a bound for the function $\rho_X(t,s) = \tau_\varphi(X(t) - X(s))$ which is given in terms of $d(t,s)$. Then Theorem 1.1 implies the following result, which is more convenient for practical use.

Theorem 1.2. *Let $X(t)$, $t \in T$, be a φ-sub-Gaussian process and T be supplied with a metrics d. Assume that*

(i) the metric space (T, d) is separable, the process X is separable on (T, d);

(ii) $\varepsilon_0 := \sup_{t \in T} \tau_\varphi(X(t)) < \infty$;

(iii) there exists a a monotonically increasing continuous function $\sigma(h)$, $0 < h \leq \sup_{t,s \in T} d(s,t)$, such that $\sigma(h) \to 0$ as $h \to 0$ and

$$\sup_{\substack{d(t,s) \leq h, \\ t,s \in T}} \tau_\varphi(X(t) - X(s)) \leq \sigma(h), \tag{1.7}$$

(iv) for a non-negative, monotone increasing function $r(x)$, $x \geq 1$ such that $r(e^x), x \geq 0$, is convex, it holds for $0 < \varepsilon \leq \gamma_0$

$$\widetilde{I}_r(\varepsilon) := \int_0^{\varepsilon} r(N_d(\sigma^{(-1)}(v))\, dv < \infty, \tag{1.8}$$

where $N_d(v)$, $v > 0$, is the massiveness of the metric space (T, d), $\gamma_0 = \sigma(\sup_{t,s \in T} d(s,t))$.

Then statements (1.4) and (1.5) of Theorem 1.1 hold for $0 < \theta < 1$ such that $\theta\varepsilon_0 < \gamma_0$ with $A(\theta\varepsilon_0)$ changed for the appropriate $\widetilde{A}(\theta\varepsilon_0)$:

$$\mathsf{E} \exp\left\{\lambda \sup_{t\in T} |X(t)|\right\} \leq 2 \exp\left\{\varphi\left(\frac{\lambda\varepsilon_0}{1-\theta}\right)\right\} \widetilde{A}(\theta\varepsilon_0) \tag{1.9}$$

and

$$P\{\sup_{t\in T} |X(t)| \geq u\} \leq 2 \exp\Big\{ - \varphi^*\Big(\frac{u(1-\theta)}{\varepsilon_0}\Big)\Big\}\widetilde{A}(\theta\varepsilon_0), \tag{1.10}$$

where

$$\widetilde{A}(\theta\varepsilon_0) = r^{(-1)}\left(\frac{\widetilde{I}_r(\theta\varepsilon_0)}{\theta\varepsilon_0}\right). \tag{1.11}$$

Proof. Theorem 1.2 follows immediately from Theorem 1.1. Indeed, once we have from (1.7) that $\sup_{\substack{d(t,s)\leq h,\\ t,s\in T}} \rho_X(t,s) \leq \sigma(h)$, the smallest number of elements in an ε-covering of (T, ρ_X) can be bounded by the smallest number of elements in a $\sigma^{(-1)}(\varepsilon)$-covering of (T, d): $N_{\rho_X}(\varepsilon) \leq N_d(\sigma^{(-1)}(\varepsilon))$, which implies the estimate $I_r(\varepsilon) \leq \widetilde{I}_r(\varepsilon)$, as $\varepsilon \leq \gamma_0$, and the statement of the theorem follows. □

Theorem 1.2 with a particular metric space (T, d) has been applied in Hopkalo and Sakhno (2021); Kozachenko *et al.* (2020); Sakhno (2023b) for studying solutions to partial differential equations with random initial condition, in Sakhno (2023a) for evaluation of suprema of spherical random fields, in Kozachenko and Olenko (2016) for developing uniform approximation schemes for φ-sub-Gaussian processes. This theorem allows to calculate the bounds for the distribution of suprema in the closed form.

Now we have stated all prerequisites necessary for our study in this paper, Theorem 1.2 will serve as the main tool.

The plan of the paper is as follows.

In Section 2 we consider the parameter set T of the form $T = [a_1, b_1] \times [a_2, b_2]$ with various metrics, in particular, we consider $d(t, s) = \max_{i=1,2} |t_i - s_i|$ and the so-called anisotropic metrics $d(t, s) = \sum_{i=1,2} |t_i - s_i|^{H_i}$, $H_i \in (0, 1]$, $i = 1, 2$, and corresponding bounds for the function $\rho_X(t, s) = \tau_\varphi(X(t) - X(s))$. We will specify Theorem 1.2 for these different cases.

In Section 3 we state the results on the rate of growth of φ-sub-Gaussian random fields over unbounded domains.

Section 4 presents examples of applications of the results obtained. In particular, we consider processes related to partial differential equations with random initial conditions and stochastic partial differential equations. These applications actually served as a motivation for our study.

2. Estimates for the distribution of suprema

We will study φ-sub-Gaussian random fields $X(t), t \in T$, defined over the parameter set $T = \{(t_1, t_2), a_i \leq t_i \leq b_i, i = 1, 2\}$. Consider the following metrics on T:

$$d_1(t,s) = \max_{i=1,2} |t_i - s_i|; \tag{2.1}$$

$$d_2(t,s) = \sum_{i=1,2} \frac{|t_i - s_i|}{a_i}, \ a_i > 0, i = 1,2; \tag{2.2}$$

$$d_3(t,s) = \sum_{i=1,2} |t_i - s_i|^{H_i}, \ H_i \in (0,1], i = 1,2; \tag{2.3}$$

$$d_4(t,s) = \sum_{i=1,2} \frac{|t_i - s_i|^{H_i}}{a_i}, \ a_i > 0, \ H_i \in (0,1], i = 1,2. \tag{2.4}$$

Note that for the metrics (2.3) and (2.4) we suppose $H_i \in (0,1]$, $i = 1,2$, but excluding the case $H_1 = H_2 = 1$. Consider the function $\rho_X(t,s) = \tau_\varphi(X(t) - X(s))$, $t,s \in T$. We state the bounds for the distribution of supremum of the field X under the condition that the function $\rho_X(t,s)$ can be bouded as follows:

$$\rho_X(t,s) \le \sigma(d_i(t,s)),$$

or

$$\sup_{t,s\in T,\, d_i(t,s)\le h} \rho_X(t,s) \le \sigma(h),$$

where $d_i(t,s)$ is one of the metrics (2.1)-(2.4), σ is a monotonically increasing function. More precisely, we consider the case $\sigma(h) = ch^\beta, c > 0, \beta \in (0,1]$.

Theorem 2.1. *Let $X(t)$, $t \in T$, be a φ-sub-Gaussian random field and T be supplied with one of the metrics d_i from the list (2.1)-(2.4). Assume that*

(i) the field X is separable on (T, d_i);

(ii) $\varepsilon_0 := \sup_{t\in T} \tau_\varphi(X(t)) < \infty$;

(iii) $\sup_{t,s\in T,\, d_i(t,s)\le h} \tau_\varphi(X(t) - X(s)) \le ch^\beta.$

Then for all $\theta \in (0,1)$ such that $\theta < \theta_i$ and $u > 0$

$$P\{\sup_{t\in T} |X(t)| \ge u\} \le 2\exp\Big\{ -\varphi^*\Big(\frac{u(1-\theta)}{\varepsilon_0}\Big)\Big\} A_i(\theta\varepsilon_0),$$

where expressions for $A_i(\theta\varepsilon_0)$ and θ_i correspond to different choices of metrics d_i in conditions of theorem and are given as follows:

$i = 1: \ A_1(\theta\varepsilon_0) = \varkappa_1 (ce)^{2/\beta} (\theta\varepsilon_0)^{-2/\beta},$

$$\varkappa_1 = \tfrac{1}{2}\min(T_1,T_2)(T_1+T_2), \ \theta_1 = \tfrac{c}{\varepsilon_0}\Big(\tfrac{1}{2}\min(T_1,T_2)\Big)^\beta;$$

$i = 2: \ A_2(\theta\varepsilon_0) = \varkappa_2 (ce)^{2/\beta} (\theta\varepsilon_0)^{-2/\beta},$

$$\varkappa_2 = \min(\tfrac{T_1}{a_1}, \tfrac{T_2}{a_2})(\tfrac{T_1}{a_1} + \tfrac{T_2}{a_2}), \ \theta_2 = \tfrac{3c}{\varepsilon_0}\Big(\min\Big(\tfrac{T_1}{a_1}, \tfrac{T_2}{a_2}\Big)\Big)^\beta;$$

$i = 3: \ A_3(\theta\varepsilon_0) = T_1 T_2 (2^\beta ce)^q (\theta\varepsilon_0)^{-q},$

$$q = \tfrac{1}{\beta H_1} + \tfrac{1}{\beta H_2}, \ \theta_3 = \tfrac{2^\beta c}{\varepsilon_0}\min_{i=1,2}\Big(\Big(\tfrac{T_i}{2}\Big)^{H_i\beta}\Big);$$

$i = 4: \ A_4(\theta\varepsilon_0) = T_1 T_2 a_1^{-1/H_1} a_2^{-1/H_2} (2^\beta ce)^q (\theta\varepsilon_0)^{-q},$

$$q = \tfrac{1}{\beta H_1} + \tfrac{1}{\beta H_2}, \ \theta_4 = \tfrac{2^\beta c}{\varepsilon_0}\min_{i=1,2}\Big(\Big(\tfrac{T_i}{2}\Big)^{H_i\beta} a_i^{-\beta}\Big).$$

Proof. To state the estimates for supremum for different metrics d_i, we apply Theorem 1.2. First we need to estimate the metric massiveness $N_{d_i}, i = 1, \dots, 4$. We can write the following estimates

$$N_{d_1}(\varepsilon) \le \Big(\frac{T_1}{2\varepsilon} + 1\Big)\Big(\frac{T_2}{2\varepsilon} + 1\Big);$$
$$N_{d_2}(\varepsilon) \le 2\Big(\frac{T_1}{2a_1\varepsilon} + \frac{3}{2}\Big)\Big(\frac{T_2}{2a_2\varepsilon} + \frac{3}{2}\Big).$$

The estimates for N_{d_1} and N_{d_2} can be found, for example, in Buldygin and Kozachenko (2000) and Kozachenko and Makogin (2014) correspondingly.

For the case of metrics d_3, we note that a rectangle $[-(\frac{\varepsilon}{2})^{\frac{1}{H_1}}, (\frac{\varepsilon}{2})^{\frac{1}{H_1}}] \times [-(\frac{\varepsilon}{2})^{\frac{1}{H_2}}, (\frac{\varepsilon}{2})^{\frac{1}{H_2}}]$ is contained in the ball in metrics d_3 with center $(0,0)$ and radius ε which is given as $B(\varepsilon) = \{(x_1,x_2) : |x_1|^{H_1} + |x_2|^{H_2} \le \varepsilon\}$. Therefore,

$$N_{d_3}(\varepsilon) \le \prod_{i=1,2}\Big(\frac{T_i}{2(\frac{\varepsilon}{2})^{\frac{1}{H_i}}} + 1\Big) = \prod_{i=1,2}\Big(\frac{2^{\frac{1}{H_i}}T_i}{2\varepsilon^{\frac{1}{H_i}}} + 1\Big).$$

Analogously,

$$N_{d_4}(\varepsilon) \le \prod_{i=1,2}\Big(\frac{T_i}{2(\frac{\varepsilon a_i}{2})^{\frac{1}{H_i}}} + 1\Big) = \prod_{i=1,2}\Big(\frac{2^{\frac{1}{H_i}}T_i}{2(\varepsilon a_i)^{\frac{1}{H_i}}} + 1\Big).$$

We present the proof for the cases of metrics d_2 and d_3, other cases are treated similarly. Note that the bound for the case of d_1 was stated in Sakhno (2023b).

Apply Theorem 1.2 with $\sigma(h) = ch^\beta$, $\sigma^{(-1)}(u) = (\frac{u}{c})^{1/\beta}$ and choose $r(v) = v^\alpha - 1$, then $r^{(-1)}(v) = (v+1)^{1/\alpha}$.

Consider case $d = d_2$:

$$I_r(\delta) = \int_0^\delta \Big(2^\alpha \prod_{i=1,2}\Big(\frac{c^{\frac{1}{\beta}}T_i}{2a_i u^{\frac{1}{\beta}}} + \frac{3}{2}\Big)^\alpha - 1\Big)du,$$

consider $\delta \in (0, \theta\varepsilon_0)$ and choose $\theta \in \Big(0, \frac{3c}{\varepsilon_0}\Big(\min_{i=1,2}(\frac{T_i}{a_i})\Big)^\beta\Big)$.

Then we can write

$$I_r(\delta) \le \int_0^\delta \Big(2^\alpha\Big(\min\Big(\frac{T_1}{a_1}, \frac{T_2}{a_2}\Big)\frac{c^{\frac{1}{\beta}}}{u^{\frac{1}{\beta}}}\Big)^\alpha\Big(\frac{1}{2}\Big(\frac{T_1}{a_1} + \frac{T_2}{a_2}\Big)\frac{c^{\frac{1}{\beta}}}{u^{\frac{1}{\beta}}}\Big)^\alpha - 1\Big)du = \varkappa_2^\alpha c^{\frac{2\alpha}{\beta}}(1 - \frac{2\alpha}{\beta})\delta^{1-\frac{2\alpha}{\beta}} - \delta,$$

where $\varkappa_2 = \min(\frac{T_1}{a_1}, \frac{T_2}{a_2})(\frac{T_1}{a_1} + \frac{T_2}{a_2})$, and

$$r^{(-1)}\Big(\frac{I_r(\theta\varepsilon_0)}{\theta\varepsilon_0}\Big) \le \varkappa_2 c^{\frac{2}{\beta}}(1 - \frac{2\alpha}{\beta})^{-\frac{1}{\alpha}}(\theta\varepsilon_0)^{-\frac{2}{\beta}}. \tag{2.5}$$

Let $\alpha \to 0$, then $(1-\frac{2\alpha}{\beta})^{-\frac{1}{\alpha}} \to e^{\frac{2}{\beta}}$ and we obtain in the right hand side of (2.5) $\varkappa_2(ce)^{\frac{2}{\beta}}(\theta\varepsilon_0)^{-\frac{2}{\beta}}$.

Consider now $d = d_3$:

$$I_r(\delta) = \int_0^\delta \Big(\prod_{i=1,2}\Big(\frac{T_i 2^{\frac{1}{H_i}} c^{\frac{1}{\beta H_i}}}{2u^{\frac{1}{\beta H_i}}} + 1\Big)^\alpha - 1\Big)du = \int_0^\delta \Big(\prod_{i=1,2}\Big(\frac{T_i(2^\beta c)^{\frac{1}{\beta H_i}}}{2u^{\frac{1}{\beta H_i}}} + 1\Big)^\alpha - 1\Big)du.$$

Denote $\tilde{c} = 2^\beta c$, $\widetilde{H_i} = \beta H_i$. Choose θ such that $\min\Big(\frac{T_i}{2}\Big)^{\widetilde{H_i}}\frac{\tilde{c}}{\theta\varepsilon_0} > 1$, then $\frac{T_i}{2}\frac{\tilde{c}^{\frac{1}{\widetilde{H_i}}}}{(\theta\varepsilon_0)^{\frac{1}{\widetilde{H_i}}}} > 1, i = 1,2$, and we can write the estimate

$$I_r(\delta) \le \int_0^\delta \Big(\prod_{i=1,2}\Big(\frac{T_i\tilde{c}^{\frac{1}{\widetilde{H_i}}}}{u^{\frac{1}{\widetilde{H_i}}}}\Big)^\alpha - 1\Big)du = (T_1T_2)^\alpha \tilde{c}^{\alpha q}(1-\alpha q)^{-1}\delta^{1-\alpha q} - \delta;$$

where $q = \frac{1}{\widetilde{H}_1} + \frac{1}{\widetilde{H}_2} = \frac{1}{\beta H_1} + \frac{1}{\beta H_2}$, and, analogously to the previous case, we come to the bound

$$r^{(-1)}\Big(\frac{I_r(\theta\varepsilon_0)}{\theta\varepsilon_0}\Big) \leq T_1T_2(\tilde{c}e)^q(\theta\varepsilon_0)^{-q} = T_1T_2(2^\beta ce)^q(\theta\varepsilon_0)^{-q}.$$

□

Remark 2.1. Note that the most studied in the literature is the case of metrics d_1, see, for example, Kozachenko *et al.* (2020), Hopkalo and Sakhno (2021), where under this metrics different bounds for the function $\rho_X(t,s) = \tau_\varphi(X(t)-X(s))$ where considered and corresponding bounds for the distributions of suprema were stated. In Theorem 2.1 (following Theorem 5.4 in Sakhno (2023b)) the improved bound is presented in comparison with the analogous results stated in Kozachenko *et al.* (2020) (Corollary 3.1) and Hopkalo and Sakhno (2021) (Corollary 2). The bounds for the distribution of suprema for self-similar Gaussian random fields were stated in Kozachenko and Makogin (2014) using the entropy approach and bounds on the increments in terms of the metrics d_3. Note that they used the estimate for $N_{d_3}(\varepsilon)$, which coincides for $H_1 = H_2 = 1$ and $a_1 = a_2 = 1$ with the estimate for $N_{d_2}(\varepsilon)$, and its form is very convenient to treat the case of self-similar random fields but is rather complicated for derivations for the general case. Our result for the case of metric d_3 appears in the form which is simpler but different from the corresponding bound in Kozachenko and Makogin (2014), first due to different estimate for N_{d_3}, but also since it is derived from Theorem 1.2, and derivations in Kozachenko and Makogin (2014) are based on another result. We postpone for further research the comparison of these bounds, in particular, by simulation studies.

3. Estimates for the rate of growth over unbounded domain

In this section we consider a φ-sub-Gaussian field $X(t,x)$, $(t,x) \in V$, defined over unbounded domain $V = [0,+\infty) \times [-A,A]$.

Let $f(t) > 0$, $t \geq 0$, be a continuous strictly increasing function and $f(t) \to \infty$ as $t \to \infty$.

Introduce the sequence $b_0 = 0$, $b_{k+1} > b_k$, $b_k \to \infty$, $k \to \infty$.

We will use the following notations:

$l_k = b_{k+1} - b_k$, $V_k = [b_k, b_{k+1}] \times [-A,A]$, $k = 0,1,\ldots$, $f_k = f(b_k)$,

$\varepsilon_k = \sup_{(t,x)\in V_k} \tau_\varphi(X(t,x))$, and suppose that $0 < \varepsilon_k < \infty$;

$\gamma_k = \sigma_k(l_k)$, where σ_k are introduced in the next theorem, $\tilde{\theta} = \inf_k \frac{\gamma_k}{\varepsilon_k}$.

Theorem 3.1 below is an extension of the result stated in Sakhno and Vasylyk (2021)(Theorem 1), see also Hopkalo and Sakhno (2021)(Theorem 4). The proof presented in Sakhno and Vasylyk (2021)(Theorem 1) for the case of metrics $d = d_1$ works for a general metrics d as well.

Theorem 3.1. *Let $X(t,x)$, $(t,x) \in V$, $V = [0,+\infty)\times[-A,A]$, be a φ-sub-Gaussian separable random field. Suppose futher that:*

(i) there exist the increasing continuous functions $\sigma_k(h), h > 0$, such that $\sigma_k(h) \to 0$ as $h \to 0$,

$$\sup_{\substack{(t_i,x_i)\in V_k, i=1,2 \\ d((t_1,x_1),(t_2,x_2))\leq h}} \tau_\varphi(X(t_1,x_1) - X(t_2,x_2)) \leq \sigma_k(h)$$

and for $k = 0,1,\ldots$

$$I_{r,k}(\gamma_k) = \int_0^{\gamma_k} r(N_{V_k}(\sigma_k^{(-1)}(u)))du < \infty;$$

(ii) $C = \sum_{k=0}^{\infty} \frac{\varepsilon_k}{f_k} < \infty$;

(*iii*) *for any* $\theta \in (0,1)$

$$S(\theta, r) = \sum_{k=0}^{\infty} \frac{\varepsilon_k}{f_k} \log\left(r^{(-1)}\Big(\frac{I_{r,k}(\theta\varepsilon_k)}{\theta\varepsilon_k}\Big)\right) < \infty. \tag{3.1}$$

Then

(*i*) *for any* $\theta \in (0, \min(1, \tilde{\theta}))$ *and any* $\lambda > 0$

$$\mathsf{E}\exp\Big\{\lambda \sup_{(t,x)\in V} \frac{|X(t,x)|}{f(t)}\Big\} \le 2\exp\Big\{\varphi\Big(\frac{\lambda C}{1-\theta}\Big)\Big\}\exp\Big\{\frac{S(\theta,r)}{C}\Big\}; \tag{3.2}$$

(*ii*) *for any* $\theta \in (0, \min(1, \tilde{\theta}))$ *and any* $u > 0$

$$P\Big\{\sup_{(t,x)\in V} \frac{|X(t,x)|}{f(t)} > u\Big\} \le 2\exp\Big\{-\varphi^*\Big(\frac{u(1-\theta)}{C}\Big)\Big\}\exp\Big\{\frac{S(\theta,r)}{C}\Big\}. \tag{3.3}$$

We consider the specification of Theorem 3.1 for the cases of metrics d_1 and d_3. Case d_1 was studied in Sakhno and Vasylyk (2021) and the following result was stated.

Theorem 3.2 (Sakhno and Vasylyk (2021))**.** *Let conditions of Theorem 3.1 hold with* $\sigma_k(h) = c_k h^{\beta}$, $c_k > 0$, $0 < \beta \le 1$, $d = d_1$, $l_k \ge 2A$, *and with condition* (*iii*) *replaced by the following one*

(*iv*) *There exists* $0 < \gamma \le 1$ *such that*

$$S_1 = \sum_{k=0}^{\infty} \frac{\varepsilon_k^{1-\frac{2\gamma}{\beta}} l_k^{2\gamma} c_k^{\frac{2\gamma}{\beta}}}{f_k} < \infty.$$

Then

(*i*) *for any* $\theta \in (0, \min(1, \tilde{\theta}))$ *and any* $\lambda > 0$

$$\mathsf{E}\exp\Big\{\lambda \sup_{(t,x)\in V} \frac{|X(t,x)|}{f(t)}\Big\} \le 2^{\frac{4}{\beta}}\exp\Big\{\varphi\Big(\frac{\lambda C}{1-\theta}\Big)\Big\}\widetilde{A}_1(\theta);$$

(*ii*) *for any* $\theta \in (0, \min(1, \tilde{\theta}))$ *and any* $u > 0$

$$P\Big\{\sup_{(t,x)\in V} \frac{|X(t,x))|}{f(t)} > u\Big\} \le 2^{\frac{4}{\beta}}\exp\Big\{-\varphi^*\Big(\frac{u(1-\theta)}{C}\Big)\Big\}\widetilde{A}_1(\theta),$$

where

$$\widetilde{A}_1(\theta) = \exp\Big\{\frac{S_1}{\gamma C}\Big(\frac{2^{\frac{4}{\beta}-2}}{\theta^{\frac{2}{\beta}}}\Big)^{\gamma}\Big\}.$$

Note that the proof is based on Theorem 1.2 and the following estimate for the integral

$$I_r(\delta) := \int_0^{\delta} r\Big(\prod_{i=1,2}\Big(\frac{T_i}{2\sigma^{(-1)}(v)} + 1\Big)\Big)\,dv,$$

where $\sigma(u) = cu^{\beta}$, $\sigma^{(-1)}(u) = (u/c)^{1/\beta}$ and $r(v) = v^{\alpha} - 1$:

$$\begin{aligned} I_r(\delta) &= \int_0^{\delta}\Big[\prod_{i=1,2}\Big(\frac{T_i c^{1/\beta}}{2u^{1/\beta}} + 1\Big)^{\alpha} - 1\Big]\,du \\ &\le \int_0^{\delta}\Big(\frac{\varkappa c^{1/\beta}}{2u^{1/\beta}}\Big)^{2\alpha}\,du = \Big(\frac{\varkappa}{2}\Big)^{\alpha} c^{2\alpha/\beta}\Big(1 - \frac{2\alpha}{\beta}\Big)^{-1}\delta^{1-2\alpha/\beta}, \end{aligned}$$

$\varkappa = \max(T_1, T_2)$, $\alpha < 1/2$. Correspondingly, we can write the estimate

$$r^{(-1)}\Big(\frac{I_r(\delta)}{\delta}\Big) \le 2^{\frac{4}{\beta}-1}\Big(\frac{\varkappa^2 c^{\frac{2}{\beta}} 2^{2(\frac{2}{\beta}-1)}}{\delta^{\frac{2}{\beta}}} + 1\Big),$$

which is more convenient for estimation of the expression (3.1) (needed to apply Theorem 3.1) than the estimate from Theorem 2.1. We refer for more details to Hopkalo and Sakhno (2021), Sakhno and Vasylyk (2021).

We now state the analogous result for the case of metrics d_3.

Theorem 3.3. *Let conditions of Theorem 3.1 hold with $\sigma_k(h) = c_k h^\beta$, $c_k > 0$, $0 < \beta \le 1$, $d = d_3$, and with condition (iii) replaced by the following one*

(v) There exists $0 < \gamma \le 1$ such that

$$S_3 = \sum_{k=0}^{\infty} \frac{\varepsilon_k^{1-\frac{2\gamma}{\beta H}} \Big(l_k^{H_1} + (2A)^{H_2}\Big)^{\frac{2\gamma}{H}} c_k^{\frac{2\gamma}{\beta H}}}{f_k} < \infty,$$

where $H = \min(H_1, H_2)$.

Then

(i) for any $\theta \in (0, \min(1, \tilde{\theta}))$ and any $\lambda > 0$

$$\mathsf{E}\exp\Big\{\lambda \sup_{(t,x)\in V} \frac{|X(t,x)|}{f(t)}\Big\} \le 2^{\frac{4}{\beta H}} \exp\Big\{\varphi\Big(\frac{\lambda C}{1-\theta}\Big)\Big\}\widetilde{A}_3(\theta);$$

(ii) for any $\theta \in (0, \min(1, \tilde{\theta}))$ and any $u > 0$

$$P\Big\{\sup_{(t,x)\in V} \frac{|X(t,x))|}{f(t)} > u\Big\} \le 2^{\frac{4}{\beta H}} \exp\Big\{-\varphi^*\Big(\frac{u(1-\theta)}{C}\Big)\Big\}\widetilde{A}_3(\theta),$$

where

$$\widetilde{A}_3(\theta) = \exp\Big\{\frac{S_3}{\gamma C}\Big(\frac{2^{\frac{4}{\beta H}+\frac{2}{H}-2}}{\theta^{\frac{2}{\beta H}}}\Big)^\gamma\Big\}.$$

Proof. The proof follows the same lines as that of Corollary 1 in Sakhno and Vasylyk (2021). First, we need to derive another bound for the integral

$$I_r(\delta) = \int_0^\delta \Big(\prod_{i=1,2}\Big(\frac{T_i 2^{\frac{1}{H_i}} c^{\frac{1}{\beta H_i}}}{2u^{\frac{1}{\beta H_i}}} + 1\Big)^\alpha - 1\Big)du, \tag{3.4}$$

which will be used to evaluate the expression (3.1).

Consider $\delta < \theta\varepsilon_0$. Therefore, $u < \theta\varepsilon_0 < \gamma_0$, where $\gamma_0 = \sigma(\max d_3(t,s)) = c(T_1^{H_1} + T_2^{H_2})^\beta$. It follows that $u < c(T_1^{H_1} + T_2^{H_2})^\beta$ and $\frac{u}{c(T_1^{H_1}+T_2^{H_2})^\beta} < 1$.

We can also write the estimate $T_i \le (T_1^{H_1} + T_2^{H_2})^{\frac{1}{H_i}}$, $i = 1, 2$.

Suppose $H_1 < H_2$. We can evaluate (3.4) as follows:

$$
\begin{aligned}
I_r(\delta) &= \int_0^{\delta} \Big(\prod_{i=1,2} \Big(\frac{T_i 2^{\frac{1}{H_i}} c^{\frac{1}{\beta H_i}}}{2c^{\frac{1}{\beta H_i}} (T_1^{H_1} + T_2^{H_2})^{\frac{1}{H_i}} (uc^{-1}(T_1^{H_1} + T_2^{H_2})^{-\beta})^{\frac{1}{\beta H_i}}} + 1 \Big)^{\alpha} - 1 \Big) du \\
&\le \int_0^{\delta} \Big(\Big(\frac{T_1 2^{\frac{1}{H_1}} c^{\frac{1}{\beta H_i}}}{2u^{\frac{1}{\beta H_1}}} + 1 \Big)^{\alpha} \Big(\frac{T_2 2^{\frac{1}{H_2}} c^{\frac{1}{\beta H_2}}}{2(T_1^{H_1} + T_2^{H_2})^{\frac{1}{H_2} - \frac{1}{H_1}} u^{\frac{1}{\beta H_1}}} + 1 \Big)^{\alpha} - 1 \Big) du \\
&\le \int_0^{\delta} \Big(\Big(\frac{(T_1^{H_1} + T_2^{H_2})^{\frac{1}{H_1}} 2^{\frac{1}{H_1}} c^{\frac{1}{\beta H_1}}}{2u^{\frac{1}{\beta H_1}}} + 1 \Big)^{2\alpha} - 1 \Big) du \\
&\le \int_0^{\delta} \Big(\frac{(T_1^{H_1} + T_2^{H_2})^{\frac{1}{H_1}} 2^{\frac{1}{H_1}} c^{\frac{1}{\beta H_1}}}{2u^{\frac{1}{\beta H_1}}} \Big)^{2\alpha} du \\
&= (T_1^{H_1} + T_2^{H_2})^{\frac{2\alpha}{H_1}} c^{\frac{2\alpha}{\beta H_1}} 2^{2\alpha(\frac{1}{H_1} - 1)} \Big(1 - \frac{2\alpha}{\beta H_1} \Big)^{-1} \delta^{1 - 2\alpha/(\beta H_1)}.
\end{aligned}
$$

This implies the estimate

$$
\begin{aligned}
r^{(-1)}\Big(\frac{I_r(\delta)}{\delta} \Big) &\le \Big(\tau^{2\alpha} \Big(1 - \frac{2\alpha}{\beta H_1} \Big)^{-1} \delta^{-2\alpha/(\beta H_1)} + 1 \Big)^{\frac{1}{\alpha}} \le \Big(\tau^{2\alpha} \delta^{-\frac{2\alpha}{\beta H_1}} \Big(1 - \frac{2\alpha}{\beta H_1} \Big)^{-1} + 1 \Big)^{\frac{1}{\alpha}} \\
&\le 2^{\frac{1}{\alpha} - 1} \Big(\tau^2 \delta^{-\frac{2}{\beta H_1}} \Big(1 - \frac{2\alpha}{\beta H_1} \Big)^{-\frac{1}{\alpha}} + 1 \Big),
\end{aligned}
$$

where $\tau = (T_1^{H_1} + T_2^{H_2})^{\frac{1}{H_1}} c^{\frac{1}{\beta H_1}} 2^{\frac{1}{H_1} - 1}$. Let $\alpha = \frac{\beta H_1}{4}$, then we obtain

$$
r^{(-1)}\Big(\frac{I_r(\delta)}{\delta} \Big) \le 2^{\frac{4}{\beta H_1} - 1} \Big(\tau^2 2^{\frac{4}{\beta H_1}} \delta^{-\frac{2}{\beta H_1}} + 1 \Big). \tag{3.5}
$$

Therefore, under the conditions of theorem we can write the corresponding estimate for $I_{r,k}(\theta\varepsilon_k)$ and then we obtain

$$
r^{(-1)}\Big(\frac{I_{r,k}(\theta\varepsilon_k)}{\theta\varepsilon_k} \Big) \le 2^{\frac{4}{\beta H_1} - 1} \Big(\frac{(l_k^{H_1} + (2A)^{H_2})^{\frac{2}{H_1}} c_k^{\frac{2}{\beta H_1}} 2^{\frac{4}{\beta H_1} + \frac{2}{H_1} - 2}}{(\theta\varepsilon_k)^{\frac{2}{\beta H_1}}} + 1 \Big).
$$

Applying the inequality $\log(1+x) \le \frac{x^\gamma}{\gamma}$, for $0 < \gamma \le 1$ and $x \ge 0$, we can write the estimate:

$$
S(\theta, r) \le C \log(2^{\frac{4}{\beta H_1} - 1}) + \frac{1}{\gamma} \sum_{k=0}^{\infty} \frac{\varepsilon_k^{1 - \frac{2\gamma}{\beta H_1}}}{f_k} (l_k^{H_1} + (2A)^{H_2})^{\frac{2\gamma}{H_1}} c_k^{\frac{2\gamma}{\beta H_1}} \frac{2^{\gamma(\frac{4}{\beta H_1} + \frac{2}{H_1} - 2)}}{\theta^{\frac{2\gamma}{\beta H_1}}},
$$

from which we obtain the expression for $\widetilde{A}_3(\theta)$. □

Remark 3.1. Note that in Kozachenko and Makogin (2014) several results were obtained for the rate of growth for self-similar Gaussian random fields using the bounds on the increments in terms of metrics d_3. We state a more general result suitable for wider class of fields. To compare the results futher investigation will be needed.

4. Applications

In this section we present several examples of random fields for which the results of previous sections can be applied. We will concentrate on the results on the distributions of suprema implied by Theorem 2.1, the rate of growth of trajectorries for these models can be evaluated analogously, using theorems presented in Section 3.

Example 4.1. Consider Funaki's model Funaki (1983); Mueller and Tribe (2002) for random string in $\mathbb{R}^n$ specified by the following stochastic heat equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + \dot{W}, \tag{4.1}$$

where $\dot{W} = \dot{W}(x,t)$ is a $\mathbb{R}^n$-valued space-time white noise (with independent components) and $u(t,x), t \geq 0, x \in \mathbb{R}$ is a continuous $\mathbb{R}^n$-valued process.

Let initial condition be given by $u_0(x), x \in \mathbb{R}$, which is two-sided $\mathbb{R}^n$-valued Brownian motion satisfying $u_0(0) = 0$ and $\mathsf{E}[(u_0(x) - u_0(y))^2] = |x - y|$, and which is independent of the white noise $\dot{W}$. Such initial process can be created as

$$u_0(x) = \int_0^\infty \int_{\mathbb{R}} (G(r, x-z) - G(r,z))\widetilde{W}(dz, dr),$$

where a space-time white noise is independent of $\dot{W}$, $G(t,x) = \frac{1}{\sqrt{2\pi t}} \exp\{-\frac{x^2}{4t}\}$ is the fundamental solution of the heat equation.

Then the solution to (4.1) is given by

$$u(t,x) = \int_0^\infty \int_{\mathbb{R}} (G(t+r, x-z) - G(t+r,z))\widetilde{W}(dz,dr) + \int_0^t \int_{\mathbb{R}} G(r, x-z)\widetilde{W}(dz,dr).$$

We refer for rigorous details to Mueller and Tribe (2002). In Mueller and Tribe (2002) a continuous version of this process is called the stationary pinned string and the following result was obtained

Proposition 4.1. *(Mueller and Tribe (2002), Proposition 1) The components* $u^{(i)}(t,x), t \geq 0, x \in \mathbb{R}$, *of the stationary pinned string are zero mean Gaussian fields satisfying*

$$\mathsf{E}[u^{(i)}(t,x) - u^{(i)}(s,y)]^2 \leq 2(|x-y| + |t-s|^{1/2}).$$

Therefore, we have

$$\rho_{u^{(i)}}(t,x) = \left(\mathsf{E}[u^{(i)}(t,x) - u^{(i)}(s,y)]^2\right)^{1/2} \leq \sqrt{2}(|x-y| + |t-s|^{1/2})^{1/2} = \sqrt{2} d((t,x),(s,y))^{1/2},$$

and we can apply Theorem 2.1 with the metrics $d((t,x),(s,y)) = |x-y| + |t-s|^{1/2}$.

Consider $u^{(i)}(t,x), (t,x) \in T = [a_1, b_1] \times [a_2, b_2]$. We obtain:

$$P\{\sup_{t,x \in T} |u^{(i)}(t,x)| \geq v\} \leq 2\exp\Big\{-\frac{v^2(1-\theta^2)}{2\varepsilon_0^2}\Big\} T_1 T_2 (2e)^3 (\theta\varepsilon_0)^{-3},$$

where $\varepsilon_0 = \sup_{t,x\in T}\left(\mathsf{E}(u^{(i)}(t,x))^2\right)^{1/2}$.

For the next examples we will need an additional definition.

Definition 4.1. Kozachenko and Koval'chuk (1998) A family Δ of φ-sub-Gaussian random variables is called strictly φ-sub-Gaussian if there exists a constant C_Δ such that for all countable sets I of random variables $\zeta_i \in \Delta$, $i \in I$, the inequality holds: $\tau_\varphi\big(\sum_{i\in I}\lambda_i\zeta_i\big) \leq C_\Delta\big(\mathsf{E}\left(\sum_{i\in I}\lambda_i\zeta_i\right)^2\big)^{1/2}$. The constant C_Δ is called the determining constant of the family Δ. Random process $\zeta(t), t \in T$, is called strictly φ-sub-Gaussian if the family of random variables $\{\zeta(t), t \in T\}$ is strictly φ-sub-Gaussian.

Let K be a deterministic kernel and $X(t) = \int_T K(t,s)\, d\xi(s)$, where $\xi(t), t \in T$, is a strictly φ-sub-Gaussian process and the integral is defined in the mean-square sense. Then $X(t), t \in T$, is strictly φ-sub-Gaussian process with the same determining constant (see Kozachenko and Koval'chuk (1998)).

Example 4.2. Heat equation with random initial condition. Consider the Cauchy problem for the heat equation

$$\frac{\partial u}{\partial t} = \mu \Delta u, t > 0, x \in \mathbb{R}, \mu > 0, \tag{4.2}$$

subject to the random initial condition

$$u(0, x) = \eta(x), x \in \mathbb{R}, \tag{4.3}$$

where $\eta(x), x \in \mathbb{R}$ satisfies the following assumption: $\eta(x)$ is a real, measurable, mean-square continuous stationary stochastic process, which is a strictly φ-sub-Gaussian with the determining constant c_η.

Let $B(x), x \in \mathbb{R}$, be a covariance function of the process $\eta(x), x \in \mathbb{R}$, with the representation

$$B(x) = \int_{\mathbb{R}} \cos(\lambda x) dF(\lambda), \tag{4.4}$$

where $F(\lambda)$ is a spectral measure, and for the process itself we can write the spectral representation

$$\eta(x) = \int_{\mathbb{R}} e^{i\lambda x} Z(d\lambda), \tag{4.5}$$

The stochastic integral is considered as $L_2(\Omega)$ integral. Orthogonal random measure Z is such that $\mathsf{E}|Z(d\lambda)|^2 = F(d\lambda)$.

The mean-square solution to the problem (4.2)-(4.3) can be writen in the form

$$u(t, x) = \int_{\mathbb{R}} \exp\Big\{ i\lambda x - \mu t \lambda^2 \Big\} Z(d\lambda) \tag{4.6}$$

(see, Hopkalo and Sakhno (2021) and references therein) and the covariance function

$$Cov\Big(u(t, x), u(s, y)\Big) = \int_{\mathbb{R}} \exp\Big\{ i\lambda(x - y) - \mu\lambda^2(t + s) \Big\} F(d\lambda).$$

Proposition 4.2. *Under the condition* $c^2 := \int_{\mathbb{R}} \lambda^{2\varepsilon} F(d\lambda) < \infty$ *for some* $\varepsilon \in (0, 1]$, *the field* (4.6) *satisfies:*

$$\tau_\varphi\Big(u(t, x) - u(s, y)\Big) \le c_\eta c \Big(4^{1-\varepsilon}|x - y|^{2\varepsilon} + \mu^\varepsilon |t - s|^\varepsilon\Big)^{1/2}. \tag{4.7}$$

Proof. We have

$$\mathsf{E}\Big(u(t, x) - u(s, y)\Big)^2 = \int_{\mathbb{R}} |b(\lambda)|^2 F(d\lambda),$$

where

$$b(\lambda) = \exp\{i\lambda x\} \exp\{-\mu\lambda^2 t\} - \exp\{i\lambda y\} \exp\{-\mu\lambda^2 s\}.$$

$$|b(\lambda)|^2 \le \Big(1 - \exp\Big\{ -\mu\lambda^2|t - s| \Big\}\Big)^2 + 4\sin^2(\frac{1}{2}\lambda(x - y)) = b_1(\lambda) + b_2(\lambda)$$

$$\int_{\mathbb{R}} |b(\lambda)|^2 F(d\lambda) \le \int_{\mathbb{R}} b_1(\lambda) F(d\lambda) + \int_{\mathbb{R}} b_2(\lambda) F(d\lambda)$$

$$b_2(\lambda) = 4\sin^2(\frac{1}{2}\lambda(x - y)) \le 4\min(\frac{1}{2}|\lambda||x - y|, 1)^2$$

$$\begin{aligned}
\int_{\mathbb{R}} b_2(\lambda) F(d\lambda) &\le \int_{\mathbb{R}} \min(4, \lambda^2|x - y|^2) F(d\lambda) \\
&= \int_{\{\frac{1}{4}\lambda^2|x-y|^2 \ge 1\}} 4F(d\lambda) + 4\int_{\{\frac{1}{4}\lambda^2|x-y|^2 < 1\}} \frac{\lambda^2|x - y|^2}{4} F(d\lambda) \\
&\le \int 4\Big(\frac{\lambda^2|x - y|^2}{4}\Big)^\varepsilon F(d\lambda) + 4\int \Big(\frac{\lambda^2|x - y|^2}{4}\Big)^\varepsilon F(d\lambda) \\
&= 4^{1-\varepsilon}|x - y|^{2\varepsilon} \int_{\mathbb{R}} \lambda^{2\varepsilon} F(d\lambda),
\end{aligned}$$

where $\varepsilon \in (0,1]$. Let us suppose $\int_{\mathbb{R}} \lambda^{2\varepsilon} F(d\lambda) < \infty$.

$$b_1(\lambda) = \Big(1 - \exp\Big\{-\mu\lambda^2|t-s|\Big\}\Big)^2 \le \Big(\min(\mu\lambda^2|t-s|, 1)\Big)^2$$

$$\begin{aligned}\int_{\mathbb{R}} |b_1(\lambda)| F(d\lambda) &\le \int_{\mathbb{R}} \min(\mu^2\lambda^4|t-s|^2, 1) F(d\lambda)\\ &= \int_{\{\mu^2\lambda^4|t-s|^2\ge 1\}} F(d\lambda) + \int_{\{\mu^2\lambda^4|t-s|^2<1\}} \mu^2\lambda^4|t-s|^2 F(d\lambda)\\ &\le \int \Big(\mu^2\lambda^4|t-s|^2\Big)^{\tilde{\varepsilon}} F(d\lambda) + \int \Big(\mu^2\lambda^4|t-s|^2\Big)^{\tilde{\varepsilon}} F(d\lambda)\\ &= \mu^{2\tilde{\varepsilon}}|t-s|^{2\tilde{\varepsilon}} \int_{\mathbb{R}} \lambda^{4\tilde{\varepsilon}} F(d\lambda),\end{aligned}$$

where $\tilde{\varepsilon} \in (0,1)$. Let us choose $\tilde{\varepsilon} = \frac{\varepsilon}{2}$, then we can write the bound for $\varepsilon \in (0,1]$

$$\Big(\mathsf{E}\Big(u(t,x) - u(s,y)\Big)^2\Big)^{1/2} = \Big(\int_{\mathbb{R}} \lambda^{2\varepsilon} F(d\lambda)\Big)^{1/2} \Big(4^{1-\varepsilon}|x-y|^{2\varepsilon} + \mu^{\varepsilon}|t-s|^{\varepsilon}\Big)^{1/2},$$

under the condition that $\int_{\mathbb{R}} \lambda^{2\varepsilon} F(d\lambda) < \infty$.

Since the initial condition process η is assumed to be strictly φ-sub-Gaussian, we have $\tau_\varphi\Big(u(t,x) - u(s,y)\Big) \le c_\eta \Big(\mathsf{E}\Big(u(t,x) - u(s,y)\Big)^2\Big)^{1/2}$, and therefore, bound (4.7) follows. □

Consider $u(t,x)$, $(t,x) \in T = [a_1, b_1] \times [a_2, b_2]$. Denote $T_i = b_i - a_i$, $i = 1, 2$.

Under the conditions of Proposition 4.2 we can apply Theorem 2.1 to obtain the bound

$$P\{\sup_{t,x\in T} |u(t,x)| \ge v\} \le 2\exp\Big\{-\varphi^*\Big(\frac{u(1-\theta)}{\varepsilon_0}\Big)\Big\} T_1 T_2 2^{5/(2\varepsilon)-1} \mu (c_\eta c e)^{3/\varepsilon} (\theta\varepsilon_0)^{-3/\varepsilon},$$

where $c = \Big(\int_{\mathbb{R}} \lambda^{2\varepsilon} F(d\lambda)\Big)^{1/2}$.

Example 4.3. Consider the Airy equation

$$\frac{\partial u}{\partial t} = -\frac{\partial^3 u}{\partial x^3}, t > 0, x \in \mathbb{R},$$

subject to the random initial condition

$$u(0,x) = \eta(x), x \in \mathbb{R},$$

where $\eta(x)$ is the stationary φ-sub-Gaussian stochastic process as in the previous example.

The mean-square solution can be written in the following form (see, Sakhno (2023b) and references therein):

$$u(t,x) = \int_{\mathbb{R}} \exp\Big\{i\lambda x - i\lambda^3 t\Big\} Z(d\lambda) \tag{4.8}$$

and the covariance function is given as

$$Cov\Big(u(t,x), u(s,y)\Big) = \int_{\mathbb{R}} \exp\Big\{i\lambda(x-y) + i\lambda^3(t-s)\Big\} F(d\lambda).$$

Proposition 4.3. *Under the condition* $\int_{\mathbb{R}} |\lambda|^3 F(d\lambda) < \infty$, *the field* (4.8) *satisfies:*

$$\tau_\varphi\Big(u(t,x) - u(s,y)\Big) \le c_\eta \sqrt{2}\Big(c_1|x-y| + c_2|t-s|\Big)^{1/2}, \tag{4.9}$$

where $c_1 = \int_{\mathbb{R}} |\lambda| F(d\lambda)$, $c_2 = \int_{\mathbb{R}} |\lambda|^3 F(d\lambda)$.

Proof. Consider

$$\mathsf{E}\Big(u(t,x)-u(s,y)\Big)^2 = \int_{\mathbb{R}} |\tilde{b}(\lambda)|^2 F(d\lambda),$$

where

$$|\tilde{b}(\lambda)|^2 = 4\sin^2\Big(\frac{\lambda(x-y)+\lambda^3(t-s)}{2}\Big) \le$$
$$\le 4\min\Big(1, \frac{1}{2}(|\lambda||x-y|+|\lambda|^3|t-s|)\Big)^2 \le 4\Big(\frac{1}{2}(|\lambda||x-y|+|\lambda|^3|t-s|)\Big)^{2\varepsilon}$$

for $\forall\varepsilon\in(0,1]$. Choose $\varepsilon=\frac{1}{2}$, then we have $|\tilde{b}(\lambda)|^2 \le 2\Big(|\lambda||x-y|+|\lambda|^3|t-s|\Big)$, and

$$\tau_\varphi\Big(u(t,x)-u(s,y)\Big) \le c_\eta\Big(\int_{\mathbb{R}} |\tilde{b}(\lambda)|^2 F(d\lambda)\Big)^{1/2}$$
$$\le \sqrt{2}c_\eta\Big(|x-y|\int_{\mathbb{R}}|\lambda|F(d\lambda)+|t-s|\int_{\mathbb{R}}|\lambda|^3F(d\lambda)\Big)^{1/2} =$$
$$= \sqrt{2}c_\eta\Big(c_1|x-y|+c_2|t-s|\Big)\Big)^{1/2},$$

where c_1 and c_2 as in the statement of proposition. □

Consider $u(t,x)$, $(t,x)\in T=[a_1,b_1]\times[a_2,b_2]$. Denote $T_i=b_i-a_i$, $i=1,2$.
In view of Proposition 4.3, under condition $\int_{\mathbb{R}}|\lambda|^3F(d\lambda)<\infty$ we can apply Theorem 2.1 for the case of the metrics $d_2((t,x),(s,y)) = c_1|x-y|+c_2|t-s|$, $c_1=\int_{\mathbb{R}}|\lambda|F(d\lambda)$, $c_2=\int_{\mathbb{R}}|\lambda|^3F(d\lambda)$, to obtain the bound:

$$P\{\sup_{t,x\in T}|u(t,x)|\ge u\} \le 2\exp\Big\{-\varphi^*\Big(\frac{u(1-\theta)}{\varepsilon_0}\Big)\Big\}\min(T_1c_1,T_2c_2)(T_1c_1+T_2c_2)(\sqrt{2}c_\eta e)^4(\theta\varepsilon_0)^{-4}.$$

In particular, in the case of Gaussian initial condition we have the estimate:

$$P\{\sup_{t,x\in T}|u(t,x)|\ge u\} \le 2\exp\Big\{-\frac{u^2(1-\theta)^2}{2\varepsilon_0^2}\Big\}\min(T_1c_1,T_2c_2)(T_1c_1+T_2c_2)(\sqrt{2}e)^4(\theta\varepsilon_0)^{-4},$$

where $\varepsilon_0=(B_\eta(0))^{1/2}$.

Example 4.4. Let $X(t), t\in\mathbb{R}^2_+$ be a centered Gaussian self-similar random field with index $(H_1,H_2)\in(0,1)^2$ and with stationary rectangular increments.
Recall that a random field X is self-similar if

$$X(a_1t_1,a_2t_2) \stackrel{d}{=} a_1^{H_1}a_2^{H_2}X(t_1,t_2), (t_1,t_2)\in\mathbb{R}^2_+,$$

for all $a_1>0, a_2>0$.
The field X has stationary rectangular increments if for any $(u_1,u_2)\in\mathbb{R}^2_+$

$$\Delta_uX(u+h)=\Delta_0X(h), h=(h_1,h_2)\in\mathbb{R}^2_+$$

where $\Delta_uX(v)=X(v_1,v_2)-X(u_1,v_2)-X(v_1,u_2)+X(u_1,u_2)$. Assume that $\mathsf{E}X^2(1)=1$.
It can be shown by using the Minkowski inequality that

$$\Big(\mathsf{E}\Big(X(t_1,t_2)-X(s_1,s_2)\Big)^2\Big)^{1/2} \le |t_1-s_1|^{H_1}t_2^{H_2}+|t_2-s_2|^{H_2}s_1^{H_1}$$

(see, for example Kozachenko and Makogin (2014), Lemma 2.4). Consider the field $X(t_1,t_2)$ over the rectangle $T=[0,T_1]\times[0,T_2]$. Then for $(t_1,t_2),(s_1,s_2)\in T$ we can write the estimate

$$\Big(\mathsf{E}\Big(X(t_1,t_2)-X(s_1,s_2)\Big)^2\Big)^{1/2} \le T_2^{H_2}|t_1-s_1|^{H_1}+|t_2-s_2|^{H_2}T_1^{H_1}=d((t_1,t_2),(s_1,s_2)).$$

Therefore, we can apply Theorem 2.1, case $i = 4$, with $\varepsilon_0 = \sup_{t\in T}(\mathsf{E}X^2(t))^{1/2} = T_1^{H_1}T_2^{H_2}$ to obtain

$$P\{\sup_{(t_1,t_2)\in[0,T_1]\times[0,T_2]}|X(t_1,t_2)| \geq u\} \leq 2\exp\Big\{-\frac{u^2(1-\theta)^2}{2T_1^{2H_1}T_2^{2H_2}}\Big\}(2e)^{\frac{1}{H_1}+\frac{1}{H_2}}\theta^{-(\frac{1}{H_1}+\frac{1}{H_2})}.$$

Note that the calculations for the above result can also be done by using the selfsimilarity of X and the following equality

$$P\{\sup_{(t_1,t_2)\in[0,T_1]\times[0,T_2]}|X(t_1,t_2)| \geq u\} = P\{\sup_{(t_1,t_2)\in[0,1]^2}|X(t_1,t_2)| \geq u/(T_1^{H_1}T_2^{H_2})\}.$$

Example 4.5. One particular way to construct a φ-sub-Gaussian stochastic process was presented in Kozachenko and Koval'chuk (1998) (see also Vasylyk *et al.* (2008)). Let $\{\xi_k,\ k = \overline{1,\infty}\}$ be a family of independent φ-sub-Gaussian random variables and φ be a such function that $\varphi(\sqrt{x})$, $x > 0$, is convex. If there exists $C > 0$ such that $\tau_\varphi(\xi_k) \leq C(\mathsf{E}\xi_k^2)^{1/2}$ for any $k \geq 1$, and for a sequence of nonrandom functions $f_k(t)$, $t \in T$, $k \geq 1$, the series $\sum_{k=1}^{\infty}\mathsf{E}\xi_k^2 f_k^2(t)$ converges for all $t \in T$, then $X(t) = \sum_{k=1}^{\infty}\xi_k f_k(t)$, $t \in T$, is a strictly φ-sub-Gaussian stochastic process with determining constant C and $\tau_\varphi^2(X(t) - X(s)) \leq C^2\mathsf{E}(X(t) - X(s))^2$, $t, s \in T$. Let $(\mathbf{T}, d)$ be as in Theorem 1.2, suppose additionally that functions f_k are such that for some $c_k > 0$, $k \geq 1$, and strictly increasing continuous function $\sigma(h)$, $h \geq 0$, $\sigma(0) = 0$, we have $\sup_{d(t,s)<h}|f_k(t) - f_k(s)| \leq c_k\sigma(h)$ and $\sum_{k=1}^{\infty}\mathsf{E}\xi_k^2 c_k^2 < \infty$. Then condition (1.7) holds. Therefore, Theorem 1.2 and its specification for different metrics in Theorem 2.1 are applicable and the bound for the distribution of supremum of the process X will depend on the bounds for increments of functions f_k.

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Affiliation:

Olha Hopkalo
Department of Economic Cybernetics, Faculty of Economics
Taras Shevchenko National University of Kyiv
Volodymyrska Str. 64, Kyiv 01601, Ukraine
E-mail: olha.hopkalo@knu.ua

Lyudmyla Sakhno
Department of Probability Theory, Statistics and Actuarial Mathematics
Faculty of Mechanics and Mathematics
Taras Shevchenko National University of Kyiv
Volodymyrska Str. 64, Kyiv 01601, Ukraine
E-mail: lms@univ.kiev.ua

Olga Vasylyk
Department of Mathematical Analysis and Probability Theory
National Technical University of Ukraine "Igor Sikorsky Kyiv Polytechnic Institute"
Peremogy Ave. 37, Kyiv 03056, Ukraine
E-mail: vasylyk@matan.kpi.ua

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