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Anti-Sum-Asymmetry Models and Orthogonal Decomposition of Anti-Sum-Symmetry Model for Ordinal Square Contingency Tables

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Abstract

For the analysis of $C \times C$ square contingency tables, we usually estimate using a statistical model an unknown probability distribution with high confidence from obtained observations. The statistical model that fits the data well and is easy to interpret is preferred. The anti-sum-symmetry (ASS) and anti-conditional sum-symmetry (ACSS) models have a structure that the ratio of the probability with which the sum of row and column levels is t, for $t = 2, \ldots, C$, and the probability with which the sum of row and column levels is $2(C+1)-t$ is always one and constant, respectively. This study proposes two kinds of models that the ratio of those changes exponentially depending on the sum of row and column levels. This study also gives the decomposition theorems of the ASS model using the proposed models. Moreover, we show that the value of the likelihood ratio chi-squared statistics for the ASS model is asymptotically equivalent to the sum of those for the decomposed models. We evaluate the advantage of the proposed models by applying they to a single data set of real-world grip strength data.

Keywords: anti-diagonals cell, asymmetry, exponentially, grip strength data, test statistic.

1. Introduction

Square contingency tables with same row and column ordinal classifications are usually obtained by cross-classifying for matched-pairs data of the ordinal categorical variables. For the analysis of $C \times C$ square contingency tables, we usually estimate using a statistical model an unknown probability distribution with high confidence from an obtained observation. The statistical model that fits the data well and is easy to interpret is preferred.

The symmetry (S) model (Bowker 1948), is the origin of the statistical model for the square contingency table. The S model has a structure that the ratio of the probability that observations will fall in the (i, j)th cell, for $i < j$, of the table, and the probability that observations will fall in the (j, i)th is always one. Thereafter, various models have been introduced as an extension of the S model, for example, the marginal homogeneity (MH) model (Stuart 1955), the quasi-symmetry (QS) (Caussinus 1965), the conditional symmetry (CS) model (McCullagh 1978), the diagonals-parameter symmetry (DPS) model (Goodman 1979), the linear diagonals-parameter symmetry (LDPS) model (Agresti 1983), and the two-ratios-parameter symmetry (TRPS) model (Tomizawa 1987). Tahata and Tomizawa (2014) has overviewed various statistical models including the above models.

Caussinus (1965) gave the origin of decomposition theorem—the S model holds if and only if both the MH and the QS models hold—for statistical models for square contingency tables. Moreover, Tomizawa and Tahata (2007) showed that the value of the likelihood ratio chi-squared statistics for the S model is asymptotically equivalent to the sum of those for the MH and the QS models. Read (1977) showed the decomposition theorem in which the S model holds if and only if both the CS and the global symmetry (GS) models hold, and the value of the likelihood ratio chi-squared statistic for the S model is equal to the sum of those for the CS and GS models. Tomizawa and Kato (2003) showed the decomposition theorem in which the S model holds if and only if both the DPS and the marginal diagonal sub-symmetry (MDS) models hold, and the value of the likelihood ratio chi-squared statistic for the S model is equal to the sum of those for the DPS and MDS models. Yamamoto, Iwashita, and Tomizawa (2007) gave the decomposition theorem in which the S model holds if and only if both the LDPS and the marginal mean equality (ME) models hold. Moreover, Tahata, Yamamoto, and Tomizawa (2008) showed that the value of the likelihood ratio chi-squared statistics for the S model is asymptotically equivalent to the sum of those for the LDPS and ME models. Tahata and Tomizawa (2009) gave the decomposition theorem in which the S model holds if and only if all the TRPS, GS, and ME models hold, and showed that the value of the likelihood ratio chi-squared statistics for the S model is asymptotically equivalent to the sum of those for the TRPS and GSME models. Note that the GSME model simultaneously satisfies both the GS and ME models.

Consider the data in Table 1. Tables 1a and 1b are the data sets of grip strength test examined aged 15–69 for men and women, respectively; source: National Health and Nutrition Examination Survey 2011–2012 (https://wwwn.cdc.gov/Nchs/Nhanes/2011-2012/MGX_G.htm). The row and column variables are the right and left hands grip strength level with the categories ordered from the highest level (1) to the lowest level (5), respectively. These levels are categorized based on Muscle Strength Procedure Manual of National Health and Nutrition Examination Survey.

Table 1: The table below is the data of grip strength test examined aged 15–69 for men and women; source: National Health and Nutrition Examination Survey 2011–2012 (https://wwwn.cdc.gov/Nchs/Nhanes/2011-2012/MGX_G.htm)

	Left hand						Left hand					
Right hand	(1)	(2)	(3)	(4)	(5)	Total	(1)	(2)	(3)	(4)	(5)	Total
(a) For men							(b) For women					
(1)	215	124	46	14	2	401	380	229	49	17	2	677
(2)	37	143	165	74	16	435	34	204	198	80	23	539
(3)	7	45	156	166	51	425	5	55	159	161	74	454
(4)	2	20	62	226	210	520	1	11	37	106	168	323
(5)	1	2	16	61	495	575	0	2	12	45	256	315
Total	262	334	445	541	774	2356	420	501	455	409	523	2308

Yamamoto, Aizawa, and Tomizawa (2016b) stated that it is natural to evaluate an individual's grip strength level as the sum of the levels of both right and left hands for these data of grip strength test. Yamamoto, Tanaka, and Tomizawa (2013) introduced the sum-symmetry (SS) and the conditional sum-symmetry (CSS) models that the ratio of the probability with which the sum of row and column levels is t $(t = 3, 4, \ldots, 2C - 1)$ when the row level is less than the column level, and the probability with which the sum of those is t when the row level is greater than the column level is always one and constant, respectively. Moreover, Yamamoto, Aizawa, and Tomizawa (2016a) introduced the exponential sum-symmetry (ESS) model that the ratio of those changes exponentially depending on the sum of row and column levels, and the two-parameter sum-symmetry (TPSS) model which includes the CSS and ESS models in special cases. Note that the SS, CSS, ESS, and TPSS models evaluate whether the structures of the sum of row and column levels with respect to the main-diagonals cells of the table are symmetric or asymmetric.

On the other hand, for the data of grip strength test, Ando (2021) mentioned that it may be more natural to evaluate whether the structures of the sum of row and column levels with respect to the anti-diagonals cells of the table rather than main-diagonals cells are symmetric or asymmetric. This is because, many people are right-handed and the grip strength of dominant hand is usually higher than one of non-dominant hand. Moreover, Ando (2021) introduced the anti-sum-symmetry (ASS) and anti-conditional sum-symmetry (ACSS) models that the ratio of the probability with which the sum of row and column levels is t ($t = 2, 3, \dots, C$), and the probability with which the sum of those is $2(C + 1) - t$ is always 1 and constant, respectively.

Let $F_{1(t)}$ ($t = 2, 3, \dots, C$) be the observed frequency with which the sum of row and column levels is t, and $F_{2(t)}$ ($t = 2, 3, \dots, C$) be the observed frequency with which the sum of those is $2(C + 1) - t$. Table 2 shows the ratio of $F_{1(t)}$ and $F_{2(t)}$ ($t = 2, 3, 4, 5$) in Table 1. From Table 2, it is likely that the values of $F_{1(t)}/F_{2(t)}$ ($t = 2, 3, 4, 5$) for men changes exponentially depending on the sum of row and column levels, and the values of $F_{1(t)}/F_{2(t)}$ ($t = 2, 3, 4, 5$) for women are almost constant. In fact, Ando (2021) showed that the ACSS model fits well for the data set in Table 1b.

Table 2: The ratio of $F_{1(t)}$ and $F_{2(t)}$ ($t = 2, 3, 4, 5$) in Table 1

	t				t			
	2	3	4	5	2	3	4	5
(a) For men					(b) For women			
$F_{1(t)}$	215	161	196	226	380	263	258	271
$F_{2(t)}$	495	271	293	246	256	213	192	223
$F_{1(t)}/F_{2(t)}$	0.434	0.594	0.723	0.919	1.484	1.235	1.344	1.215

This study proposes two kinds of models that the ratio of the probability with which the sum of row and column levels is t ($t = 2, 3, \dots, C$), and the probability with which the sum of those is $2(C + 1) - t$ changes exponentially depending on the sum of row and column levels. This study gives decomposition theorems of the ASS model using the proposed models. Moreover, we show that the value of the likelihood ratio chi-squared statistics for the ASS model is asymptotically equivalent to the sum of those for the decomposed models.

The remainder of this paper is organized as follows. Section 2 introduces two kinds of models in square contingency tables, and gives decomposition theorems of the ASS model using the proposed models. Section 3 shows the orthogonality for test statistic of the ASS model using the proposed models. Section 4 shows the advantage of the proposed models by applied to the two data sets of real-world grip strength data. Section 5 closes with concluding remarks.

2. Proposed models and decomposition of the ASS model

Let X and Y be row and column variables, respectively. And let

$$A_t = \Pr(X + Y = t, X + Y < C + 1) \quad \text{and}$$
$$B_t = \Pr(X + Y = 2(C + 1) - t, X + Y > C + 1) \quad \text{for } t = 2, \dots, C.$$

Ando (2021) proposed the ASS model defined by

$$A_t = B_t \quad \text{for } t = 2, \dots, C,$$

and the ACSS model defined by

$$A_t = \Gamma B_t \quad \text{for } t = 2, \ldots, C,$$

where the parameter Γ is unspecified. The ACSS model with $\Gamma = 1$ is equivalent to the ASS model.

We propose the anti-exponential sum-symmetry (AESS) defined by

$$A_t = \Delta^{C+1-t} B_t \quad \text{for } t = 2, \ldots, C,$$

where the parameter Δ is unspecified. The AESS model with $\Delta = 1$ is equivalent to the ASS model. Under the AESS model, (i) if $\Delta > 1$, then $A_t > B_t$ for $t = 2, \ldots, C$, and (ii) $\Delta < 1$, then $A_t < B_t$ for $t = 2, \ldots, C$. For the grip strength data such as Table 1, under the AESS model, we can interpret that (i) when $\Delta > 1$, the median for the individual's grip strength is less than the midpoint $C + 1$ in the range $[2, 2C]$ of the sum of row and column levels, (ii) when $\Delta < 1$, that is greater than the midpoint $C + 1$. Moreover, the larger the difference between the sum of row and column levels and the midpoint $C + 1$, the larger the degree of asymmetry exponentially.

Moreover, as a model which includes the ACSS and AESS models in special cases, we propose the anti-two-parameters sum-symmetry (ATPSS) defined by

$$A_t = \Gamma \Delta^{C+1-t} B_t \quad \text{for } t = 2, \ldots, C.$$

The ATPSS models with $\Delta = 1$, $\Gamma = 1$, and $\Delta = \Gamma = 1$ are equivalent to the ACSS, AESS, and ASS models, respectively. Under the ATPSS model, we can interpret that (i) if $\Gamma > 1$ and $\Delta > 1$, then $A_t > B_t$ for $t = 2, \ldots, C$, (ii) $\Gamma < 1$ and $\Delta < 1$, then $A_t < B_t$ for $t = 2, \ldots, C$, and (iii) if $\Gamma < 1$ and $\Delta > 1$ (or $\Gamma > 1$ and $\Delta < 1$), then it is likely that $A_t < B_t$ (or $A_t > B_t$) for $t = 2, \ldots, c$ and $A_t > B_t$ (or $A_t < B_t$) for $t = c, \ldots, C$.

The numbers of degrees of freedom for testing goodness-of-fit of the ASS, ACSS, AESS, and ATPSS models are $C - 1$, $C - 2$, $C - 2$, and $C - 3$, respectively.

Iki (2016) and Kurakami, Negishi, and Tomizawa (2017) proposed the anti-global symmetry (AGS) model defined by

$$\Pr(X + Y < C + 1) = \Pr(X + Y > C + 1).$$

The AGS model is more parsimonious than the ASS model from the following equalities:

$$\Pr(X + Y < C + 1) = \sum_{t=2}^{C} A_t \quad \text{and} \quad \Pr(X + Y > C + 1) = \sum_{t=2}^{C} B_t.$$

The number of degrees of freedom for testing goodness-of-fit of the AGS model is 1. Note that the number of degrees of freedom for the ASS model is equal to the sum of those for the AESS and AGS models.

We obtain the following decomposition theorem.

Theorem 1. *The ASS model holds if and only if both the AESS and AGS models hold.*

Proof. It is clear satisfied the necessary condition: If the ASS model holds, then both the AESS and AGS models hold. We need to show that the sufficient condition also holds: If both the AESS and AGS models hold, then the ASS model holds. Since the AESS model holds, the following equality holds:

$$\sum_{t=2}^{C} A_t = \sum_{t=2}^{C} \Delta^{C+1-t} B_t.$$

Therefore, if $\Delta > 1$ then $\sum_{t=2}^{C} A_t > \sum_{t=2}^{C} B_t$, and if $\Delta < 1$ then $\sum_{t=2}^{C} A_t < \sum_{t=2}^{C} B_t$. Since the AGS model (i.e., $\sum_{t=2}^{C} A_t = \sum_{t=2}^{C} B_t$) holds, we obtain $\Delta = 1$ (i.e., $A_t = B_t$ for $t = 2, \ldots, C$). The proof is complete. □

Kurakami *et al.* (2017) introduced the model that the mean of sum of X and Y is equal to the midpoint $C+1$. We shall refer to this model as the mean-midpoint equality (MME) model. The MME model is defined by $E(X+Y) = C+1$. The MME model is also expressed as follows:

$$\sum\sum_{i+j\neq C+1}(i+j)\pi_{ij} = (C+1)\sum\sum_{i+j\neq C+1}\pi_{ij},$$

where π_{ij} $(= \Pr(X=i, Y=j))$ is the probability that an observation will fall in the (i,j)th cell of the table $(i=1,\dots,C; j=1,\dots,C)$. The number of degrees of freedom for testing goodness-of-fit of the MME model is 1. Note that the number of degrees of freedom for the ASS model is equal to the sum of those for the AESS and MME models.

We obtain the following decomposition theorem.

Theorem 2. *The ASS model holds if and only if both the AESS and MME models hold.*

Proof. We will first show the necessary condition hold: If the ASS model holds, then both the AESS and MME models hold. Assume that ASS model holds. Then, the AESS model obviously holds, because the AESS model with $\Delta = 1$ is equivalent to the ASS model. Since the ASS model (i.e., $A_t = B_t$ for $t = 2,\dots,C$) holds, we obtain the following equality:

$$\begin{aligned}
& \sum_{t=2}^{C} A_t = \sum_{t=2}^{C} B_t \\
\Longleftrightarrow\ & \sum_{t=2}^{C}\sum\sum_{\substack{i+j<C+1\\ i+j=t}}\pi_{ij} = \sum_{t=2}^{C}\sum\sum_{\substack{i+j>C+1\\ i+j=2(C+1)-t}}\pi_{ij} \\
\Longleftrightarrow\ & \sum_{t=2}^{C}(t+2(C+1)-t)\sum\sum_{\substack{i+j<C+1\\ i+j=t}}\pi_{ij} = \sum_{t=2}^{C}2(C+1)\sum\sum_{\substack{i+j>C+1\\ i+j=2(C+1)-t}}\pi_{ij} \\
\Longleftrightarrow\ & \sum_{t=2}^{C}\left[\sum\sum_{\substack{i+j<C+1\\ i+j=t}}(i+j)\pi_{ij} + \sum\sum_{\substack{i+j>C+1\\ i+j=2(C+1)-t}}(i+j)\pi_{ij}\right] \\
& \qquad = (C+1)\sum_{t=2}^{C}\left[\sum\sum_{\substack{i+j<C+1\\ i+j=t}}\pi_{ij} + \sum\sum_{\substack{i+j>C+1\\ i+j=2(C+1)-t}}\pi_{ij}\right] \\
\Longleftrightarrow\ & \sum\sum_{i+j\neq C+1}(i+j)\pi_{ij} = (C+1)\sum\sum_{i+j\neq C+1}\pi_{ij}.
\end{aligned}$$

Therefore, the necessary condition holds.

We will next show that the sufficient condition also holds: If both the AESS and MME models hold, then the ASS model holds. Since the MME model holds, the following equality holds:

$$\begin{aligned}
& \sum\sum_{i+j\neq C+1}(i+j)\pi_{ij} = (C+1)\sum\sum_{i+j\neq C+1}\pi_{ij} \\
\Longleftrightarrow\ & \sum_{t=2}^{C}[tA_t + (2(C+1)-t)B_t] = (C+1)\sum_{t=2}^{C}(A_t + B_t) \qquad (1)
\end{aligned}$$

Since the AESS model holds, the equality (1) is expressed as follows:

$$\begin{aligned}
&\sum_{t=2}^{C}\left[t\Delta^{C+1-t}+2(C+1)-t\right]B_t=(C+1)\sum_{t=2}^{C}(\Delta^{C+1-t}+1)B_t\\
\iff\ &\sum_{t=2}^{C}\left[t(\Delta^{C+1-t}-1)+2(C+1)\right]B_t=\sum_{t=2}^{C}\left[(C+1)(\Delta^{C+1-t}-1)+2(C+1)\right]B_t\\
\iff\ &t(\Delta^{C+1-t}-1)=(C+1)(\Delta^{C+1-t}-1)\quad\text{for } t=2,\ldots,C.
\end{aligned}$$

Therefore, we obtain $\Delta = 1$ (i.e., $A_t = B_t$ for $t = 2, \ldots, C$). The proof is complete. □

Theorems 1 and 2 are useful for searching the cause that the ASS model does not hold for the presented data.

We obtain the following decomposition theorem of the ASS model using the ATPSS model. Note that the number of degrees of freedom for the ASS model is equal to the sum of those for the ATPSS, AGS, and MME models.

Theorem 3. *The ASS model holds if and only if all the ATPSS, AGS, and MME models hold.*

Proof. From Theorems 1 and 2, it is clear satisfied the necessary condition: If the ASS model holds, then all the ATPSS, AGS, and MME models hold. We need to show that the sufficient condition also holds: If both the ATPSS, AGS, and MME models hold, then the ASS model holds.

Since the ATPSS model holds, the following equality holds:

$$\begin{aligned}
&\sum_{t=2}^{C}A_t=\sum_{t=2}^{C}\Gamma\Delta^{C+1-t}B_t\\
\iff\ &\Gamma=\frac{\sum_{t=2}^{C}A_t}{\sum_{t=2}^{C}\Delta^{C+1-t}B_t}. \qquad (2)
\end{aligned}$$

Since the AGS model holds, the equality (2) is also expressed as follows:

$$\Gamma=\frac{\sum_{t=2}^{C}B_t}{\sum_{t=2}^{C}\Delta^{C+1-t}B_t}. \qquad (3)$$

On the other hand, since the MME model holds, the following equality holds:

$$\begin{aligned}
&\sum_{i+j\neq C+1}\sum(i+j)\pi_{ij}=(C+1)\sum_{i+j\neq C+1}\sum\pi_{ij}\\
\iff\ &\sum_{t=2}^{C}\left[tA_t+(2(C+1)-t)B_t\right]=(C+1)\sum_{t=2}^{C}(A_t+B_t) \qquad (4)
\end{aligned}$$

Since the ATPSS model holds, the equality (4) is also expressed as follows:

$$\sum_{t=2}^{C}\left[t\Gamma\Delta^{C+1-t}+2(C+1)-t\right]B_t=(C+1)\sum_{t=2}^{C}(\Gamma\Delta^{C+1-t}+1)B_t$$

$$\Longleftrightarrow\quad \Gamma=\frac{\sum_{t=2}^{C}(t-C-1)B_t}{\sum_{t=2}^{C}(t-C-1)\Delta^{C+1-t}B_t}. \tag{5}$$

From the equalities (3) and (5), we obtain the following equality:

$$\left(\sum_{s=2}^{C}B_s\right)\left(\sum_{t=2}^{C}(t-C-1)\Delta^{C+1-t}B_t\right)=\left(\sum_{t=2}^{C}\Delta^{C+1-t}B_t\right)\left(\sum_{s=2}^{C}(s-C-1)B_t\right)$$

$$\Longleftrightarrow\quad \sum_{t=2}^{C}\left[\left(\sum_{s=2}^{C}(t-s)B_s\right)\Delta^{C+1-t}B_t\right]=0 \tag{6}$$

Since $\sum_{s=2}^{C}\sum_{t=2}^{C}(t-s)B_sB_t=0$, the equality (6) is also expressed as follows:

$$\sum_{t=2}^{C}\Delta^{C+1-t}B_tC_t=0, \tag{7}$$

where $D_t=\sum_{s=2}^{C}(t-s)B_s$. Since $\sum_{s=2}^{C}\sum_{t=2}^{C}(t-s)B_sB_t=0$, the equality (7) is also expressed as follows:

$$(\Delta-1)\left[\sum_{t=3}^{C}(B_CD_C+\cdots+B_tD_t)\Delta^{C+1-t}\right]=0.$$

For $t=3,\ldots,C$,

$$\begin{aligned} B_CD_C+\cdots+B_tD_t &= \sum_{k=t}^{C}\sum_{s=2}^{C}(k-s)B_sB_t \\ &= \sum_{k=t}^{C}\sum_{s=2}^{t-1}(k-s)B_sB_t+\sum_{k=t}^{C}\sum_{s=t}^{C}(k-s)B_sB_t. \end{aligned} \tag{8}$$

The first term on the right-hand side of the equality (8) is positive, and the second term is zero. Thus, we obtain $\Delta=1$. Moreover, from the equalities (3) and (5), we obtain $\Gamma=1$. Therefore, we obtain $\Gamma=1$ and $\Delta=1$ (i.e., $A_t=B_t$ for $t=2,\ldots,C$). The proof is complete. □

3. Orthogonality for test statistic of the ASS model

We denote by f_{ij} the observed frequency in the (i,j) cell of the table ($i=1,\ldots,C; j=1,\ldots,C$) with sample size n $(=\sum_{i=1}^{C}\sum_{j=1}^{C}f_{ij})$. We assume multinomial sampling over the cells of the square contingency table; that is, the observed frequencies $\{f_{ij}\}$ have a multinomial distribution with parameters that are the cell probabilities $\{\pi_{ij}\}$. The maximum likelihood estimates (MLEs) of the expected frequencies under the model can be obtained using the Newton–Raphson method in the log-likelihood equation for example.

For example, in order to obtain MLEs of the expected frequencies under the AESS model, we must maximize the Lagrangian,

$$L = \sum_{i=1}^{C}\sum_{j=1}^{C} f_{ij}\log\pi_{ij} - \phi\left(\sum_{i=1}^{C}\sum_{j=1}^{C}\pi_{ij} - 1\right) - \sum_{t=2}^{C}\psi_t\left(\sum\sum_{\substack{i+j<C+1\\ i+j=t}}\pi_{ij} - \Delta^{C+1-t}\sum\sum_{\substack{i+j>C+1\\ i+j=2(C+1)-t}}\pi_{ij}\right)$$

with respect to $\{\pi_{ij}\}, \phi, \{\psi_t\}$, and Δ.

Each model can be tested for the goodness-of-fit by, for example, the likelihood ratio chi-squared statistic (denoted by G^2) with the corresponding degrees of freedom. The test statistic G^2 of the model M is given by

$$G^2(\mathrm{M}) = 2\sum_{i=1}^{C}\sum_{j=1}^{C} f_{ij}\log\left(\frac{f_{ij}}{\hat{e}_{ij}}\right),$$

where $\hat{e}_{ij}$ is the MLE of the expected frequency e_{ij} under the model M.

Assume that the model M_1 holds if and only if both the models M_2 and M_3 hold, and the following asymptotic equivalence holds:

$$G^2(\mathrm{M}_1) \simeq G^2(\mathrm{M}_2) + G^2(\mathrm{M}_3), \tag{9}$$

where the number of degrees of freedom for the model M_1 is equal to the sum of those for the models M_2 and M_3. Darroch and Silvey (1963) mentioned that (i) when the equation (9) holds, if both the models M_2 and M_3 are accepted (at the α significance level) with high probability, then the model M_1 would be accepted; however (ii) when the equation (9) does not hold, such an incompatible situation that both the models M_2 and M_3 are accepted with high probability but the M_1 model is rejected with high probability is quite possible. In fact, Darroch and Silvey (1963) and Tahata, Ando, and Tomizawa (2011) showed such interesting examples.

We point out that Theorem 2 satisfies the equation (9), although Theorem 1 does not. From the above point, we believe that Theorem 2 is superior to Theorem 1. We obtain the following theorem.

Theorem 4. *The following asymptotic equivalence holds:*

$$G^2(\mathrm{ASS}) \simeq G^2(\mathrm{AESS}) + G^2(\mathrm{MME}).$$

Proof. Let $\boldsymbol{\pi} = (\pi_{11}, \pi_{12}, \ldots, \pi_{1C}, \ldots, \pi_{C1}, \pi_{C2}, \ldots, \pi_{CC})^T$, "$\boldsymbol{A}^T$" denote the transpose of matrix (or vector) $\boldsymbol{A}$. Thus, $\boldsymbol{\pi}$ is an $C^2 \times 1$ vector. Since $\Delta^{C-1} = \pi_{11}/\pi_{CC}$ under the AESS model, the AESS model is expressed as

$$\boldsymbol{h}_1(\boldsymbol{\pi}) = \mathbf{0}_{C-2},$$

where

$$\boldsymbol{h}_1(\boldsymbol{\pi}) = (h_{13}(\boldsymbol{\pi}), h_{14}(\boldsymbol{\pi}), \ldots, h_{1C}(\boldsymbol{\pi}))^T$$

with

$$h_{1t}(\boldsymbol{\pi}) = (\pi_{CC})^{\frac{C+1-t}{C-1}}\sum\sum_{\substack{i+j<C+1\\ i+j=t}}\pi_{ij} - (\pi_{11})^{\frac{C+1-t}{C-1}}\sum\sum_{\substack{i+j>C+1\\ i+j=2(C+1)-t}}\pi_{ij} \quad \text{for } t = 3, \ldots, C,$$

and $\mathbf{0}_d$ is an $d \times 1$ vector with all components zero.

The MME model is expressed as

$$\boldsymbol{h}_2(\boldsymbol{\pi}) = \mathbf{0}_1,$$

where

$$\boldsymbol{h}_2(\boldsymbol{\pi}) = \sum_{i+j\neq C+1}\sum (i+j)\pi_{ij} - (C+1)\sum_{i+j\neq C+1}\sum \pi_{ij}.$$

From Theorem 2, the ASS model is expressed as

$$\boldsymbol{h}_3(\boldsymbol{\pi}) = (\boldsymbol{h}_1(\boldsymbol{\pi})^T, \boldsymbol{h}_2(\boldsymbol{\pi})^T)^T = \boldsymbol{0}_{C-1}.$$

Let $\boldsymbol{H}_s(\boldsymbol{\pi})$ $(s = 1,2,3)$ denote the matrix of partial derivatives of $\boldsymbol{h}_s(\boldsymbol{\pi})$ with respect to $\boldsymbol{\pi}$ (i.e., $\boldsymbol{H}_s(\boldsymbol{\pi}) = \partial\boldsymbol{h}_s(\boldsymbol{\pi})/\partial\boldsymbol{\pi}^T$). Let $\boldsymbol{\Sigma}(\boldsymbol{\pi})$ be $\mathrm{diag}(\boldsymbol{\pi}) - \boldsymbol{\pi}\boldsymbol{\pi}^T$, where $\mathrm{diag}(\boldsymbol{\pi})$ is a diagonal matrix with ith component of $\boldsymbol{\pi}$ as ith diagonal element. We denote $\boldsymbol{p}$ as $\boldsymbol{\pi}$ with $\{\pi_{ij}\}$ replaced by $\{p_{ij}\}$, where $p_{ij} = f_{ij}/n$. Using the delta method, $\sqrt{n}(\boldsymbol{h}_3(\boldsymbol{p}) - \boldsymbol{h}_3(\boldsymbol{\pi}))$ has asymptotically a normal distribution with mean $\boldsymbol{0}_{C-1}$ and covariance matrix

$$\boldsymbol{H}_3(\boldsymbol{\pi})\boldsymbol{\Sigma}(\boldsymbol{\pi})\boldsymbol{H}_3(\boldsymbol{\pi})^T = \begin{bmatrix} \boldsymbol{H}_1(\boldsymbol{\pi})\boldsymbol{\Sigma}(\boldsymbol{\pi})\boldsymbol{H}_1(\boldsymbol{\pi})^T & \boldsymbol{H}_1(\boldsymbol{\pi})\boldsymbol{\Sigma}(\boldsymbol{\pi})\boldsymbol{H}_2(\boldsymbol{\pi})^T \\ \boldsymbol{H}_2(\boldsymbol{\pi})\boldsymbol{\Sigma}(\boldsymbol{\pi})\boldsymbol{H}_1(\boldsymbol{\pi})^T & \boldsymbol{H}_2(\boldsymbol{\pi})\boldsymbol{\Sigma}(\boldsymbol{\pi})\boldsymbol{H}_2(\boldsymbol{\pi})^T \end{bmatrix}.$$

All elements of $\boldsymbol{H}_1(\boldsymbol{\pi})\boldsymbol{\Sigma}(\boldsymbol{\pi})\boldsymbol{H}_2(\boldsymbol{\pi})^T$ is equal to 0 under the ASS model. This is because, we obtain the following equalities:

$$\frac{\partial h_{1t}(\boldsymbol{\pi})}{\partial\boldsymbol{\pi}^T}\mathrm{diag}(\boldsymbol{\pi})\frac{\partial\boldsymbol{h}_2(\boldsymbol{\pi})}{\partial\boldsymbol{\pi}} = 0 \quad \text{for } t = 3,\ldots,C,$$

$$\boldsymbol{\pi}^T\frac{\partial\boldsymbol{h}_2(\boldsymbol{\pi})}{\partial\boldsymbol{\pi}} = \sum_{i+j\neq C+1}\sum (i+j)\pi_{ij} - (C+1)\sum_{i+j\neq C+1}\sum \pi_{ij}.$$

Therefore, we obtain

$$\begin{aligned}&\boldsymbol{h}_3(\boldsymbol{\pi})^T[\boldsymbol{H}_3(\boldsymbol{\pi})\boldsymbol{\Sigma}(\boldsymbol{\pi})\boldsymbol{H}_3(\boldsymbol{\pi})^T]^{-1}\boldsymbol{h}_3(\boldsymbol{\pi}) \\ &\quad = \boldsymbol{h}_1(\boldsymbol{\pi})^T[\boldsymbol{H}_1(\boldsymbol{\pi})\boldsymbol{\Sigma}(\boldsymbol{\pi})\boldsymbol{H}_1(\boldsymbol{\pi})^T]^{-1}\boldsymbol{h}_1(\boldsymbol{\pi}) + \boldsymbol{h}_2(\boldsymbol{\pi})^T[\boldsymbol{H}_2(\boldsymbol{\pi})\boldsymbol{\Sigma}(\boldsymbol{\pi})\boldsymbol{H}_2(\boldsymbol{\pi})^T]^{-1}\boldsymbol{h}_2(\boldsymbol{\pi}).\end{aligned}$$

Since the Wald statistic is asymptotic equivalent to the likelihood ratio statistic, see for example, Rao (1973, Sec. 6e. 3), Darroch and Silvey (1963), and Aitchison (1962), we obtain Theorem 4. The proof is complete. □

We need to consider modifying Theorem 3, this is because, Theorem 3 does not satisfy the equation (9).

We consider a model that simultaneously satisfies both the AGS and MME models defined by

$$\Pr(X+Y < C+1) = \Pr(X+Y > C+1) \quad \text{and} \quad E(X+Y) = C+1.$$

We shall refer to this model as the AGSME model.

From Theorem 3, we obtain the following corollary.

Corollary 1. *The ASS model holds if and only if both the ATPSS and AGSME models hold.*

We obtain the following theorem.

Theorem 5. *The following asymptotic equivalence holds:*

$$G^2(\mathrm{ASS}) \simeq G^2(\mathrm{ATPSS}) + G^2(\mathrm{AGSME}).$$

Proof. Under the ATPSS model,

$$\Gamma = \left(\frac{\pi_{CC}}{\pi_{11}}\right)^{\frac{1}{C-2}}\left(\frac{A_C}{B_C}\right)^{\frac{C-1}{C-2}} \quad \text{and} \quad \Delta = \left(\frac{\pi_{11}}{\pi_{CC}}\frac{B_C}{A_C}\right)^{\frac{1}{C-2}}.$$

The ATPSS model is expressed as

$$\boldsymbol{h}_4(\boldsymbol{\pi}) = \boldsymbol{0}_{C-3},$$

where

$$\boldsymbol{h}_4(\boldsymbol{\pi}) = (h_{43}(\boldsymbol{\pi}), h_{44}(\boldsymbol{\pi}), \ldots, h_{4C-1}(\boldsymbol{\pi}))^T,$$

with

$$h_{4t}(\boldsymbol{\pi}) = (\pi_{11})^{\frac{t-C}{C-2}}(B_C)^{\frac{t-2}{C-2}}A_t - (\pi_{CC})^{\frac{t-C}{C-2}}(A_C)^{\frac{t-2}{C-2}}B_t \quad \text{for } t = 3, \ldots, C-1.$$

The AGSME model is expressed as

$$\boldsymbol{h}_5(\boldsymbol{\pi}) = \boldsymbol{0}_2,$$

where

$$\boldsymbol{h}_5(\boldsymbol{\pi}) = (h_{51}(\boldsymbol{\pi}), h_{52}(\boldsymbol{\pi}))^T,$$

with

$$h_{51}(\boldsymbol{\pi}) = \sum_{i+j<C+1}\sum \pi_{ij} - \sum_{i+j>C+1}\sum \pi_{ij},$$

$$h_{52}(\boldsymbol{\pi}) = \sum_{i+j\neq C+1}\sum (i+j)\pi_{ij} - (C+1)\sum_{i+j\neq C+1}\sum \pi_{ij}.$$

From Corollary 1, the ASS model is expressed as

$$\boldsymbol{h}_6(\boldsymbol{\pi}) = (\boldsymbol{h}_4(\boldsymbol{\pi})^T, \boldsymbol{h}_5(\boldsymbol{\pi})^T)^T = \boldsymbol{0}_{C-1}.$$

Let $\boldsymbol{H}_s(\boldsymbol{\pi})$ $(s = 4, 5, 6)$ denote the matrix of partial derivatives of $\boldsymbol{h}_s(\boldsymbol{\pi})$ with respect to $\boldsymbol{\pi}$ (i.e., $\boldsymbol{H}_s(\boldsymbol{\pi}) = \partial \boldsymbol{h}_s(\boldsymbol{\pi})/\partial \boldsymbol{\pi}^T$). Using the delta method, $\sqrt{n}(\boldsymbol{h}_6(\boldsymbol{p}) - \boldsymbol{h}_6(\boldsymbol{\pi}))$ has asymptotically a normal distribution with mean $\boldsymbol{0}_{C-1}$ and covariance matrix

$$\boldsymbol{H}_6(\boldsymbol{\pi})\boldsymbol{\Sigma}(\boldsymbol{\pi})\boldsymbol{H}_6(\boldsymbol{\pi})^T = \begin{bmatrix} \boldsymbol{H}_4(\boldsymbol{\pi})\boldsymbol{\Sigma}(\boldsymbol{\pi})\boldsymbol{H}_4(\boldsymbol{\pi})^T & \boldsymbol{H}_4(\boldsymbol{\pi})\boldsymbol{\Sigma}(\boldsymbol{\pi})\boldsymbol{H}_5(\boldsymbol{\pi})^T \\ \boldsymbol{H}_5(\boldsymbol{\pi})\boldsymbol{\Sigma}(\boldsymbol{\pi})\boldsymbol{H}_4(\boldsymbol{\pi})^T & \boldsymbol{H}_5(\boldsymbol{\pi})\boldsymbol{\Sigma}(\boldsymbol{\pi})\boldsymbol{H}_5(\boldsymbol{\pi})^T \end{bmatrix}.$$

All elements of $\boldsymbol{H}_4(\boldsymbol{\pi})\boldsymbol{\Sigma}(\boldsymbol{\pi})\boldsymbol{H}_5(\boldsymbol{\pi})^T$ is equal to 0 under the ASS model. This is because, we obtain the following equalities:

$$\frac{\partial h_{4t}(\boldsymbol{\pi})}{\partial \boldsymbol{\pi}^T}\mathrm{diag}(\boldsymbol{\pi})\frac{\partial h_{51}(\boldsymbol{\pi})}{\partial \boldsymbol{\pi}} = 0 \quad \text{for } t = 3, \ldots, C-1,$$

$$\frac{\partial h_{4t}(\boldsymbol{\pi})}{\partial \boldsymbol{\pi}^T}\mathrm{diag}(\boldsymbol{\pi})\frac{\partial h_{52}(\boldsymbol{\pi})}{\partial \boldsymbol{\pi}} = 0 \quad \text{for } t = 3, \ldots, C-1,$$

$$\boldsymbol{\pi}^T\frac{\partial h_{51}(\boldsymbol{\pi})}{\partial \boldsymbol{\pi}} = \sum_{i+j<C+1}\sum \pi_{ij} - \sum_{i+j>C+1}\sum \pi_{ij},$$

$$\boldsymbol{\pi}^T\frac{\partial h_{52}(\boldsymbol{\pi})}{\partial \boldsymbol{\pi}} = \sum_{i+j\neq C+1}\sum (i+j)\pi_{ij} - (C+1)\sum_{i+j\neq C+1}\sum \pi_{ij}.$$

Therefore, we obtain

$$\begin{aligned} &\boldsymbol{h}_6(\boldsymbol{\pi})^T[\boldsymbol{H}_6(\boldsymbol{\pi})\boldsymbol{\Sigma}(\boldsymbol{\pi})\boldsymbol{H}_6(\boldsymbol{\pi})^T]^{-1}\boldsymbol{h}_6(\boldsymbol{\pi}) \\ &\quad = \boldsymbol{h}_4(\boldsymbol{\pi})^T[\boldsymbol{H}_4(\boldsymbol{\pi})\boldsymbol{\Sigma}(\boldsymbol{\pi})\boldsymbol{H}_4(\boldsymbol{\pi})^T]^{-1}\boldsymbol{h}_4(\boldsymbol{\pi}) + \boldsymbol{h}_5(\boldsymbol{\pi})^T[\boldsymbol{H}_5(\boldsymbol{\pi})\boldsymbol{\Sigma}(\boldsymbol{\pi})\boldsymbol{H}_5(\boldsymbol{\pi})^T]^{-1}\boldsymbol{h}_5(\boldsymbol{\pi}). \end{aligned}$$

Since the Wald statistic is asymptotic equivalent to the likelihood ratio statistic, see for example, Rao (1973, Sec. 6e. 3), Darroch and Silvey (1963), and Aitchison (1962), we obtain Theorem 5. The proof is complete. □

4. Application to real-world grip strength data

First, we consider the data set in Table 1a, which presents a cross-classification of grip strength levels for right and left hands. This data set is the grip strength test examined aged 15–69 for men; source: National Health and Nutrition Examination Survey 2011–2012 (https://wwwn.cdc.gov /Nchs/Nhanes/2011-2012/MGX_G.htm).

Table 3 gives the values of G^2 for each ASS, ACSS, AESS, ATPSS, AGS, MME, AGSME model applied to the data set of Table 1a. This table shows that the AESS and ATPSS models fit well, but the other models fit poorly. The goodness of fit of the AESS model dramatically improves compared to the ASS and ACSS models. We consider comparing the goodness-of-fit between the AESS and ATPSS models for this data set. The ATPSS model constantly holds when the AESS model holds; that is, the AESS and ATPSS models are nested. For testing that the AESS model holds assuming that ATPSS model holds true, we can use the likelihood ratio statistic $G^2(\text{AESS}|\text{ATPSS}) = G^2(\text{AESS}) - G^2(\text{ATPSS})$. When the the AESS model holds, $G^2(\text{AESS}|\text{ATPSS})$ has an asymptotic chi-squared distribution with 1 degrees of freedom. The goodness-of-fit of AESS model is superior to the ATPSS model, this is because $G^2(\text{AESS}|\text{ATPSS}) = 1.44$. Table 4 shows the MLEs of the expected frequencies under the AESS and ATPSS models.

Table 3: Values of the likelihood ratio chi-square statistic G^2 for each model applied to the data set of Table 1a

Applied models	Degrees of freedom	G^2
ASS	4	162.02*
ACSS	3	38.57*
AESS	3	2.30
ATPSS	2	0.86
AGS	1	123.44*
MME	1	160.07*
AGSME	2	161.16*

* indicates significance at the 0.05 level.

Under the AESS model, the MLE of Δ is 0.82. The probability with which the sum of row and column levels is t, for $t = 2, 3, 4, 5$ is estimated to be 0.82^{6-t} times higher than the probability with which the sum of those is $12 - t$. Moreover, since $\Delta < 1$, the median for the individual's grip strength is estimated to be greater than the midpoint 6. Therefore, it may be necessary to revise the criteria of the grip strength level, since we can infer that the men's grip strength tends lower than the criteria.

Under the ATPSS model, the MLEs of Γ and Δ are 1.14 and 0.79, respectively. The probability with which the sum of row and column levels is t, for $t = 2, 3, 4, 5$ is estimated to be $1.14 \times 0.79^{6-t}$ times higher than the probability with which the sum of those is $12 - t$. All $\Gamma\Delta^{6-t}$ for $t = 2, 3, 4, 5$ are less than 1, although $\Gamma < 1$ and $\Delta > 1$. For that reason, we believe that the goodness-of-fit of AESS model may be superior to the ATPSS model. If $A_t < B_t$ (or $A_t > B_t$) for $t = 2, \ldots, c$ and $A_t > B_t$ (or $A_t < B_t$) for $t = c, \ldots, C$, then the goodness-of-fit of ATPSS model may be superior to the AESS model. This is because the AESS model cannot express such the structure.

From Theorem 2 (or 1), we can infer that the cause that the ASS model does not hold for the data set of Table 1a is the MME (or AGS) model rather than the AESS model. Moreover, from Corollary 1 (or Theorem 3), we can infer that the cause that the ASS model does not hold for the data set of Table 1a is the AGSME model (or AGS and MME models) rather than the ATPSS model.

Table 4: Maximum likelihood estimates of expected frequencies under the anti-exponential sum-symmetry (AESS) and anti-two-parameters sum-symmetry (ATPSS) models applied to the data set in Table 1a

	Left hand					
Right hand	(1)	(2)	(3)	(4)	(5)	Total
Excellent (1)	215	124	46	14	2	401
	(222.66)	(118.85)	(46.29)	(13.19)	(2)	
	(216.88)	(119.20)	(47.60)	(13.85)	(2)	
Very good (2)	37	143	165	74	16	435
	(35.46)	(143.89)	(155.48)	(74)	(16.85)	
	(35.57)	(147.97)	(163.20)	(74)	(16.16)	
Good (3)	7	45	156	166	51	425
	(7.04)	(42.40)	(156)	(174.79)	(50.79)	
	(7.24)	(44.51)	(156)	(167.67)	(49.81)	
Fair (4)	2	20	62	226	210	520
	(1.88)	(20)	(65.28)	(225.06)	(215.18)	
	(1.98)	(20)	(62.62)	(220.74)	(214.82)	
Poor (5)	1	2	16	61	495	575
	(1)	(2.11)	(15.93)	(62.50)	(487.34)	
	(1)	(2.02)	(15.63)	(62.40)	(493.12)	
Total	262	334	445	541	774	2356

Note: Estimates under the AESS and ATPSS models are shown in parentheses in the second and third lines, respectively.

Secondly, we consider the data set in Table 1b, which presents a cross-classification of grip strength levels for right and left hands. This data set is the grip strength test examined aged 15–69 for women; source: National Health and Nutrition Examination Survey 2011–2012 (https://wwwn.cdc.gov /Nchs/Nhanes/2011-2012/MGX_G.htm).

Table 5 gives the values of G^2 for each ASS, ACSS, AESS, ATPSS model applied to the data set of Table 1b. This table shows that the ACSS, AESS and ATPSS models fit well, but the ASS model fit poorly. The goodness of fit of the AESS model slightly improves compared to the ACSS model. The goodness-of-fit of AESS model is superior to the ATPSS model, this is because $G^2(\text{AESS}|\text{ATPSS}) = 1.60$.

Table 5: Values of the likelihood ratio chi-square statistic G^2 for each model applied to the data set of Table 1b

Applied models	Degrees of freedom	G^2
ASS	4	43.98*
ACSS	3	3.50
AESS	3	3.05
ATPSS	2	1.45

* indicates significance at the 0.05 level.

Under the AESS model, the MLE of Δ is 1.10. The probability with which the sum of row and column levels is t, for $t = 2, 3, 4, 5$ is estimated to be 1.10^{6-t} times higher than the probability with which the sum of those is $12 - t$. Moreover, since $\Delta > 1$, the median for the individual's grip strength is estimated to be less than the midpoint 6. Therefore, it may be necessary to revise the criteria of the grip strength level, since we can infer that the women's grip strength tends higher than the criteria. Note that the men's grip strength tends lower than the criteria.

5. Concluding remarks

This study proposed the AESS model that the ratio of the probability with which the sum of row and column levels is t ($t = 2, 3, \ldots, C$), and the probability with which the sum of those is $2(C+1) - t$ changes exponentially depending on the sum of row and column levels. Moreover, as a model which includes the ACSS and AESS models in special cases, we propose the ATPSS model. The proposed models are useful for applying to the data interested in the structure of the sum of row and column levels such as grip strength data.

This study also gave the decomposition theorems in which the ASS model holds if and only if both the AESS and AGS models hold (i.e., Theorem 1), and the ASS model holds if and only if both the AESS and MME models hold (i.e., Theorem 2). Theorems 1 and 2 are useful for searching the cause that the ASS model does not hold for the presented data as shown in Section 4. Moreover, we showed that the value of the likelihood ratio chi-squared statistics for the ASS model is asymptotically equivalent to the sum of those for the AESS and MME models (i.e., Theorem 4). From the above point, we believed that Theorem 2 is superior to Theorem 1.

Moreover, this study gave the decomposition theorems in which the ASS model holds if and only if all the ATPSS, AGS, and MME models hold (i.e., Theorem 3), and the ASS model holds if and only if both ATPSS and AGSME models (i.e., Corollary 1). We showed that the value of the likelihood ratio chi-squared statistics for the ASS model is asymptotically equivalent to the sum of those for the ATPSS and AGSME models (i.e., Theorem 5). From the above point, we believed that Corollary 1 is superior to Theorem 3.

On the other hand, Theorem 3 can search the cause that the ASS model does not hold for the presented data more detail than Corollary 1. This is because, Theorem 3 can see the structure of the ASS model in details, such that the AGS model holds but the MME model does not hold for analyzing the data, although Corollary 1 cannot see such structure. The readers may believe that Theorem 3 is superior to Corollary 1 in the aforementioned reason.

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