Austrian Journal of Statistics
*January 2023, Volume 52, 86–103.*
*http://www.ajs.or.at/*
doi:10.17713/ajs.v52i2.1395

# Maximum Product Spacings Estimator for Fuzzy Data Using Inverse Lindley Distribution

**Ankita Chaturvedi**
Banaras Hindu University

**Sanjay Kumar Singh**
Banaras Hindu University

**Umesh Singh**
Banaras Hindu University

**Abstract**

The article addresses the problem of parameter estimation of the inverse Lindley distribution when the observations are fuzzy. The estimation of the unknown model parameter was performed using both classical and Bayesian methods. In the classical approach, the estimation of the population parameter is performed using the maximum likelihood (ML) method and the maximum product of distances (MPS) method. In the Bayesian setup, the estimation is obtained using the squared error loss function (SELF) with the Markov Chain Monte Carlo (MCMC) technique. Asymptotic confidence intervals and highest posterior density (HPD) credible intervals for the unknown parameter are also obtained. The performances of the estimators are compared based on their MSEs. Finally, a real data set is analyzed for numerical illustration of the above estimation methods.

*Keywords*: inverse Lindley distribution, maximum likelihood (ML), maximum product of spacings (MPS), asymptotic confidence intervals, HPD credible intervals, Markov chain Monte Carlo (MCMC).

## 1. Introduction

In statistics, estimation refers to the process of drawing inferences about a population. These inferences arebased on the information obtained from a sample that is drawn from the same population. Numerous estimation methods are available in literature along with their advantages and disadvantages. Some well-known estimation methods are the method of maximum likelihood, method of least square, method of moments, etc. Out of all these methods, method of ML is the most widely used method of estimation due to its various useful properties. But in certain situations, MLE ceases to perform satisfactorily. Various authors have noted the limitations of MLE in different contexts. For example, MLE performs efficiently only when the likelihood is bounded above. It does not perform satisfactorily for "heavy-tailed" distributions with unspecified scale and location parameters (see Pitman (1979)) and provides inconsistent results for small samples. Cheng and Traylor (1995) have remarked that MLE fails to estimate three-parameter distributions, like gamma, Weibull, and lognormal.

To overcome the drawbacks of MLE, Cheng and Amin (1983) introduced the MPS method of estimation. Several useful properties of the MPS estimator were also discussed by them. They showed that MPS provides a consistent and asymptotically efficient estimator in both situations whether MLE exists or not. When the MLE exists, MPS gives consistent estimators with asymptotic efficiency equal to MLE, and if it does not then also MPS provides consistent results. The concept of the MPS estimator was also developed simultaneously by Ranneby (1984) using the Kullback-Leibler measure of information. Anatolyev and Kosenok (2005) have shown that the MPS estimator is more efficient than MLE for small samples. Since we are using the product of spacings which is always bounded owing to the properties of CDF there is no compulsion of checking the above boundedness, as in the case of MLE. Considering all these facts, we can say that the method of MPS possesses asymptotic properties like MLE and overcomes all shortcomings of MLE.

Statistical modeling is an essential part of data handling. It covers inherent randomness in the data and has universal applications in every branch of science. We encounter several continuous variables in our daily life, which are examined in miscellaneous prospects. Every measurement of a continuous variable is assumed to be a precise number. This assumption is not appropriate since continuous phenomenons cannot be measured precisely. Countless sophisticated tools have been developed to get precise measurements. Still, the results obtained are more or less imprecise and are called fuzzy. Therefore, we can say that two types of uncertainties may be incorporated in the observed data, one is variation among the observations and the other is the imprecision of the individual observation called fuzziness. Although plenty of methods have been evolved to model the uncertainty due to variation among observations statistical literature discussing uncertainty due to imprecision or vagueness of observed data was scarcely available.However, a remarkable upsurge has been experienced during the last few decades after the introduction of the concept of fuzzy sets by Zadeh (1965) who developed it as the generalization of the classical crisp sets. After that, he discussed many related concepts and associated methodologies in his subsequent articles and popularized it (see; Zadeh (1965, 1968, 1983, 1987, 1996, 1999, 2008, 2011), Yager and Zadeh (2012), Zadeh, Fu, and Tanaka (2014)). Although initially it was introduced as a mathematical concept, soon it was adopted by different branches of science since it works in those situations where usual methods fail to provide any result or some approximation is used to convert the data into a particular form that can be analyzed regularly.

Primary sources of vagueness, un-ambiguity, non-preciseness, or simply fuzziness in observed data are experimental errors, human errors, the precision of measurement, and several other practical difficulties such as linguistic descriptions, measurement of continuous variables in the form of precise numbers, etc. In life testing experiments, sometimes due to the shortage of experimental material, their failure times are directly observed by the users after purchasing the product. We cannot expect that the user will observe and report the failure time of the product with precision. We try to explore such type of situation by the following example. We try to explore such type of situation by the following example. In survival analysis, suppose we want to know the failure time of an item from its consumer. It can be reported as “approximately between 99 hours to 101 hours”, “about 100 hours” etc. We may note that, although these observations are not precise, their representation in this form seems logical in the sense that the lifetime of an experimental unit is a continuous variable and hence cannot be an exact number. Depending upon the preciseness of the measurement, it will always have some errors.Therefore, whenever we represent the lifetime of an experimental unit as an exact number, some error gets incorporated instantaneously, as also indicated by Barbato, Germak, and Genta (2013). For example, suppose the exact failure time of an item in survival analysis is reported as 100 hours. It can be argued that no value can be measured as absolutely 100. There will always be a difference between true and reported value. Hence, the reported value is nothing but an approximation of the exact value. In fact, it corresponds to any value between 99 hours to 101 hours i.e. more than 99 hours but less than 101 hours. Moreover, it should be very clear that all the methods of estimation which are available in the literature cannot be utilized to analyze such type of imprecise data. A suitable method of incorporating the ambiguity in standard methods is the fuzzy set theory proposed by Zadeh (1965). A fuzzy number is a generalization of a regular, real number in the sense that it does not refer to one single value but rather to a connected set of possible values, where each possible value has its own weight between 0 and 1. In this sense, the theory of fuzzy sets generalizes the concept of indicator function by relating a grade of membership between $[0,1]$ defined in terms of the membership function of a fuzzy number. In mathematical form, a membership function corresponding to a fuzzy number is nothing but a functional form that can take values between $[0,1]$. These membership functions are flexible, and the transition from true to false is more gradual than the usual crisp values. A fuzzy set can be represented graphically with the help of its membership function. In mathematical terminology, a fuzzy number say $A$ denoted as $\tilde{A}$, of the set of real numbers $\mathbb{R}$ and is characterized by membership function $\xi_{\tilde{A}}(x)$ which associates with each point in $A$, a real number in interval $[0,1]$, with the value of $\xi_{\tilde{A}}(x)$ at each $x$ representing the grade of membership of $x$ in $A$. There is no fixed rule in the literature to select the membership function corresponding to a fuzzy number, and it solely depends upon the concerned problem. Any kind of priori information is very helpful in deciding the shape of the membership function. Membership functions represent the simple events usually observed in real-life situations. Therefore, simple functions are used as membership functions. The most common membership functions are triangular, trapezoidal, Gaussian, bell-shaped, etc. A $L-R$ fuzzy membership function is the one that specifies the membership function with the help of two functional forms. In generalized notations, a $L-R$ fuzzy membership function is given as follows:

$$\xi_{\widetilde{A}}(x) = \begin{cases} 0 & \text{if} \quad x < a1 \\ \beta_{\widetilde{A}}(x) & \text{if} \quad a1 \leq x < a2 \\ 1 & \text{if} \quad a2 \leq x < a3 \\ \gamma_{\widetilde{A}}(x) & \text{if} \quad a3 \leq x < a4 \\ 0 & \text{if} \quad a4 < x \end{cases}$$

where $\beta_{\widetilde{A}}(x)$ is a non-decreasing and $\gamma_{\widetilde{A}}(x)$ is a non-increasing function. For different functional forms, the above stated general membership function will reduce to the triangular, trapezoidal, Gaussian or bell shaped membership functions. For more detailed description of the basic concepts of fuzzy logic, readers are referred to; Dubois (1980), Dubois and Prade (1998), Zimmermann (2001), Buckley (2006), Lee (2006), Nguyen and Wu (2006), Viertl (2011).

In the last few decades, fuzzy logic gained popularity in every branch of science. Singpurwalla and Booker (2004) provided a detailed description of the membership function. Estimation of reliability in Bayesian setup, when the available data is fuzzy, was discussed by several authors such as Hryniewicz (1986), Viertl (1997), Huang, Zuo, and Sun (2006), Viertl (2009), Pak, Parham, and Saraj (2014b). Coppi, Gil, and Kiers (2006) presented some applications of fuzzy techniques in statistical analysis. Denœux (2011) considered the MLE based on fuzzy data using the EM algorithm. Taheri and Zarei (2011) considered the Bayesian estimation of failure rate and mean time to failure based on vague set theory in the case of complete and censored data sets. Pak, Parham, and Saraj (2013b), Pak, Parham, and Saraj (2013a), Pak, Parham, and Saraj (2014a), Pak and Chatrabgoun (2016) discussed the inferential procedures of a number of lifetime distributions under both classical and Bayesian setup using complete as well as censored fuzzy data. Recently, Chaturvedi, Singh, and Singh (2018) obtained the inferences of type-II progressively hybrid censored fuzzy data using Rayleigh distribution.

In the present piece of work, the lifetime of units understudy is denoted by the random variable $X$. It is governed by the inverse Lindley distribution,denoted as ILD$(\theta)$. An ILD$(\theta)$ variate $X$ may be obtained by inverting a Lindley$(\theta)$ variate (see; Ghitany, Atieh, and Nadarajah (2008) ) or by mixing inverse exponential distribution with scale parameter $\theta$ and inverse Gamma distribution with shape parameter 2 and scale parameter $\theta$ in the proportion $\theta/(1+\theta)$ (see; Sharma, Singh, Singh, and Agiwal (2015)). They have discussed the properties of the ILD$(\theta)$ and also justified its suitability in a real scenario using two different real data sets. The novelty of the distribution lies in the fact that it possesses a single parameter like exponential

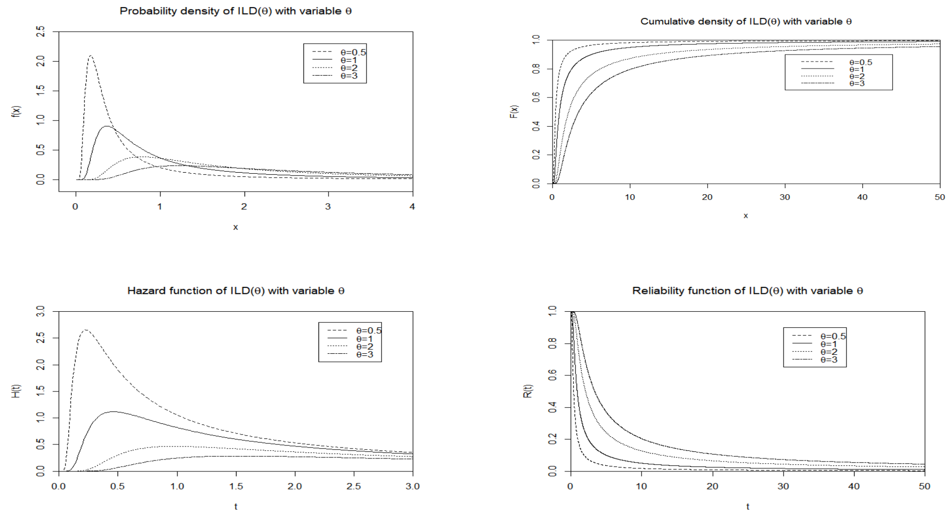


Figure 1: Probability density function, Cumulative density function, Hazard function and Reliability function of ILD($\theta$) for different values of $\theta$

distribution, but in contrast to the constant hazard rate of the exponential distribution, it has a uni-modal, non-monotone, up-side-down bathtub (UBT) shape for hazard rate. The probability density function (PDF) and cumulative distribution function (CDF) of ILD($\theta$) are given respectively as:

$$f(x;\theta) = \frac{\theta^2}{(1+\theta)}\left(\frac{1+x}{x^3}\right)e^{-\frac{\theta}{x}} \quad ; \quad x>0, \theta>0 \tag{1}$$

$$F(x;\theta) = \left[1+\frac{\theta}{(1+\theta)x}\right]e^{-\frac{\theta}{x}} \quad ; \quad x>0, \theta>0 \tag{2}$$

where $\theta$ is the scale parameter. The expressions of reliability and hazard rate of ILD($\theta$) are also available in nice closed form and hence depicts its suitability for modeling lifetime data.

$$R(x;\theta) = 1-\left[1+\frac{\theta}{(1+\theta)x}\right]e^{-\frac{\theta}{x}} \quad ; \quad x>0, \theta>0 \tag{3}$$

$$H(x;\theta) = \frac{\theta^2(1+x)}{x^2\left[x(1+\theta)\left(e^{\frac{\theta}{x}}-1\right)-\theta\right]} \quad ; \quad x>0, \theta>0 \tag{4}$$

The specific aim of the present study is to develop classical and Bayesian estimation procedures for estimating the parameters of inverse Lindley distribution in the presence of vague or imprecise data. For this purpose, we have discussed the computation procedure of MLE of the population parameter. It is worthwhile to mention that we have proposed the well-known MPS estimator for imprecise data. Approximate confidence intervals for the unknown parameter are also obtained by using the asymptotic distributions of the MLE and MPS. Further, parameter estimate is also obtained in the Bayesian setup, and it is noticed that the Bayes estimators cannot be obtained in explicit form; therefore, we have used the MCMC technique to compute the Bayes estimates and construct the HPD credible interval of the parameter.In addition to it, the estimated reliability and hazard rate are also obtained under classical and Bayesian setup. The rest of the paper is organized as follows: In section 2, MLE and MPS estimates and their corresponding approximate confidence intervals are obtained by using their property of asymptotic normality. Section 3 discusses the estimation of parameters under the Bayesian setup. An overview of the simulation study followed by a real data illustration is given in Section 4. Finally, conclusions and recommendations are provided in section 5.

## 2. Classical estimation

In classical estimation, the unknown parameter is assumed to be deterministic or non-random, which means that randomness in the measurement is solely due to noise and not parameter variations. The purpose of estimation is to obtain an approximate value of the population parameter based on observations from the parent population. In this section, we will consider two different methods of classical estimation, MLE and MPS methods.

### 2.1. Maximum likelihood estimate

Let $X_1, X_2, \ldots, X_n$ be a sequence of independent observations on a random variable $X$ having the CDF and PDF given in Eqs. (2) and (1) respectively. If realizations of $X_1, X_2, \ldots, X_n$ say $x_1, x_2, \ldots, x_n$ are observed exactly then the likelihood equation is given as:

$$l(x;\theta) = \left(\frac{\theta^2}{1+\theta}\right)^n e^{-\theta \sum_{i=1}^{n} \frac{1}{x_i}} \prod_{i=1}^{n} \left(\frac{1+x_i}{x_i^3}\right) \qquad ; \qquad x, \theta > 0 \tag{5}$$

But here, we are assuming that we have vague knowledge of the observed values of the variable. As they are not in the precise form and reported with some error, i.e., the investigator can only provide approximate lifetimes of the items as guess values by specifying a small interval around the observed value. Besides, this interval is formed by the investigator according to his belief in various values of the interval. Such a type of ambiguity in the observed data can be successfully modeled with the help of the membership function of a fuzzy number. Therefore we may assume that the observed data is fuzzy and the inherent variability is incorporated in the form of possibility distribution. Let $\widetilde{x}_1, \widetilde{x}_2, \ldots, \widetilde{x}_n$ denotes the observed fuzzy lifetimes where $\widetilde{x}_i = (a_i, b_i, c_i); i = 1, 2, \ldots, n$, with the corresponding membership functions $\xi_{\widetilde{x}_1}(.), \xi_{\widetilde{x}_2}(.), \ldots, \xi_{\widetilde{x}_n}(.)$ such that

$$\xi_{\widetilde{x}_i}(x) = \begin{cases} \frac{x-(x_i-h)}{h}; & x_i - h \le x \le x_i \\ \frac{(x_i+h)-x}{h}; & x_i \le x \le x_i + h \\ 0 \quad ; & \text{otherwise} \end{cases}$$

where $h$ can be chosen suitably according to the prior knowledge available to us. Naturally, the value of the factor $h$ will vary from problem to problem depending upon its nature and extent of error in the observed data. Therefore, the joint membership function of observed fuzzy data vector $\widetilde{x} = \{\widetilde{x}_1, \widetilde{x}_2, \ldots, \widetilde{x}_n\}$ can be written as:

$$\xi_{\widetilde{x}}(x) = \prod_{i=1}^{n} \xi_{\widetilde{x}_i}(x) \tag{6}$$

Following the definition of probability of a fuzzy event by Zadeh (1968), the likelihood function of the observed fuzzy data is written as:

$$l(\theta; \widetilde{x}) = P(\widetilde{x}; \theta) = \int f(x;\theta) \xi_{\widetilde{x}}(x) dx \tag{7}$$

Substituting the values of $f(x;\theta)$ and $\xi_{\widetilde{x}}(x)$ from Eqs. (1) and (6) the likelihood equation will become as:

$$\begin{aligned} l(\theta; \widetilde{x}) &= \prod_{i=1}^{n} \int f(x;\theta) \xi_{\widetilde{x}_i}(x) dx \\ &= \prod_{i=1}^{n} \int \left(\frac{\theta^2}{1+\theta}\right) \left(\frac{1+x}{x^3}\right) e^{-\frac{\theta}{x}} \xi_{\widetilde{x}_i}(x) dx \end{aligned} \tag{8}$$

After taking the logarithm of the likelihood function, we get

$$L(\theta;\widetilde{x}) = logl(\theta;\widetilde{x}) = 2nlog\theta - nlog(1+\theta) + \sum_{i=1}^{n} log \int \left(\frac{1+x}{x^3}\right) e^{-\frac{\theta}{x}} \xi_{\widetilde{x}_i}(x)dx \tag{9}$$

Now, we will obtain the MLE of $\theta$ by maximizing the log-likelihood equation. Therefore, we differentiate the above equation with respect to parameter $\theta$ and equate the derivative to zero.

$$\frac{\partial L(\theta;\widetilde{x})}{\partial \theta} = \frac{2n}{\theta} - \frac{n}{(1+\theta)} - \sum_{i=1}^{n} \frac{\int \left(\frac{1+x}{x^4}\right) e^{-\frac{\theta}{x}} \xi_{\widetilde{x}_i}(x)dx}{\int \left(\frac{1+x}{x^3}\right) e^{-\frac{\theta}{x}} \xi_{\widetilde{x}_i}(x)dx} = 0 \tag{10}$$

Solving the above equation for $\theta$, we obtain the estimated value of parameter $\theta$. However, it is visible that the above equation fails to return an explicit solution. An appropriate approach for the parameter estimation of such equations may be to use iterative numerical methods. We devise the Newton-Raphson algorithm to estimate the parameter using a suitable initial guess.

## 2.2. Maximum product of spacings

The concept of the product of spacings is quite similar to that of maximum likelihood. To obtain the likelihood equation, we write the joint density of the sample observations. Similarly, we can obtain the product of spacings by taking the geometric mean of the spacings of ordered sample observations. Let $f(x)$ and $F(x)$ be the PDF and CDF of the distribution respectively and density $f(x)$ is strictly positive in the interval $(m,n)$ and zero everywhere else i.e. $m < x_{(1)} < x_{(2)} < \cdots < x_{(n)} < n$. Here, we are assuming that the r.v. $X$ is a continuous variable and support of $X$ is $(0,\infty)$ which makes $m = 0$ and $n = \infty$. Accordingly, boundary points of PDF and CDF will become as:

$$F(x) = 0 = f(x) \quad \forall \quad x < m \quad \text{and} \quad F(x) = 1; f(x) = 0 \quad \forall \quad x > n$$

Let $x_{i:n}$ denote the $i^{th}$ order statistics. Then the $ith$ spacing $D_i$ is defined as follows:

$$D_i = F(x_{i:n}) - F(x_{i-1:n}) \quad ; \quad i = 1, 2, \ldots, (n+1)$$

The product of spacings (PS) is defined as the geometric mean of $D_i$'s i.e. $G = \left[\prod_{i=1}^{n+1} D_i\right]^{(1/n+1)}$ with initial conditions $F(x_{0:n}) = 0$ and $F(x_{n+1:n}) = 1$. It is important to note that each spacing is nothing but the difference of CDF of ordered sample observations. Therefore, it is always bounded and provides a solution. A problem that arises naturally in practical situations is the problem of tied observations i.e. equal magnitude of two or more data points. In this situation, the product of spacings will necessarily reduce to zero, and we cannot obtain the estimated value of the parameter. Several authors have discussed the problem of tied observations and its remedies in detail such as; Shao, Hahn *et al.* (1999), Cheng and Stephens (1989), Singh, Singh, and Singh (2014), Cheng and Traylor (1995).

Here, we have to obtain the value of CDF using vague or fuzzy observations. This vagueness in the observed data is incorporated using fuzzy numbers by defining a triangular membership function corresponding to each observation based on some prior knowledge of the nature of imprecision.Therefore, CDF $F(\theta;\widetilde{x})$ corresponding to an observed fuzzy numbers $\widetilde{x} = (a,b,c)$, may be written as:

$$\begin{aligned} F(\theta;\widetilde{x}) &= P(X \le a) + \int P(a \le X \le x)\xi_{\widetilde{x}_i}(x)dx \\ &= F(a) + \int (F(x) - F(a))\xi_{\widetilde{x}_i}(x)dx \end{aligned} \tag{11}$$

Now, substituting the value of $F(x)$ and $F(a)$ in Eq. (11) from Eq. (2) the above equation can be written as:

$$F(\theta;\widetilde{x}) = \left[1+\frac{\theta}{(1+\theta)a}\right]e^{-\frac{\theta}{a}} + \int\left[\left(1+\frac{\theta}{(1+\theta)x}\right)e^{-\frac{\theta}{x}} - \left(1+\frac{\theta}{(1+\theta)a}\right)e^{-\frac{\theta}{a}}\right]\xi_{\widetilde{x}_i}(x)dx$$

As we have already mentioned, the product of spacings is nothing but the geometric mean of the spacings obtained by successive observations of an ordered sample from the given distribution.

$$G(\theta;\widetilde{x}) = \left[\prod_{i=1}^{n+1} D_i\right]^{(1/n+1)} \tag{12}$$

Now, that value of parameter $\theta$ which maximizes Eq. (12) will be the MPS estimator of parameter $\theta$. Therefore, taking logarithm on both sides of Eq. (12), we get the following expression:

$$\begin{aligned} S(\theta;\widetilde{x}) = logG(\theta;\widetilde{x}) &= \frac{1}{n+1}\sum_{i=1}^{n+1} logD_i \\ &= \frac{1}{n+1}\sum_{i=1}^{n+1} log\left[F(\theta;\widetilde{x}_i) - F(\theta;\widetilde{x}_{i-1})\right] \end{aligned} \tag{13}$$

where,

$$\begin{aligned} F(\theta;\widetilde{x}_i) - F(\theta;\widetilde{x}_{i-1}) = &\left[F(a_i) + \int (F(x_i) - F(a_i))\xi_{\widetilde{x}_i}(x)dx\right] \\ &- \left[F(a_{i-1}) + \int (F(x_{i-1}) - F(a_{i-1}))\xi_{\widetilde{x}_{i-1}}(x)dx\right] \end{aligned}$$

Differentiating Eq.(13) with respect to parameter $\theta$ we get the following normal equation in $\theta$:

$$\begin{aligned} \frac{\partial \ln G(\theta;\widetilde{x}_i)}{\partial\theta} = \sum_{i=1}^{n+1}\frac{1}{F(\theta;\widetilde{x}_i) - F(\theta;\widetilde{x}_{i-1})} &\left[\left\{\omega(\theta,a_i) + \int\left\{\omega(\theta,x_i) - \omega(\theta,a_i)\right\}\xi_{\widetilde{x}_i}(x)dx\right\}\right. \\ &\left. - \left\{\omega(\theta,a_{i-1}) + \int\left\{\omega(\theta,x_{i-1}) - \omega(\theta,a_{i-1})\right\}\xi_{\widetilde{x}_{i-1}}(x)dx\right\}\right] \end{aligned} \tag{14}$$

where,

$$\omega(\theta,x) = \frac{\partial}{\partial\theta}\left(\left\{1+\frac{\theta}{(1+\theta)x}\right\}e^{-\theta/x}\right) = -\frac{e^{-\theta/x}[\theta^2(x+1)+\theta(2x+1)]}{x^2(1+\theta)^2}$$

By solving the above non-linear equation in $\theta$, we get the MPS estimate of the population parameter $\theta$. Moreover, it is noticeable that no closed solution of the above equation exists. Therefore, we will use numerical methods to get the solution.

### 2.3. Asymptotic confidence interval

We cannot obtain the exact interval estimates based on MLE and MPS estimators, i.e., $\hat{\theta}_{ML}$ and $\hat{\theta}_{MPS}$ easily since the distribution of MLE and MPS is not available in explicit form. It is worthwhile to mention that the MPS estimator also shows asymptotic properties like MLE [see: Anatolyev and Kosenok (2005); Ghosh and Jammalamadaka (2001); Cheng and Amin (1983)] i.e.

$$(\hat{\theta}_{MPS} - \theta) \xrightarrow{D} N(0, I^{-1}(\theta))$$

Therefore, we will use large sample theory to construct asymptotic confidence intervals for population parameter $\theta$ using MLE and MPS estimators. As we know that the observed Fisher information is the second-order partial derivative of the likelihood function, we can obtain the same for MPS by simply replacing the likelihood function with the product spacings function. We can write the approximate $100(1-\alpha)\%$ asymptotic confidence interval for parameter $\theta$ using MLE and MPS estimates as follows:

$$CI_{ML} = \left[\hat{\theta}_{ML} \pm z_{\alpha/2}\sqrt{I^{-1}(\hat{\theta}_{ML})}\right] \tag{15}$$

$$CI_{MPS} = \left[\hat{\theta}_{MPS} \pm z_{\alpha/2}\sqrt{I^{-1}(\hat{\theta}_{MPS})}\right] \tag{16}$$

where the observed Fisher information $I(\hat{\theta}_{ML})$ is defined as:

$$I(\hat{\theta}_{ML}) = -\left(\frac{\partial^2 L(\theta;\widetilde{x})}{\partial\theta^2}\right)_{\theta=\hat{\theta}_{ML}}$$

$$\frac{\partial^2 L(\theta;\widetilde{x})}{\partial\theta^2} = \frac{-2n}{\theta^2} + \frac{n}{(1+\theta)^2} + \sum_{i=1}^{n}\left[\frac{\int\left(\frac{1+x}{x^5}\right)e^{-\frac{\theta}{x}}\xi_{\widetilde{x}_i}(x)dx}{\int\left(\frac{1+x}{x^3}\right)e^{-\frac{\theta}{x}}\xi_{\widetilde{x}_i}(x)dx} - \left(\frac{\int\left(\frac{1+x}{x^4}\right)e^{-\frac{\theta}{x}}\xi_{\widetilde{x}_i}(x)dx}{\int\left(\frac{1+x}{x^3}\right)e^{-\frac{\theta}{x}}\xi_{\widetilde{x}_i}(x)dx}\right)^2\right]$$

Similarly, we can obtain the observed Fisher information $I(\hat{\theta}_{MPS})$ as follows:

$$I(\hat{\theta}_{MPS}) = -\left(\frac{\partial^2 S(\theta;\widetilde{x})}{\partial\theta^2}\right)_{\theta=\hat{\theta}_{MPS}}$$

Let

$$\begin{aligned} N(\theta,x) = \Bigg[&\left\{\omega(\theta,a_i) + \int\left\{\omega(\theta,x_i)-\omega(\theta,a_i)\right\}\xi_{\widetilde{x}_i}(x)dx\right\} \\ &- \left\{\omega(\theta,a_{i-1}) + \int\left\{\omega(\theta,x_{i-1})-\omega(\theta,a_{i-1})\right\}\xi_{\widetilde{x}_{i-1}}(x)dx\right\}\Bigg] \end{aligned}$$

Then, the second order partial derivative of the logarithm of product spacings function $S$ is given as:

$$\frac{\partial^2 S(\theta;\widetilde{x})}{\partial\theta^2} = \sum_{i=1}^{n+1}\left[\frac{N'(\theta,x)}{F(\theta;\widetilde{x}_i)-F(\theta;\widetilde{x}_{i-1})} - \left(\frac{N(\theta,x)}{F(\theta;\widetilde{x}_i)-F(\theta;\widetilde{x}_{i-1})}\right)^2\right]$$

where

$$\begin{aligned} N'(\theta,x) &= \frac{\partial N(\theta,x)}{\partial\theta} \\ &= \Bigg[\left\{\omega'(\theta,a_i) + \int\left\{\omega'(\theta,x_i)-\omega'(\theta,a_i)\right\}\xi_{\widetilde{x}_i}(x)dx\right\} \\ &\quad - \left\{\omega'(\theta,a_{i-1}) + \int\left\{\omega'(\theta,x_{i-1})-\omega'(\theta,a_{i-1})\right\}\xi_{\widetilde{x}_{i-1}}(x)dx\right\}\Bigg] \end{aligned}$$

and

$$\omega'(\theta,x) = \frac{\partial\omega(\theta,x)}{\partial\theta} = \frac{e^{-\theta/x}[\theta(x+1)\{\theta(1+\theta)-2x\} + (2x+1)\{\theta(1+\theta)-x(1-\theta)\}]}{x^3(1+\theta)^3}$$

### 2.4. Estimation of reliability and hazard function

After developing the methodology of obtaining the MLE and MPS using fuzzy data, we can easily obtain the estimated reliability and hazard function by simply substituting the estimated parameter value in their functional forms. It is important to note that we can obtain the estimated reliability and hazard by directly replacing the population parameter with an estimated one since both MLE and MPS possesses the invariance property. Cheng and Amin (1983) and Coolen and Newby (1990) had mentioned that MPS also shows the invariance property like MLE. Therefore, the estimated reliability and hazard can be written as:

$$R(t;\hat{\theta}) = 1 - \left[1 + \frac{\hat{\theta}}{(1+\hat{\theta})t}\right] e^{-\frac{\hat{\theta}}{t}} \tag{17}$$

$$H(t;\hat{\theta}) = \frac{\hat{\theta}^2(1+t)}{t^2\left[t(1+\hat{\theta})\left(e^{\frac{\hat{\theta}}{t}} - 1\right) - \hat{\theta}\right]} \tag{18}$$

where $\hat{\theta}$ is the estimate of parameter $\theta$ i.e. $\hat{\theta}_{ML}$ or $\hat{\theta}_{MPS}$. On substituting the estimated values of parameter $\hat{\theta}_{ML}$ and $\hat{\theta}_{MPS}$ in the above expressions it will provide estimated reliability and hazard function at any time point $t$ corresponding to that estimated value of $\theta$.

## 3. Bayesian estimation

In this section, we discuss the Bayesian estimation of population parameter $\theta$ using fuzzy data. From the Bayesian point of view, the parameter $\theta$ is a random variable. Hence we need to specify a distribution to model its variation, known as the prior distribution of the population parameter.We use the information contained in the sample observations in the form of likelihood to update the existing prior information about unknown parameter $\theta$ using the Bayes theorem. The primary reason behind choosing gamma prior is that the support of the gamma random variable is the same as the support of parameter $\theta$, i.e., $(0, \infty)$ and it is quite flexible in nature. The probability density function of prior with hyperparameters $p$ and $q$ is as follows:

$$\pi(\theta) = \frac{p^q}{\Gamma(q)} e^{-p\theta}\theta^{(q-1)} \quad ; \quad \theta > 0, q > 0, p > 0 \tag{19}$$

Combining the prior given in Eq. (19) and the likelihood given in Eq. (8) one can easily obtain the posterior distribution of the parameter $\theta$ for the given data.

$$\begin{aligned} \pi(\theta|\widetilde{x}) &= \frac{l(\theta;\widetilde{x})\pi(\theta)}{\int_0^\infty l(\theta;\widetilde{x})\pi(\theta)d\theta} \\ &= \frac{J}{\int_0^\infty J d\theta} \end{aligned} \tag{20}$$

where,

$$J = e^{-p\theta}\theta^{2n+q-1}(1+\theta)^{-n}\prod_{i=1}^{n}\int\left(\frac{1+x}{x^3}\right)e^{-\frac{\theta}{x}}\xi_{\widetilde{x}_i}(x)dx$$

For estimation of the parameter under the Bayesian paradigm, we have employed quadratic loss function, which is defined as follows:

$$l_S(\hat{\theta}, \theta) = \epsilon(\hat{\theta} - \theta)^2$$

where $\hat{\theta}$ is the estimated value of parameter $\theta$ and $\epsilon$ may be a function of parameter $\theta$. If $\epsilon = 1$, we have SELF. It is the most common symmetric loss function. It treats overestimation and underestimation of parameters equally. Also, it penalizes large errors much more as compared to small errors. Suppose h(.) is a function of $\theta$, then the Bayes estimate of h(.) under SELF is equal to the expectation of the posterior distribution.

$$\begin{aligned} \hat{h}(\theta) &= E_\pi[h(\theta)] \\ &= \frac{\int_0^\infty h(\theta)Jd\theta}{\int_0^\infty Jd\theta} \end{aligned} \tag{21}$$

Here, we consider Bayes estimator of $\theta$ under SELF which is well known to be the posterior mean. It is clearly visible that Eq.(21) is not mathematically facile and hence no closed form for the estimator is available. Therefore, we will employ the MCMC method to generate samples from the non-facile posterior distribution using the Metropolis-Hastings (MH) algorithm. The MH algorithm was proposed by Hastings (1970) and it is most general and simplest MCMC algorithm (see; Roberts and Smith (1994), Chib and Greenberg (1995), Robert (2004)). In this algorithm we generate sample values of parameter $\theta$ using a proposal distribution with stationary distribution $\pi(\theta)$. The major steps of the algorithm are stated below:

Step 1: Start with an initial guess value $\theta^0$ and set $j = 1$.

Step 2: Generate a new candidate parameter value $\theta^*$ at the $j$th stage from proposal density $q(\theta^{(j)}|\theta^{(j-1)})$.

Step 3: Accept candidate $\theta^*$ as

$$\theta^{(j)} = \begin{cases} \theta^* & \text{with probability} \quad \rho(\theta^*, \theta^{(j-1)}) \\ \theta^{(j-1)} & \text{with probability} \quad 1 - \rho(\theta^*, \theta^{(j-1)}) \end{cases}$$

where

$$\rho(\theta^*, \theta^{(j-1)}) = \min\left\{ \frac{\pi(\theta^*|\widetilde{u})q(\theta^{(j-1)}|\theta^*)}{\pi(\theta^{(j-1)}|\widetilde{u})q(\theta^*|\theta^{(j-1)})}, 1 \right\}$$

Step 4: Calculate $R^{(j)}(t) = 1 - \left[1 + \frac{\theta^{(j)}}{(1+\theta^{(j)})x}\right] e^{-\frac{\theta^{(j)}}{x}}$ and $H^{(j)}(t) = \frac{{\theta^{(j)}}^2(1+x)}{x^2\left[x(1+\theta^{(j)})\left(e^{\frac{\theta^{(j)}}{x}}-1\right)-\theta^{(j)}\right]}$ (see; Pak *et al.* (2014b))

Step 5: Repeat steps 2-4, $M$ times and obtain $\theta^{(j)}, R^{(j)}(t)$ and $H^{(j)}(t); j = 1, 2, \ldots, M$

Bayes estimates of the parameter $\theta, R(t)$ and $H(t)$ under SELF will be as follows:

$$\hat{\theta}_B = E_\pi(\theta \mid \widetilde{u}) = \frac{1}{M - M_0} \sum_{j=M_0+1}^{M} \theta^{(j)}$$

$$\hat{R}(t)_B = E_\pi(R(t) \mid \widetilde{u}) = \frac{1}{M - M_0} \sum_{j=M_0+1}^{M} R^{(j)}(t)$$

$$\hat{H}(t)_B = E_\pi(H(t) \mid \widetilde{u}) = \frac{1}{M - M_0} \sum_{j=M_0+1}^{M} H^{(j)}(t)$$

where $M_0$ is burn-in period. To construct the HPD credible intervals for $\theta$, order the simulated samples as $\theta^{(1)} \leq \theta^{(2)} \leq \ldots \leq \theta^{(M)}$. Then construct all the $100(1-\alpha)\%$ credible intervals of $\theta$ as

$$\left(\theta^{[1]}, \theta^{[M(1-\alpha)+1]}\right), \ldots, \left(\theta^{[M\alpha]}, \theta^{[M]}\right)$$

Here $[X]$ denotes the largest integer less than or equal to $X$. Then the HPD credible interval of $\theta$ is that credible interval which has the shortest length.

# 4. Simulation study

In the present section, we will analyze the behavior of estimators based on Monte-Carlo simulation studies. In a Monte-Carlo simulation study, we simulate artificial samples from the desired distribution and try to observe the behavior of estimators based on these samples.In practical situations, only one sample is available, and based on this sample, we can estimate parameter values, but we cannot investigate the behavior of the estimators. The samples that are generated from the simulation can be considered as those that are obtained when samples are collected more than once. Here we have chosen an arbitrary value of parameter $\theta$ to generate samples from the ILD($\theta$). Generation of the random sample by the inverse transformation method is not practicable in this case. However, in the case of breakdown of inverse transformation method, we can utilize several other sample generation methods as described by Ghitany *et al.* (2008), Jodrá (2010) and Sharma *et al.* (2015). We have to fuzzify the generated samples to compute the estimated value of the population parameter using the approach discussed in previous sections of this article. To accomplish this goal, we have generated samples of different sizes $n = (10, 15, 20, 25, 30)$ from the distribution under consideration for a fixed value of parameter $\theta$. These samples are utilized for estimation of the population parameter, reliability function, and hazard rate. We have repeated this process a sufficient number of times to calculate the estimated mean square errors (MSEs) and compare the performance of estimators in terms of these estimated MSEs.We have also obtained average estimates of 95% asymptotic confidence interval and corresponding coverage probabilities.In the Bayesian approach, we have used non-informative prior due to lack of sufficient prior information. HPD intervals are reported along with their corresponding coverage probabilities. After analyzing the effect of variation in the sample size, we have investigated variation in values of population parameter $\theta$ for a fixed sample size $n$. We have simulated the samples using the procedure mentioned above for $\theta = (0.5, 1, 2, 3)$, and then the estimate of the parameter was calculated.Further, we have also obtained the MSEs of estimated values of parameter, reliability, and hazard rate to assess the performance of estimators under classical and Bayesian setup.In both situations stated above (i.e., for variation of parameter and variation of sample size), we have considered non-informative priors and SELF. Note that it is stated in the previous section that the prior distribution of population parameter $\theta$ is gamma with hyperparameters $p$ and $q$. As we know, for different values of hyperparameters $p$ and $q$, it is an informative prior.To compare the Bayes estimator with the classical one, we should assume that we do not have any prior knowledge about the parameter i.e., the prior distribution of population parameter $\theta$ is non-informative.To make the gamma prior non-informative, we can put the value of hyperparameters $p = q = 0$, but it will make the prior improper. Therefore, to preserve the attributes of proper prior, we have substituted very small non-negative values (close to zero say, 0.0001) of hyperparameters $p$ and $q$.We have also considered the different values of hyperparameters $p$ and $q$ and reported the results for informative prior.We have chosen the values of hyperparameters by equating the mean of the gamma distribution with the actual value of parameter and variance with three different choices of the amount of variabilities, i.e., $(1, 10, 50)$. Therefore for every specific value of parameter $\theta$ there are three priors which are denoted in rest of the paper as $p_1, p_2$ and $p_3$. The simple logic behind choosing the hyperparameters in the above-said manner is that the experimenter can easily guess the expected value of parameters with some degree of belief, and based on that, Singh (2011) proposed a method of choosing the hyperparameters. All the results are summarized in table [1-6]. Table [1] provides the average estimated value and MSE of parameter $\theta$ for different methods of estimation. It also provides the average length of the confidence interval and corresponding coverage probabilities. Estimated values of reliability and hazard function along with MSEs for a fixed point of time $t = 1$ are given in table [2].Both the table [1] and [2] provides estimated parameters for a fixed value of $\theta = 1$ and varying $n$. Similarly, tables [3] and [4] provide the estimated values of the same set of parameters but for a fixed $n = 20$ and varying values of the parameter $\theta$.We can see that the MSE of parameter $\theta$ and lengths of the asymptotic confidence interval and HPD intervals corresponding to all the three estimators decreases with the increase in $n$.A similar pattern is visible in MSEs of estimated values of $R(t)$ and $H(t)$. For a fixed value of $n$, an increase in the actual value of the parameter increases the MSE of the parameter.Also, an increasing pattern is visible in the length of confidence intervals with the increase in true $\theta$. It is worthwhile to mention that the MPS method is the best among all the three estimators in terms of the MSE of estimated parameter values. In the case of non-informative priors, the performance of Bayes and MLE are more or less alike in terms of their MSEs. Table [5] and [6] provides the comparison of Bayes estimators under SELF using informative priors. As quoted above, for every specific value of parameter we have considered three priors $p_1, p_2$ and $p_3$.Table [5] shows the effect of variation of sample size $n$ on the MSEs of estimated parameter, reliability, hazard, and the average length of the confidence interval for fixed $\theta = 1$. On the other hand, table [6] captures the effect variation of parameter $\theta$ for fixed sample size $n = 20$. From both of the tables it is clearly visible that the MSEs of $\theta, R(t)$ and $H(t)$ is minimum for $p_1$ and maximum for $p_3$. It is worthwhile to mention that, although the performance of MPS is better than the MLE and Bayes estimator in the case of non-informative prior, Bayes estimator always provides better results for an informative prior with suitable hyperparameters.

Table 1: Average estimates, the corresponding MSEs, length of confidence intervals and coverage probabilities of the parameters $\theta$ for different sample size $n$ and a fixed value of parameter $\theta = 1$

| n | MLE | | | | MPS | | | | Bayes | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $\theta$ | MSE | Length | C.P. | $\theta$ | MSE | Length | C.P. | $\theta$ | MSE | Length | C.P. |
| 10 | 1.0731 | 0.0851 | 1.0154 | 0.9500 | 0.9497 | 0.0637 | 0.8803 | 0.9460 | 1.0901 | 0.0981 | 0.9910 | 0.9480 |
| 15 | 1.0298 | 0.0427 | 0.7919 | 0.9540 | 0.9301 | 0.0364 | 0.6996 | 0.9560 | 1.0445 | 0.0464 | 0.7812 | 0.9500 |
| 20 | 1.0305 | 0.0330 | 0.6859 | 0.9480 | 0.9333 | 0.0307 | 0.6066 | 0.9490 | 1.0422 | 0.0352 | 0.6766 | 0.9470 |
| 25 | 1.0291 | 0.0250 | 0.6123 | 0.9560 | 0.9433 | 0.0224 | 0.5475 | 0.9550 | 1.0401 | 0.0271 | 0.6051 | 0.9530 |
| 30 | 1.0283 | 0.0212 | 0.5584 | 0.9640 | 0.9498 | 0.0187 | 0.5023 | 0.9630 | 1.0396 | 0.0234 | 0.5522 | 0.9470 |

Table 2: Average estimates and corresponding MSEs of the reliability and hazard function using MLE and MPS respectively at specified time say $(t = 1)$ for different sample size $n$ and fixed value of parameter $\theta = 1$

| n | MLE | | | | MPS | | | | Bayes | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | R(t) | MSE | H(t) | MSE | R(t) | MSE | H(t) | MSE | R(t) | MSE | H(t) | MSE |
| 10 | 0.4704 | 0.0131 | 0.7959 | 0.0130 | 0.4171 | 0.0122 | 0.8476 | 0.0114 | 0.4774 | 0.0131 | 0.7891 | 0.0132 |
| 15 | 0.4560 | 0.0078 | 0.8113 | 0.0073 | 0.4112 | 0.0080 | 0.8539 | 0.0070 | 0.4623 | 0.0083 | 0.8053 | 0.0078 |
| 20 | 0.4577 | 0.0062 | 0.8103 | 0.0057 | 0.4135 | 0.0068 | 0.8521 | 0.0059 | 0.4628 | 0.0064 | 0.8054 | 0.0060 |
| 25 | 0.4581 | 0.0047 | 0.8103 | 0.0043 | 0.4191 | 0.0049 | 0.8471 | 0.0043 | 0.4629 | 0.0050 | 0.8057 | 0.0047 |
| 30 | 0.4583 | 0.0040 | 0.8103 | 0.0037 | 0.4226 | 0.0041 | 0.8440 | 0.0036 | 0.4631 | 0.0044 | 0.8056 | 0.0040 |

Table 3: Average estimates, corresponding MSEs, length of confidence intervals and coverage probabilities of the parameters $\theta$ for different values of the parameter $\theta$ and sample size $n = 20$

| Theta | MLE | | | | MPS | | | | Bayes | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $\theta$ | MSE | Length | C.P. | $\theta$ | MSE | Length | C.P. | $\theta$ | MSE | Length | C.P. |
| 0.5 | 0.5159 | 0.0078 | 0.3301 | 0.9490 | 0.4714 | 0.0068 | 0.2968 | 0.9500 | 0.5190 | 0.0081 | 0.3243 | 0.9430 |
| 1 | 1.0305 | 0.0330 | 0.6859 | 0.9480 | 0.9333 | 0.0307 | 0.6066 | 0.9470 | 1.0422 | 0.0352 | 0.6766 | 0.9470 |
| 2 | 2.0551 | 0.1370 | 1.4515 | 0.9600 | 1.8663 | 0.1220 | 1.2874 | 0.9610 | 2.0795 | 0.1395 | 1.4293 | 0.9600 |
| 3 | 3.1051 | 0.3570 | 2.2829 | 0.9530 | 2.7880 | 0.3247 | 2.0116 | 0.9520 | 3.0548 | 0.2576 | 2.1627 | 0.9630 |

Table 4: Average estimates and corresponding MSEs of the reliability and hazard function using MLE and MPS respectively at specified time say ($t = 1$) for different values of parameter $\theta$ and sample size $n = 20$

| $\theta$ | MLE | | | | MPS | | | | Bayes | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | R(t) | MSE | H(t) | MSE | R(t) | MSE | H(t) | MSE | R(t) | MSE | H(t) | MSE |
| 0.5 | 0.2000 | 0.0022 | 1.0494 | 0.0020 | 0.1763 | 0.0019 | 1.0721 | 0.0018 | 0.2017 | 0.0023 | 1.0478 | 0.0020 |
| 1 | 0.4577 | 0.0062 | 0.8103 | 0.0057 | 0.4135 | 0.0068 | 0.8521 | 0.0059 | 0.4628 | 0.0064 | 0.8054 | 0.0060 |
| 2 | 0.7745 | 0.0049 | 0.4594 | 0.0093 | 0.7345 | 0.0069 | 0.5121 | 0.0108 | 0.7795 | 0.0048 | 0.4526 | 0.0092 |
| 3 | 0.9097 | 0.0020 | 0.2435 | 0.0075 | 0.8807 | 0.0038 | 0.2967 | 0.0112 | 0.9079 | 0.0018 | 0.2481 | 0.0064 |

Table 5: Average Bayes estimates of $\theta$, reliability and hazard along with corresponding MSEs, average length of 95% HPD intervals and coverage probabilities of $\theta$ based on simulated data for informative prior with $n = (10, 15, 20, 25)$ and fixed parameter $\theta = 1$

| Prior | n | $\theta$ | MSE | Length | C.P. | R(t) | MSE | H(t) | MSE |
|---|---|---|---|---|---|---|---|---|---|
| $p_1$ | 10 | 1.0791 | 0.0777 | 0.9640 | 0.9470 | 0.4740 | 0.0122 | 0.7928 | 0.0121 |
| (p=1,q=1) | 15 | 1.0538 | 0.0500 | 0.7881 | 0.9560 | 0.4661 | 0.0084 | 0.8015 | 0.0081 |
| | 20 | 1.0441 | 0.0343 | 0.6698 | 0.9550 | 0.4637 | 0.0062 | 0.8045 | 0.0058 |
| | 25 | 1.0373 | 0.0287 | 0.6031 | 0.9430 | 0.4614 | 0.0053 | 0.8070 | 0.0049 |
| $p_2$ | 10 | 1.0800 | 0.0964 | 0.9882 | 0.9440 | 0.4720 | 0.0142 | 0.7938 | 0.0144 |
| (p=0.1,q=0.1) | 15 | 1.0612 | 0.0523 | 0.7951 | 0.9450 | 0.4691 | 0.0088 | 0.7986 | 0.0085 |
| | 20 | 1.0483 | 0.0381 | 0.6802 | 0.9430 | 0.4651 | 0.0068 | 0.8030 | 0.0064 |
| | 25 | 1.0340 | 0.0290 | 0.6024 | 0.9470 | 0.4598 | 0.0055 | 0.8085 | 0.0050 |
| $p_3$ | 10 | 1.0886 | 0.0981 | 0.9947 | 0.9290 | 0.4757 | 0.0145 | 0.7903 | 0.0147 |
| (p=0.02,q=0.02) | 15 | 1.0671 | 0.0535 | 0.7896 | 0.9520 | 0.4717 | 0.0091 | 0.7961 | 0.0087 |
| | 20 | 1.0411 | 0.0383 | 0.6767 | 0.9380 | 0.4618 | 0.0069 | 0.8061 | 0.0065 |
| | 25 | 1.0413 | 0.0313 | 0.6008 | 0.9440 | 0.4629 | 0.0056 | 0.8055 | 0.0053 |

Table 6: Average Bayes estimates of $\theta$, reliability and hazard along with corresponding MSEs, average length of 95% HPD intervals and coverage probabilities of $\theta$ based on simulated data for informative prior with $\theta = (0.5, 1, 2)$ and fixed $n = 20$

| $\theta$ | Prior | | $\theta$ | MSE | Length | C.P. | R(t) | MSE | H(t) | MSE |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.5 | $p_1$: | (p=0.25,q=0.5) | 0.5204 | 0.0084 | 0.3244 | 0.9420 | 0.2024 | 0.0024 | 1.0471 | 0.0021 |
| | $p_2$: | (p=0.025,q=0.05) | 0.5202 | 0.0079 | 0.3251 | 0.9560 | 0.2023 | 0.0023 | 1.0472 | 0.0020 |
| | $p_3$: | (p=0.005,q=0.01) | 0.5154 | 0.0077 | 0.3216 | 0.9450 | 0.1997 | 0.0022 | 1.0496 | 0.0020 |
| 1 | $p_1$: | (p=1,q=1) | 1.0441 | 0.0343 | 0.6698 | 0.9550 | 0.4637 | 0.0062 | 0.8045 | 0.0058 |
| | $p_2$: | (p=0.1,q=0.1) | 1.0483 | 0.0381 | 0.6802 | 0.9430 | 0.4651 | 0.0068 | 0.8030 | 0.0064 |
| | $p_3$: | (p=0.02,q=0.02) | 1.0411 | 0.0383 | 0.6767 | 0.9380 | 0.4618 | 0.0069 | 0.8061 | 0.0065 |
| 2 | $p_1$: | (p=4,q=2) | 2.0980 | 0.1326 | 1.3660 | 0.9540 | 0.7841 | 0.0044 | 0.4469 | 0.0086 |
| | $p_2$ : | (p=0.4,q=0.2) | 2.1303 | 0.1789 | 1.4638 | 0.9540 | 0.7877 | 0.0053 | 0.4406 | 0.0107 |
| | $p_3$: | (p=0.08,q=0.04) | 2.1186 | 0.1936 | 1.4651 | 0.9370 | 0.7840 | 0.0058 | 0.4448 | 0.0117 |

## 5. Real data analysis

In this section, we will check the performance of developed estimators by analyzing a real data set. This set consists of survival times in days of 72 guinea pigs after they were infected with different doses of virulent tubercle bacilli in a medical experiment. This data set was initially reported by Bjerkedal *et al.* (1960) and was formerly used by several authors in different contexts. Here, we are primarily concerned with the animals in the same cage that were under the same regimen. The regimen number is the common logarithm of the number of bacillary units in 0.5 ml of challenge solution. Corresponding to regimen 6.6, there were 72 observations listed below:

12, 15, 22, 24, 24, 32, 32, 33, 34, 38, 38, 43, 44, 48, 52, 53, 54, 54, 55, 56, 57, 58, 58, 59, 60, 60, 60, 60, 61, 62, 63, 65, 65, 67, 68, 70, 70, 72, 73, 75, 76, 76, 81, 83, 84, 85, 87, 91, 95, 96, 98, 99, 109, 110, 121, 127, 129, 131, 143, 146, 146, 175, 175, 211, 233, 258, 258, 263, 297, 341, 341, 376.

Recently, Basu, Singh, and Singh (2017) checked the suitability of inverse Lindley distribution to represent this data set. As mentioned earlier, the survival times of infected pigs, which is nothing but a continuous variate, cannot be reported precisely. Therefore, each reported observation is fuzzy and represented with the help of its corresponding membership function. The functional form of the membership function corresponding to each realization of $x$ is as follows:

$$\xi_{\widetilde{x}_i}(x) = \begin{cases} \frac{x-(x_i-h_i)}{h_i}; & x_i - h_i \leq x \leq x_i \\ \frac{(x_i+h_i)-x}{h_i}; & x_i \leq x \leq x_i + h_i \\ \quad 0 \quad ; & \text{otherwise} \end{cases}$$

where $h_i = 0.05x_i$. The data so obtained is then employed to obtain the MLE and MPS estimates of population parameters. The MLE and MPS estimate calculated from the data set are $\hat{\theta}_{ML} = 0.1137, \hat{\theta}_{MPS} = 0.1042$ and corresponding 95% asymptotic confidence intervals are obtained as $(0.0928, 0.1347)$ and $(0.0860, 0.1223)$. To obtain the estimators in Bayesian setup we have considered non-informative priors due to lack of any other prior knowledge except the observed data as suggested by Berger (2013). We have considered three different initial values of chain as $\hat{\theta}$, $\hat{\theta} - \sqrt{V(\hat{\theta})}$ and $\hat{\theta} + \sqrt{V(\hat{\theta})}$ to run three separate MCMC chains. Figure (2) shows the trace plot and density plot of all the three chains and it reveals that the simulated samples of MCMC are well mixed and the nature of posterior is positively skewed since most of the density is concentrated in a small area. Utilizing these MCMC samples, we computed Bayes estimate as $\hat{\theta}_{Bayes} = 0.1136$ under SELF using non-informative prior. The 95% HPD interval estimates for $\theta$ is obtained as $(0.0928, 0.1354)$.

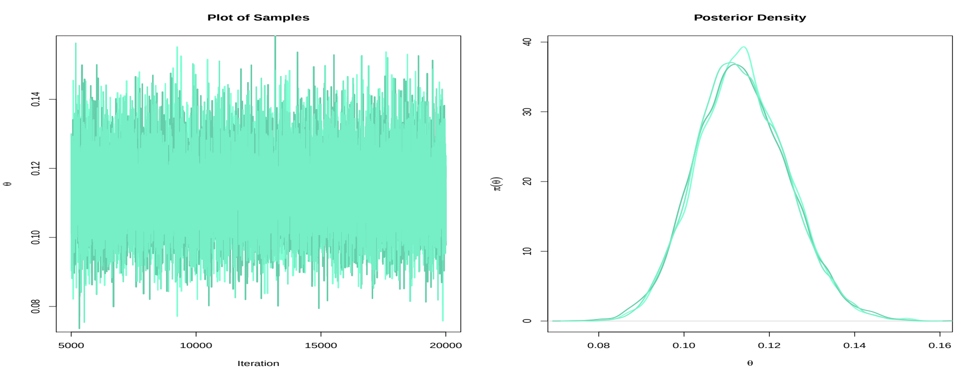


Figure 2: Iteration and density plot of MCMC samples for guinea pigs data

## 6. Conclusion

In this article, our main concern was to develop the MPS estimator for the non precise observed data. For this purpose, we have considered inverse Lindley distribution which is a very useful survival model. To compare the performance of MPS with the other estimators, we have also obtained the MLE and Bayes estimator of the population parameter. In addition to the point estimate of the parameter, estimated reliability, hazard, and confidence interval is also obtained in both the classical and the Bayesian approaches. Estimation under Bayesian setup is performed using both informative and non-informative gamma priors using SELF. We have noted that the MPS estimator performs better than the MLE and non-informative Bayes estimator in terms of MSE. Further, we have seen that although the Bayes estimator with the non-informative gamma prior is similar to MLE, for informative prior, it is superior to MLE and MPS estimator in terms of inherent variability and length of the confidence interval.

The present article presents a combination of statistical methods and fuzzy approach. The methodology developed in this paper will be helpful to researchers and statistician who encounters fuzzy data during the experiments. Expansion of MPS estimator under Bayesian setup for fuzzy data will be extremely fruitful research. Also dealing the problems of fuzzy data via fuzzy models would help us to further explore this idea and will increase our level of understanding in both statistical and fuzzy approaches.

## References

Anatolyev S, Kosenok G (2005). "An Alternative to Maximum Likelihood Based on Spacings." *Econometric Theory*, **21**(2), 472–476. doi:10.1017/S0266466605050255.

Barbato G, Germak A, Genta G (2013). *Measurements for Decision Making*. Società Editrice Esculapio.

Basu S, Singh SK, Singh U (2017). "Parameter Estimation of Inverse Lindley Distribution for Type-I Censored Data." *Computational Statistics*, **32**(1), 367–385. doi:10.1007/s00180-016-0704-0.

Berger JO (2013). *Statistical Decision Theory and Bayesian Analysis*. Springer Science & Business Media.

Bjerkedal T, *et al.* (1960). "Acquisition of Resistance in Guinea Pies Infected with Different Doses of Virulent Tubercle Bacilli." *American Journal of Hygiene*, **72**(1), 130–48. doi:10.1093/oxfordjournals.aje.a120129.

Buckley JJ (2006). *Fuzzy Probability and Statistics*. Springer.

Chaturvedi A, Singh SK, Singh U (2018). "Statistical Inferences of Type-II Progressively Hybrid Censored Fuzzy Data with Rayleigh Distribution." *Austrian Journal of Statistics*, **47**(3), 40–62. doi:10.17713/ajs.v47i3.752.

Cheng RCH, Amin NAK (1983). "Estimating Parameters in Continuous Univariate Distributions with a Shifted Origin." *Journal of the Royal Statistical Society. Series B (Methodological)*, pp. 394–403. doi:10.1111/j.2517-6161.1983.tb01268.x.

Cheng RCH, Stephens MA (1989). "A Goodness-of-Fit Test Using Moran's Statistic with Estimated Parameters." *Biometrika*, **76**(2), 385–392. doi:10.1093/biomet/76.2.385.

Cheng RCH, Traylor L (1995). "Non-Regular Maximum Likelihood Problems." *Journal of the Royal Statistical Society. Series B (Methodological)*, pp. 3–44. doi:10.1111/j.2517-6161.1995.tb02013.x.

Chib S, Greenberg E (1995). "Understanding the Metropolis-Hastings Algorithm." *The American Statistician*, **49**(4), 327–335. doi:10.2307/2684568.

Coolen FPA, Newby MJ (1990). *A Note on the Use of the Product of Spacings in Bayesian Inference*. Department of Mathematics and Computing Science, University of Technology.

Coppi R, Gil MA, Kiers HAL (2006). "The Fuzzy Approach to Statistical Analysis." *Computational Statistics & Data Analysis*, **51**(1), 1–14. doi:10.1016/j.csda.2006.05.012.

Denœux T (2011). "Maximum Likelihood Estimation from Fuzzy Data Using the EM Algorithm." *Fuzzy Sets and Systems*, **183**(1), 72–91. doi:10.1016/j.fss.2011.05.022.

Dubois D, Prade H (1998). "Possibility Theory: Qualitative and Quantitative Aspects." In *Quantified Representation of Uncertainty and Imprecision*, pp. 169–226. Springer. doi:10.1007/978-94-017-1735-9_6.

Dubois DJ (1980). *Fuzzy Sets and Systems: Theory and Applications*, volume 144. Academic press.

Ghitany ME, Atieh B, Nadarajah S (2008). "Lindley Distribution and Its Application." *Mathematics and Computers in Simulation*, **78**(4), 493–506. doi:10.1016/j.matcom.2007.06.007.

Ghosh K, Jammalamadaka SR (2001). "A General Estimation Method Using Spacings." *Journal of Statistical Planning and Inference*, **93**(1-2), 71–82. doi:10.1016/S0378-3758(00)00160-9.

Hastings WK (1970). "Monte Carlo Sampling Methods Using Markov Chains and Their Applications." doi:10.2307/2334940.

Hryniewicz O (1986). "Evaluation of System's Reliability with a Fuzzy Prior Information." *IFAC Proceedings Volumes*, **19**, 451–454. doi:10.1016/B978-0-08-033452-3.50087-9.

Huang HZ, Zuo MJ, Sun ZQ (2006). "Bayesian Reliability Analysis for Fuzzy Lifetime Data." *Fuzzy Sets and Systems*, **157**(12), 1674–1686. doi:10.1016/j.fss.2005.11.009.

Jodrá P (2010). "Computer Generation of Random Variables with Lindley or Poisson–Lindley Distribution via the Lambert W Function." *Mathematics and Computers in Simulation*, **81**(4), 851–859. doi:10.1016/j.matcom.2010.09.006.

Lee KH (2006). *First Course on Fuzzy Theory and Applications*, volume 27. Springer Science & Business Media.

Nguyen H, Wu B (2006). *Fundamentals of Statistics with Fuzzy Data*.

Pak A, Chatrabgoun O (2016). "Inference for Exponential Parameter under Progressive Type-II Censoring from Imprecise Lifetime." *Electronic Journal of Applied Statistical Analysis*, **9**(1), 227–245. doi:0.1285/i20705948v9n1p227.

Pak A, Parham GA, Saraj M (2013a). "Inference for the Weibull Distribution Based on Fuzzy Data." *Revista Colombiana de Estadística*, **36**(2), 339–358. doi:10.15446/rce.

Pak A, Parham GA, Saraj M (2013b). "On Estimation of Rayleigh Scale Parameter Under Doubly Type-II Censoring From Imprecise Data." *Journal of Data Science*, **11**(2), 305–322. doi:10.6339/JDS.2013.11(2).1144.

Pak A, Parham GA, Saraj M (2014a). "Inference for the Rayleigh Distribution Based on Progressive Type-II Fuzzy Censored Data." *Journal of Modern Applied Statistical Methods*, **13**(1), 19. doi:10.22237/jmasm/1398917880.

Pak A, Parham GA, Saraj M (2014b). "Reliability Estimation in Rayleigh Distribution Based on Fuzzy Lifetime Data." *International Journal of System Assurance Engineering and Management*, **5**(4), 487–494. doi:10.1007/s13198-013-0190-5.

Pitman EJG (1979). *Some Basic Theory for Statistical Inference: Monographs on Applied Probability and Statistics*. Chapman and Hall/CRC. doi:10.1201/9781351076777.

Ranneby B (1984). "The Maximum Spacing Method. An Estimation Method Related to the Maximum Likelihood Method." *Scandinavian Journal of Statistics*, pp. 93–112.

Robert CP (2004). "Casella: Monte Carlo Statistical Methods." *Springerverlag, New York*, **3**. doi:10.2307/1270959.

Roberts GO, Smith AFM (1994). "Simple Conditions for the Convergence of the Gibbs Sampler and Metropolis-Hastings Algorithms." *Stochastic Processes and Their Applications*, **49**(2), 207–216. doi:10.1016/0304-4149(94)90134-1.

Shao Y, Hahn MG, *et al.* (1999). "Maximum Product of Spacings Method: A Unified Formulation with Illustration of Strong Consistency." *Illinois Journal of Mathematics*, **43**(3), 489–499. doi:10.1215/ijm/1255985105.

Sharma VK, Singh SK, Singh U, Agiwal V (2015). "The Inverse Lindley Distribution: A Stress-Strength Reliability Model with Application to Head and Neck Cancer Data." *Journal of Industrial and Production Engineering*, **32**(3), 162–173. doi:10.1080/21681015.2015.1025901.

Singh SK (2011). "Estimation of Parameters and Reliability Function of Exponentiated Exponential Distribution: Bayesian Approach under General Entropy Loss Function." *Pakistan Journal of Statistics and Operation Research*, **7**(2), 217–232. doi:10.18187/pjsor.v7i2.239.

Singh U, Singh SK, Singh RK (2014). "A Comparative Study of Traditional Estimation Methods and Maximum Product Spacings Method in Generalized Inverted Exponential Distribution." *Journal of Statistics Applications & Probability*, **3**(2), 153. doi:10.12785/jsap/030206.

Singpurwalla ND, Booker JM (2004). "Membership Functions and Probability Measures of Fuzzy Sets." *Journal of the American Statistical Association*, **99**(467), 867–877. doi:10.1198/016214504000001196.

Taheri SM, Zarei R (2011). "Bayesian System Reliability Assessment under the Vague Environment." *Applied Soft Computing*, **11**(2), 1614–1622. doi:10.1016/j.asoc.2010.04.021.

Viertl R (1997). "On Statistical Inference for Non-Precise Data." *Environmetrics: The official journal of the International Environmetrics Society*, **8**(5), 541–568. doi:10.1002/(SICI)1099-095X(199709/10)8:5<541::AID-ENV269>3.0.CO;2-U.

Viertl R (2009). "On Reliability Estimation Based on Fuzzy Lifetime Data." *Journal of Statistical Planning and Inference*, **139**(5), 1750–1755. doi:10.1016/j.jspi.2008.05.048.

Viertl R (2011). *Statistical Methods for Fuzzy Data*. John Wiley & Sons.

Yager RR, Zadeh LA (2012). *An Introduction to Fuzzy Logic Applications in Intelligent Systems*, volume 165. Springer Science & Business Media.

Zadeh LA (1965). "Fuzzy Sets." *Information and Control*, **8**(3), 338–353. doi:10.1142/9789814261302_0021.

Zadeh LA (1968). "Probability Measures of Fuzzy Events." *Journal of Mathematical Analysis and Applications*, **23**(2), 421–427. doi:10.1016/0022-247X(68)90078-4.

Zadeh LA (1983). "The Role of Fuzzy Logic in the Management of Uncertainty in Expert Systems." *Fuzzy Sets and Systems*, **11**(1-3), 199–227. doi:10.1016/S0165-0114(83)80081-5.

Zadeh LA (1987). "Fuzzy Sets, Usuality and Common Sense Reasoning." In *Matters of Intelligence*, pp. 289–309. Springer. doi:10.1007/978-94-009-3833-5_13.

Zadeh LA (1996). "Fuzzy Logic= Computing with Words." *IEEE Transactions on Fuzzy Systems*, **4**(2), 103–111. doi:10.1007/978-3-7908-1873-4_1.

Zadeh LA (1999). "Fuzzy Sets as a Basis for a Theory of Possibility." *Fuzzy Sets and Systems*, **100**(1), 9–34. doi:10.1016/0165-0114(78)90029-5.

Zadeh LA (2008). "Is there a Need for Fuzzy Logic?" *Information Sciences*, **178**(13), 2751–2779. doi:10.1016/j.ins.2008.02.012.

Zadeh LA (2011). "Generalized Theory of Uncertainty: Principal Concepts and Ideas." In *Fundamental Uncertainty*, pp. 104–150. Springer. doi:10.1057/9780230305687_6.

Zadeh LA, Fu KS, Tanaka K (2014). *Fuzzy Sets and Their Applications to Cognitive and Decision Processes: Proceedings of the US–Japan Seminar on Fuzzy Sets and Their Applications, held at the University of California, Berkeley, California, July 1-4, 1974*. Academic press.

Zimmermann HJ (2001). *Fuzzy Set Theory and Its Applications*. Springer.

**Affiliation:**

Ankita Chaturvedi
Department of Statistics, Institute of Science
Banaras Hindu University
Varanasi, India-221005
E-mail: ankita.c187@gmail.com

Austrian Journal of Statistics http://www.ajs.or.at/
published by the Austrian Society of Statistics http://www.osg.or.at/
Volume 52 *Submitted:* 2021-08-03
January 2023 *Accepted:* 2022-04-25