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Multicomponent Stress-strength Reliability with Exponentiated Teissier Distribution

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Abstract

This article deals with the problem of reliability in a multicomponent stress-strength (MSS) model when both stress and strength variables are from exponentiated Teissier (ET) distributions. The reliability of the system is determined using both classical and Bayesian methods, based on two scenarios where the common scale parameter is unknown or known. In the first scenario, where the common scale parameter is unknown, the maximum likelihood estimation (MLE) and the approximate Bayes estimation are derived. In the second scenario, where the scale parameter is known, the MLE, the uniformly minimum variance unbiased estimator (UMVUE) and the exact Bayes estimation are obtained. In the both scenarios, the asymptotic confidence interval and the highest probability density credible interval are established. Furthermore, two other asymptotic confidence intervals are computed based on the Logit and Arcsin transformations. Monte Carlo simulations are implemented to compare the different proposed methods. Finally, one real example is presented in support of suggested procedures.

Keywords: multicomponent stress-strength reliability, exponentiated Teissier distribution, Bayes estimation.

1. Introduction

Among the existing well-known distributions for modelling lifetime data, the Gamma, Lognormal, generalized Exponential and Weibull distributions are most popular. However, their applicability are restricted to monotonically decreasing, monotonically increasing or constant hazard rate functions. Particularly, the distributions mentioned do not have bathtub shape for their hazard rate functions. This is a disadvantage because in many real systems the bathtub shaped hazard rate is observed. In an effort to analyze lifetime data sets in the best-case scenario, the exponentiated Teissier (ET) distribution introduced by Sharma, Singh, and Shekhawat (2020), is very flexible. The ET distribution is obtained by exponentiating the Teissier distribution (Teissier 1934). The Teissier distribution was proposed to the model frequency of mortality due to ageing only. From a practical perspective, there are many cases that the Teissier distribution may not be appropriate, for example the data set may show nonincreasing hazard rate behavior. The ET distribution has monotonically decreasing, monotonically increasing and bathtub shape hazard rate functions, so it significantly increases the flexibility of the Teissier distribution, as well as can be taken as a good alternative to Gamma, Lognormal, generalized Exponential and Weibull distributions. The cumulative distribution function (cdf) of the ET distribution with two parameters δ and η is given by

$$F_X\left(x;\delta,\eta\right)=\left[1-\exp\left(\eta x-e^{\eta x}+1\right)\right]^{\delta},\ \ x>0, \tag{1}$$

and corresponding probability density function (pdf) is

$$f_X\left(x;\delta,\eta\right)=\delta\eta\left(e^{\eta x}-1\right)\exp\left(\eta x-e^{\eta x}+1\right)\left[1-\exp\left(\eta x-e^{\eta x}+1\right)\right]^{\delta-1},\quad x>0, \tag{2}$$

where $\delta>0$ controls the shape of distribution and $\eta>0$ is scale parameter. The ET distribution belongs to proportional reversed hazard models that are an important class of distribution families in reliability analysis (Gupta, Gupta, and Gupta 1998). Clearly, the role of the shape parameter δ in Equation (1) is to give more flexibility to the Teissier distribution. Hereafter, we use the notation $ET\left(\delta,\eta\right)$ for random variable X has an ET distribution with parameters δ and η. Sharma *et al.* (2020) discussed the several distributional properties of the ET distribution and used it as a model for three real examples include failure times of electronic devices, stress-rupture lifetimes and rainfall data. They showed that data sets modeled using the ET distribution can be better model than those based on the Weibull, generalized exponential, Gamma, generalized Lindley and exponentiated Chen distributions. In the reliability context, the stress-strength models have gained a great deal of consideration over recent decades due to its wide utilization in numerous fields. In these models, our major focus is the assessment of $R=P\left(Y<X\right)$, where X presents the random strength exposed to the random stress Y. In engineering applications, if X presents the strength of a building and Y presents the resultant of the destructive forces acting on it, such as an earthquake, then R can be interpreted as the safety factor of a building. In aquaculture, if X is the growth value of fish in a treatment group and Y is the growth value of a control group, then R shows the effectiveness of treatment. This fundamental idea was firstly studied by Birnbaum (1956). Thereafter, the problem of estimating R has been discussed by a great number of researchers. Of the recent efforts pertaining to stress-strength models, to name a few, are Al-Mutairi, Ghitany, and Kundu (2013), Genc (2013), Singh, Singh, and Sharma (2014), Rezaei, Noughabi, and Nadarajah (2015), Akgül and Şenoğlu (2017), Mahdizadeh and Zamanzade (2018), Abravesh, Ganji, and Mostafaiy (2019), Jose and Drisya (2020), Sadeghpour, Nezakati, and Salehi (2021) and Biswas, Chakraborty, and Mukherjee (2021).
In recent years, inference for the reliability of MSS system has received much attention among researchers. This system contains k identical and independent strength components and it operates when at least $s\,(1\leq s\leq k)$ of the components work properly against a common stress. It is commonly known as s-out-of-k: G system. MSS models appear in many practical situations, such as communication systems, industrial operations, military technologies and so on. For example, consider an airplane with four engines that flies when at least two engines work satisfactorily. Thus, the airplane operation is a 2-out-of-4: G system. As another example, the kidney function in the human body is a 1-out-of-2: G system, since a person can survive with at least one healthy kidney. Assume $X_1,X_2,...,X_k$ are independent random variables with common cdf of $F\left(.\right)$ and exposed to the common stress Y with cdf of $G\left(.\right)$. Thus, the reliability in a MSS model is given by

$$\begin{aligned}R_{s,k}&=P\left[\text{at least } s \text{ of } \left(X_1,X_2,...,X_k\right)\ \text{ exceed }\ Y\right]\\&=\sum_{i=s}^{k}\binom{k}{i}\int_0^{\infty}\left[1-F_X\left(y\right)\right]^i\left[F_X\left(y\right)\right]^{k-i}dG\left(y\right).\end{aligned} \tag{3}$$

The mentioned model was firstly examined by Bhattacharyya and Johnson (1974). Thereafter, many authors have shown considerable interests in the MSS model. Some recent efforts regard to the issue, to mention a few, can be found in Nadar and Kızılaslan (2015), Pak, Kumar Gupta, and Bagheri Khoolenjani (2018), Akgül (2019), Kohansal and Shoaee (2019), Maurya and Tripathi (2020), Kayal, Tripathi, Dey, and Wu (2020), Mahto, Tripathi, and Kızılaslan (2020) and Jovanović, Milošević, and Obradović (2020).

The ET distribution is capable of modeling data sets that show monotonically decreasing, monotonically increasing or bathtub shaped hazard rate functions. So, it may be remarkably used for modeling real life data sets over the existing two-parameter distributions. This justifies the motivation for applied studies based on the ET distribution, such as the stress-strength reliability.

It is important to mention that, to our knowledge, no work has been carried out on the MSS model under the ET distribution. The focus of this article is to establish classical and Bayesian inferences on the reliability of the MSS model when the stress and the strength both follow ET distributions. The rest of the content of this paper is organized as follows. In Section 2, when the common scale parameter is unknown, the MLE and the asymptotic confidence interval (ACI) of $R_{s,k}$ are obtained. Also, two other ACIs are constructed based on the Logit and Arcsin transformations. The Bayes estimation of $R_{s,k}$ is obtained by using the Markov Chain Monte Carlo (MCMC) method under the square loss (SE) function. Furthermore, the highest probability density (HPD) credible interval is provided in this section. In Section 3, when the scale parameter is known, the MLE, UMVUE and ACI of $R_{s,k}$ are investigated. Also, the Bayes estimator of $R_{s,k}$ is determined explicitly, in this section. Moreover, the HPD credible interval is provided. In Section 4, proposed methods are compared via Monte Carlo simulations. In Section 5, analysis of the real example is provided for a demonstration of the findings. Finally, concluding remarks are considered in Section 6.

2. Estimation of $R_{s,k}$ when the common scale parameter is unknown

2.1. MLE of $R_{s,k}$

Suppose $X_1, X_2, ..., X_k$ be independent strength random variables which follow $ET(\delta_1, \eta)$ and Y be stress random variable follows $ET(\delta_2, \eta)$. Hence, the reliability of MSS model, using from Equations (1), (2) and (3) is obtained as

$$R_{s,k} = \sum_{i=s}^{k} \binom{k}{i} \delta_2 \int_0^{\infty} \left[1 - \{1 - \exp(\eta y - e^{\eta y} + 1)\}^{\delta_1}\right]^i [1 - \exp(\eta y - e^{\eta y} + 1)]^{\delta_1 (k-i)}$$
$$\times \eta (e^{\eta y} - 1) \exp(\eta y - e^{\eta y} + 1) [1 - \exp(\eta y - e^{\eta y} + 1)]^{\delta_2 - 1} dy.$$

By using the change of variable $t = [1 - \exp(\eta y - e^{\eta y} + 1)]^{\delta_1}$ and solving the above integral, we get

$$R_{s,k} = \sum_{i=s}^{k} \sum_{j=0}^{i} \binom{k}{i} \binom{i}{j} \frac{(-1)^j \delta_2}{\delta_1 (j + k - i) + \delta_2}. \tag{4}$$

It is interesting to note that the above expression dose not depend on the parameter η. In the following we compute the MLE of $R_{s,k}$. For reach this aim, assume that $x_1, x_2 ..., x_m$ and $y_1, y_2 ..., y_n$ are the random samples from the ET distribution with the sets of parameters (δ_1, η) and (δ_2, η), respectively. Let $\lambda \equiv (\delta_1, \delta_2, \eta)$, thus the likelihood function based on a given observed sample is

$$L(\lambda) = {\delta_1}^m {\delta_2}^n \eta^{m+n} \prod_{i=1}^{m} (e^{\eta x_i} - 1) \prod_{i=1}^{n} (e^{\eta y_i} - 1) \prod_{i=1}^{m} \exp(\eta x_i - e^{\eta x_i} + 1) \prod_{i=1}^{n} \exp(\eta y_i - e^{\eta y_i} + 1)$$
$$\times \prod_{i=1}^{m} [1 - \exp(\eta x_i - e^{\eta x_i} + 1)]^{\delta_1 - 1} \prod_{i=1}^{n} [1 - \exp(\eta y_i - e^{\eta y_i} + 1)]^{\delta_2 - 1}, \tag{5}$$

and the corresponding log-likelihood function is

$$l(\lambda) = m + n + m\ln\delta_1 + n\ln\delta_2 + (m+n)\ln\eta + \sum_{i=1}^{m}\ln\left(e^{\eta x_i} - 1\right) + \sum_{i=1}^{n}\ln\left(e^{\eta y_i} - 1\right) + \eta\sum_{i=1}^{m} x_i$$
$$+ \eta\sum_{i=1}^{n} y_i - \sum_{i=1}^{m} e^{\eta x_i} - \sum_{i=1}^{n} e^{\eta y_i} + (\delta_1 - 1)\sum_{i=1}^{m}\ln\left[1 - \exp\left(\eta x_i - e^{\eta x_i} + 1\right)\right]$$
$$+ (\delta_2 - 1)\sum_{i=1}^{n}\ln\left[1 - \exp\left(\eta y_i - e^{\eta y_i} + 1\right)\right].$$

The MLEs of δ_1, δ_2 and η can be computed as the solution of the following nonlinear equations:

$$\frac{\partial l(\lambda)}{\partial\delta_1} = \frac{m}{\delta_1} + \sum_{i=1}^{m}\ln\left[1 - \exp\left(\eta x_i - e^{\eta x_i} + 1\right)\right] = 0, \tag{6}$$

$$\frac{\partial l(\lambda)}{\partial\delta_2} = \frac{n}{\delta_2} + \sum_{i=1}^{n}\ln\left[1 - \exp\left(\eta y_i - e^{\eta y_i} + 1\right)\right] = 0, \tag{7}$$

$$\frac{\partial l(\lambda)}{\partial\eta} = \frac{m+n}{\eta} + \sum_{i=1}^{m}\frac{x_i\left(2 - e^{\eta x_i}\right)}{1 - e^{-\eta x_i}} + \sum_{i=1}^{n}\frac{y_i\left(2 - e^{\eta y_i}\right)}{1 - e^{-\eta y_i}} + \sum_{i=1}^{m} x_i + \sum_{i=1}^{n} y_i$$
$$+ (\delta_1 - 1)\sum_{i=1}^{m}\frac{x_i\left(e^{\eta x_i} - 1\right)}{\exp\left[-\eta x_i + e^{\eta x_i} - 1\right] - 1} + (\delta_2 - 1)\sum_{i=1}^{n}\frac{y_i\left(e^{\eta y_i} - 1\right)}{\exp\left[-\eta y_i + e^{\eta y_i} - 1\right] - 1}. \tag{8}$$

Obviously, we can readily find from Equations (6) and (7)

$$\hat{\delta}_1(\eta) = -\frac{m}{\sum_{i=1}^{m}\ln\left[1 - \exp\left(\eta x_i - e^{\eta x_i} + 1\right)\right]}, \tag{9}$$

$$\hat{\delta}_2(\eta) = -\frac{n}{\sum_{i=1}^{n}\ln\left[1 - \exp\left(\eta y_i - e^{\eta y_i} + 1\right)\right]}. \tag{10}$$

By substituting Equations (9) and (10) into Equation (8), the ML estimate of η can be found as the solution of the nonlinear equation, by using the fixed point iteration method or some other iteration techniques. So, using from Equation (4) and the invariant property of MLE, the MLE of $R_{s,k}$ is computed as

$$\hat{R}_{s,k}^{MLE} = \sum_{i=s}^{k}\sum_{j=0}^{i}\binom{k}{i}\binom{i}{j}\frac{(-1)^j\hat{\delta}_2}{\hat{\delta}_1(j+k-i)+\hat{\delta}_2}.$$

Now, considering $\lambda \equiv (\delta_1, \delta_2, \eta)$, we determine the ACI of $R_{s,k}$ using the asymptotic distribution of λ and delta method. The expected Fisher information matrix of λ is specified as follows

$$I(\lambda) = E\begin{bmatrix} -\frac{\partial^2 l}{\partial{\delta_1}^2} & -\frac{\partial^2 l}{\partial\delta_1\partial\delta_2} & -\frac{\partial^2 l}{\partial\delta_1\partial\eta} \\ -\frac{\partial^2 l}{\partial\delta_2\partial\delta_1} & -\frac{\partial^2 l}{\partial{\delta_2}^2} & -\frac{\partial^2 l}{\partial\delta_2\partial\eta} \\ -\frac{\partial^2 l}{\partial\eta\partial\delta_1} & -\frac{\partial^2 l}{\partial\eta\partial\delta_2} & -\frac{\partial^2 l}{\partial\eta^2} \end{bmatrix} = E(A),$$

where

$$a_{11} = \frac{m}{{\delta_1}^2}, \quad a_{12} = a_{21} = 0, \quad a_{13} = a_{31} = -\sum_{i=1}^{m} \frac{x_i\left(e^{\eta x_i}-1\right)\exp\left(\eta x_i - e^{\eta x_i}+1\right)}{1-\exp\left(\eta x_i - e^{\eta x_i}+1\right)},$$

$$a_{22} = \frac{n}{{\delta_2}^2}, \quad a_{23} = a_{32} = -\sum_{i=1}^{n} \frac{y_i\left(e^{\eta y_i}-1\right)\exp\left(\eta y_i - e^{\eta y_i}+1\right)}{1-\exp\left(\eta y_i - e^{\eta y_i}+1\right)},$$

$$a_{33} = \frac{m}{\eta^2} + \sum_{i=1}^{m} \frac{{x_i}^2\left(e^{\eta x_i}+2e^{-\eta x_i}-2\right)}{\left(1-e^{-\eta x_i}\right)^2} + \sum_{i=1}^{m} \frac{{x_i}^2\left\{e^{\eta x_i}\left[p\left(x_i,\eta\right)-1\right]-\left(e^{\eta x_i}-1\right)^2 p\left(x_i,\eta\right)\right\}}{\left[p\left(x_i,\eta\right)-1\right]^2}$$
$$\frac{n}{\eta^2} + \sum_{i=1}^{n} \frac{{y_i}^2\left(e^{\eta y_i}+2e^{-\eta y_i}-2\right)}{\left(1-e^{-\eta y_i}\right)^2} + \sum_{i=1}^{n} \frac{{y_i}^2\left\{e^{\eta y_i}\left[q\left(y_i,\eta\right)-1\right]-\left(e^{\eta y_i}-1\right)^2 q\left(y_i,\eta\right)\right\}}{\left[q\left(y_i,\eta\right)-1\right]^2},$$

where $p\left(x_i,\eta\right) = \exp\left[-\eta x_i + e^{\eta x_i} - 1\right]$ and $q\left(y_i,\eta\right) = \exp\left[-\eta y_i + e^{\eta y_i} - 1\right]$. Unfortunately, it is very difficult to determine the expectation of the above expressions analytically. Thus, we used the observed information matrix A as a consistent estimator of $I(\lambda)$ by omitting the expectation operator E. The MLE of $R_{s,k}$ has an asymptotically normally distribution with the mean $R_{s,k}$ and variance

$$H_{s,k} = \sum_{i=1}^{3}\sum_{j=1}^{3} \frac{\partial R_{s,k}}{\partial \lambda_i}\frac{\partial R_{s,k}}{\partial \lambda_j} A_{ij}^{-1},$$

where A_{ij}^{-1} is the (i,j)th element of the inverse of A. Also, we have

$$\frac{\partial R_{s,k}}{\partial \delta_1} = \sum_{i=s}^{k}\sum_{j=0}^{i}\binom{k}{i}\binom{i}{j}\frac{(-1)^{j+1}\delta_2\left(j+k-i\right)}{\left[\delta_1\left(j+k-i\right)+\delta_2\right]^2},$$
$$\frac{\partial R_{s,k}}{\partial \delta_2} = \sum_{i=s}^{k}\sum_{j=0}^{i}\binom{k}{i}\binom{i}{j}\frac{(-1)^{j}\delta_1\left(j+k-i\right)}{\left[\delta_1\left(j+k-i\right)+\delta_2\right]^2}.$$

Hence, the asymptotic variance is given by

$$\hat{H}_{s,k} = A_{11}^{-1}\left(\frac{\partial R_{s,k}}{\partial \delta_1}\right)^2 + 2A_{12}^{-1}\frac{\partial R_{s,k}}{\partial \delta_1}\frac{\partial R_{s,k}}{\partial \delta_2} + A_{22}^{-1}\left(\frac{\partial R_{s,k}}{\partial \delta_2}\right)^2\Bigg|_{\left(\hat{\delta}_1,\hat{\delta}_2\right)}. \tag{11}$$

and the 100(1-γ)% ACI of $R_{s,k}$ is constructed as

$$\hat{R}_{s,k}^{MLE} \pm z_{\gamma/2}\sqrt{\hat{H}_{s,k}}\,,$$

where, $z_{\gamma/2}$ is the upper $\gamma/2$th quantile of the $N(0,1)$. It should be pointed out that, the confidence interval obtained from above equation may not be within the interval (0,1). In this situation, we follow the methods of Ghitany, Al-Mutairi, and Aboukhamseen (2015) and Mukherjee and Maiti (1998) and use the Logit and Arcsin transformations for $R_{s,k}$ as $g\left(R_{s,k}\right) = \log\left[R_{s,k}/(1-R_{s,k})\right]$ and $h\left(R_{s,k}\right) = \text{Arcsin}\left(\sqrt{R_{s,k}}\right)$, respectively. Based on this transformations, the 100(1-γ)% ACI for $g\left(R_{s,k}\right)$ and $R_{s,k}$ respectively take the following form

$$\log\left(\frac{R_{s,k}}{1-R_{s,k}}\right) \pm z_{\gamma/2}\frac{\sqrt{\hat{H}_{s,k}}}{\hat{R}_{s,k}\sqrt{\left(1-\hat{R}_{s,k}\right)}} \equiv \left(L_1, U_1\right),$$

and

$$\left(\frac{e^{L_1}}{1+e^{L_1}}, \frac{e^{U_1}}{1+e^{U_1}}\right).$$

Also, the 100(1-γ)% ACI for $h\left(R_{s,k}\right)$ and $R_{s,k}$ respectively are as follows:

$$\text{Arcsin}\left(\sqrt{\hat{R}_{s,k}}\right) \pm z_{\gamma/2}\,\frac{\sqrt{\hat{H}_{s,k}}}{2\sqrt{\hat{R}_{s,k}\left(1-\hat{R}_{s,k}\right)}} \equiv \left(L_2, U_2\right),$$

and

$$\left(\sin^2 L_2, \sin^2 U_2\right).$$

Hereafter, we denote LOGT, AST and NT for ACIs which are obtained respectively based on Logit transformation, Arcsin transformation and not using either of these two transformations.

2.2. Bayes estimation of $R_{s,k}$

In this subsection, we derive the Bayes estimate and corresponding HPD credible interval of $R_{s,k}$ under the SE loss function. To achieve this aim, we suppose that the independent random variables δ_1, δ_2 and η have Gamma priors with positive parameters $(a_1, b_1), (a_2, b_2)$ and (a_3, b_3), respectively. Based on the observations, the joint posterior density function is

$$\begin{aligned}\pi\left(\delta_1, \delta_2, \eta \left|\mathbf{x}, \mathbf{y}\right.\right) &= \frac{L\left(\mathbf{x}, \mathbf{y}\left|\delta_1, \delta_2, \eta\right.\right)\pi_1\left(\delta_1\right)\pi_2\left(\delta_2\right)\pi_3\left(\eta\right)}{\int_0^\infty\int_0^\infty\int_0^\infty L\left(\mathbf{x}, \mathbf{y}\left|\delta_1, \delta_2, \eta\right.\right)\pi_1\left(\delta_1\right)\pi_2\left(\delta_2\right)\pi_3\left(\eta\right)d\delta_1 d\delta_2 d\eta} \\ &\propto {\delta_1}^{m+a_1-1}{\delta_2}^{n+a_2-1}\eta^{m+n+a_3-1}\exp\left[-\delta_1\left(b_1+U_\eta\right)-\delta_2\left(b_2+V_\eta\right)-\eta b_3+W_\eta\right],\end{aligned}$$

where $U_\eta = -\sum_{i=1}^{m}\ln\left[1-\exp\left(\eta x_i - e^{\eta x_i}+1\right)\right]$, $V_\eta = -\sum_{i=1}^{n}\ln\left[1-\exp\left(\eta y_i - e^{\eta y_i}+1\right)\right]$ and $W_\eta = U_\eta + V_\eta + \sum_{i=1}^{m}\ln\left(e^{\eta x_i}-1\right) + \sum_{i=1}^{n}\ln\left(e^{\eta y_i}-1\right) + \eta\left(\sum_{i=1}^{m}x_i + \sum_{i=1}^{n}y_i\right) - \sum_{i=1}^{m}e^{\eta x_i} - \sum_{i=1}^{n}e^{\eta y_i}$.

Then, the Bayes estimate of $R_{s,k}$, against the SE loss function is calculated by

$$\hat{R}_{s,k}^{Bayes} = E\left(R_{s,k}\left|\mathbf{x}, \mathbf{y}\right.\right) = \int_0^\infty\int_0^\infty\int_0^\infty R_{s,k}\pi\left(\delta_1, \delta_2, \eta\left|\mathbf{x}, \mathbf{y}\right.\right)d\delta_1 d\delta_2 d\eta.$$

Since the Bayes estimate of $R_{s,k}$ cannot be computed analytically, therefore we apply the MCMC technique. Based on the joint posterior density function, the posterior pdfs of δ_1, δ_2 and η are as follows, respectively:

$$\delta_1\left|\eta, \mathbf{x}, \mathbf{y}\right. \sim Gamma\left[a_1+m, b_1+U_\eta\right], \tag{12}$$

$$\delta_2\left|\eta, \mathbf{x}, \mathbf{y}\right. \sim Gamma\left[a_2+n, b_2+V_\eta\right], \tag{13}$$

$$\pi\left(\eta\left|\delta_1, \delta_2, \mathbf{x}, \mathbf{y}\right.\right) \propto \eta^{m+n+a_3-1}\exp\left[-\delta_1 U_\eta - \delta_2 V_\eta - \eta b_3 + W_\eta\right]. \tag{14}$$

We see that the posterior pdfs of δ_1 and δ_2 given in Equations (12) and (13) have Gamma distribution. Thus, using the Gibbs sampling method, we generate random sample from δ_1 and δ_2. From the other side, the posterior pdf of η given in Equation (14) does not reduce analytically to a known distribution. Hence, we apply the Metropolis-Hasting technique to generate a sample from η by the considering a normal proposal distribution. The necessary steps to achieve this goal are as follows:

Step 1: Start with an initial conjecture $\left({\delta_1}^{(0)}, {\delta_2}^{(0)}, \eta^{(0)}\right)$.

Step 2: Set $h = 1$.

Step 3: Generate ${\delta_1}^{(h)}$ from Gamma$\left(m+a_1, b_1+U_{\eta^{(h-1)}}\right)$.

Step 4: Generate ${\delta_2}^{(h)}$ from Gamma$\Big(n + a_2, b_2 + V_{\eta^{(h-1)}}\Big)$.

Step 5: Generate $\eta^{(h)}$ from $\pi\left(\eta\left|{\delta_1}^{(h-1)}, {\delta_2}^{(h-1)}, \mathbf{x}, \mathbf{y}\right.\right)$ using the Metropolis-Hasting with proposal distribution $N\left(\eta^{(h-1)}, H\right)$, where H can be computed from Equation (11).

Step 6: Compute $R^{(h)}_{s,k} = \sum\limits_{i=s}^{k}\sum\limits_{j=0}^{i}\binom{k}{i}\binom{i}{j}\frac{(-1)^j {\delta_2}^{(h)}}{{\delta_1}^{(h)}(j+k-i)+{\delta_2}^{(h)}}$.

Step 7: Set $h = h + 1$.

Step 8: Repeat steps 3-7, N times, and determine $R^{(h)}_{s,k}$ for $l = 1, 2, ..., N$.

The Bayes estimate of $R_{s,k}$, based on the MCMC method, is calculated by

$$\hat{R}^{MC}_{s,k} = \frac{1}{N - N_0}\sum_{h=N_0+1}^{N} R^{(h)}_{s,k},$$

where N_0 is the burn-in period. Also, the 100(1-γ)% HPD credible interval for $R_{s,k}$ can be computed using the method of Chen and Shao (1999) , by minimizing

$$\left({R_{s,k}}^{([(1-\gamma)(N-N_0)]+h)} - {R_{s,k}}^{(h)}\right), \qquad 1 \le h \le \gamma(N - N_0),$$

where, [.] denotes the largest integer function and the values of $R_{s,k}$ are ranked in ascending order from 1 to $(N - N_0)$.

3. Estimation of $R_{s,k}$ when the common scale parameter is known

3.1. MLE of $R_{s,k}$

Assume that $x_1, x_2..., x_m$ and $y_1, y_2..., y_n$ are the random samples from the ET distribution with the sets of parameters (δ_1, η_0) and (δ_2, η_0), respectively. Thus the log-likelihood function based on a given observed sample is

$$l\left(\delta_1, \delta_2\left|\eta_0, \mathbf{x}, \mathbf{y}\right.\right) \propto m\ln\delta_1 + n\ln\delta_2 + (\delta_1 - 1)\sum_{i=1}^{m}\ln\left[1 - \exp\left(\eta_0 x_i - e^{\eta_0 x_i} + 1\right)\right]$$
$$+ (\delta_2 - 1)\sum_{i=1}^{n}\ln\left[1 - \exp\left(\eta_0 y_i - e^{\eta_0 y_i} + 1\right)\right],$$

where the constant terms are omitted from the above equation. Based on log-likelihood function, it is easily seen that the MLEs of δ_1 and δ_2 are given as:

$$\hat{\delta}_1 = -\frac{m}{\sum\limits_{i=1}^{m}\ln\left[1 - \exp\left(\eta_0 x_i - e^{\eta_0 x_i} + 1\right)\right]},$$
$$\hat{\delta}_2 = -\frac{n}{\sum\limits_{i=1}^{n}\ln\left[1 - \exp\left(\eta_0 y_i - e^{\eta_0 y_i} + 1\right)\right]}.$$

Therefore, by the invariant property of MLE, the MLE of $R_{s,k}$ is obtained from Equation (4). Proceeding in a way similar to Section 2.1, the MLE of $R_{s,k}$ has an asymptotically normal distribution with the mean $R_{s,k}$ and variance

$$\hat{H}_{s,k} = \frac{{\delta_1}^2}{m}\left(\frac{\partial R_{s,k}}{\partial\delta_1}\right)^2 + \frac{{\delta_2}^2}{m}\left(\frac{\partial R_{s,k}}{\partial\delta_2}\right)^2\Bigg|\left(\hat{\delta}_1, \hat{\delta}_2\right).$$

and the 100(1-γ)% ACI of $R_{s,k}$ is constructed as

$$\hat{R}^{MLE}_{s,k} \pm z_{\gamma/2}\sqrt{\hat{H}_{s,k}},$$

where, $z_{\gamma/2}$ is the upper $\gamma/2$th quantile of the $N(0,1)$.

3.2. UMVUE of $R_{s,k}$

In this subsection, we derive the UMVUE of $R_{s,k}$ through an unbiased estimator of $\varphi(\delta_1,\delta_2) = (-1)^j\delta_2/[\delta_1(j+k-i)+\delta_2]$ and a complete sufficient statistic of (δ_1,δ_2). It is obvious that $(U^*,\ V^*) = \left(-\sum_{i=1}^{m}\ln\left[1-\exp\left(\eta_0 x_i - e^{\eta_0 x_i}+1\right)\right], -\sum_{i=1}^{n}\ln\left[1-\exp\left(\eta_0 y_i - e^{\eta_0 y_i}+1\right)\right]\right)$ is the complete sufficient statistic of (δ_1,δ_2). Also, the statistic U^* has Gamma distribution with parameters m and δ_1, and the statistic V^* has Gamma distribution with parameters n and δ_2. Let $U^0 = -\ln\left[1-\exp\left(\eta_0 X_1 - e^{\eta_0 X_1}+1\right)\right]$ and $V^0 = -\ln\left[1-\exp\left(\eta_0 Y_1 - e^{\eta_0 Y_1}+1\right)\right]$, we define

$$\psi\left(U^0,V^0\right) = \begin{cases} 1, & U^0 > (j+k-i)\,V^0 \\ 0, & \text{otherwise.} \end{cases}$$

It is easy to know that U^0 and V^0 come from the exponential distributions with parameters δ_1 and δ_2, respectively. Then, $\psi\left(U^0,V^0\right)$ is an unbiased estimator of $\varphi(\delta_1,\delta_2)$. Applying Lehmann's theorem, the UMVUE of $\varphi(\delta_1,\delta_2)$ is specified by

$$\begin{aligned}\hat{\varphi}_{UM}(\delta_1,\delta_2) &= E\left[\psi\left(U^0,V^0\right)|U^*=u^*,\ V^*=v^*\right] \\ &= \int_A\int f_{U^0|U^*=u^*}\left(u^0\,|u^*\right) f_{V^0|V^*=v^*}\left(v^0\,|v^*\right) du^0\,dv^0,\end{aligned}$$

where $A = \left\{\left(u^0,v^0\right):\ 0<u^0<u^*,\ 0<v^0<v^*,\ u^0>(j+k-i)\,v^0\right\}$. This integral can be discussed with regards to $h<1$ and $h>1$, where $h=(j+k-i)\,v^*/u^*$. When $h<1$, we have

$$\hat{\varphi}_{UM}(\delta_1,\delta_2) = \int_0^{v^*}\int_{(j+k-i)v^0}^{u^*} \frac{(m-1)(n-1)}{u^*v^*}\left(1-\frac{u^0}{u^*}\right)^{m-2}\left(1-\frac{v^0}{v^*}\right)^{n-2} du^0\,dv^0.$$

By using the change of variable $z = v^0/v^*$ and with some simplification, we obtain

$$\hat{\varphi}_{UM}(\delta_1,\delta_2) = \sum_{l=0}^{m-1}(-1)^l(h)^l\binom{m-1}{l}\Big/\binom{n+l-1}{l}. \tag{15}$$

Similarly, when $h>1$, we have

$$\hat{\varphi}_{UM}(\delta_1,\delta_2) = \int_0^{u^*}\int_0^{u^0/(j+k-i)} \frac{(m-1)(n-1)}{u^*v^*}\left(1-\frac{u^0}{u^*}\right)^{m-2}\left(1-\frac{v^0}{v^*}\right)^{n-2} dv^0\,du^0.$$

By using the change of variable $z = u^0/u^*$ and with some simplification, we obtain

$$\hat{\varphi}_{UM}(\delta_1,\delta_2) = 1-\sum_{l=0}^{n-1}(-1)^l(h)^{-l}\binom{n-1}{l}\Big/\binom{m+l-1}{l}. \tag{16}$$

Thus, the $\hat{\varphi}_{UM}(\delta_1,\delta_2)$ is obtained from Equations (15) and (16). Finally, the UMVUE of $R_{s,k}$ is determined by applying the linearity property of UMVUE as follows

$$\hat{R}^{UM}_{s,k} = \sum_{i=s}^{k}\sum_{j=0}^{k}\binom{k}{i}\binom{k}{j}(-1)^j\hat{\varphi}_{UM}(\delta_1,\delta_2).$$

3.3. Bayes estimation of $R_{s,k}$

In this subsection, we derive the Bayes estimate of $R_{s,k}$ along with its HPD credible interval under the SE loss function. To achieve this aim, we suppose that the independent random variables δ_1 and δ_2 have Gamma priors with positive parameters (a_1, b_1) and (a_2, b_2), respectively. Based on the observations, the joint posterior density function is

$$\begin{aligned}\pi\left(\delta_1, \delta_2 \left|\eta_0, \mathbf{x}, \mathbf{y}\right.\right) &= \frac{L\left(\eta_0, \mathbf{x}, \mathbf{y}\left|\delta_1, \delta_2\right.\right)\pi_1\left(\delta_1\right)\pi_2\left(\delta_2\right)}{\int_0^\infty\int_0^\infty L\left(\eta_0, \mathbf{x}, \mathbf{y}\left|\delta_1, \delta_2\right.\right)\pi_1\left(\delta_1\right)\pi_2\left(\delta_2\right)d\delta_1 d\delta_2}\\ &= \frac{\left(b_1+U^*\right)^{m+a_1}\left(b_2+V^*\right)^{n+a_2}}{\Gamma\left(m+a_1\right)\Gamma\left(n+a_2\right)}{\delta_1}^{m+a-1}{\delta_2}^{n+a_2-1}\exp\left[-\delta_1\left(b_1+U^*\right)-\delta_2\left(b_2+V^*\right)\right],\end{aligned}$$

where U^* and V^* being defined in Subsection 3.2. Then, the Bayes estimate of $R_{s,k}$ is calculated by

$$\hat{R}_{s,k}^{Bayes} = \sum_{i=s}^{k}\sum_{j=0}^{i}\binom{k}{i}\binom{i}{j}(-1)^j\int_0^\infty\int_0^\infty \frac{\delta_2}{\delta_1\left(j+k-i\right)+\delta_2}\pi\left(\delta_1, \delta_2\left|\eta_0, \mathbf{x}, \mathbf{y}\right.\right)d\delta_1 d\delta_2.$$

Now using the results of Kızılaslan (2017), the Bayes estimate of $R_{s,k}$ can be rewritten as

$$\hat{R}_{s,k}^{Bayes} = \begin{cases}\sum\limits_{i=s}^{k}\sum\limits_{j=0}^{k}\begin{pmatrix}k\\ i\end{pmatrix}\begin{pmatrix}k\\ j\end{pmatrix}(-1)^j(1-w)^{n+a_2}\frac{n+a_2}{q}{}_1F_2\left(q, n+a_2+1; q+1, w\right), & |w|<1\\ \sum\limits_{i=s}^{k}\sum\limits_{j=0}^{k}\begin{pmatrix}k\\ i\end{pmatrix}\begin{pmatrix}k\\ j\end{pmatrix}\frac{(-1)^j(n+a_2)}{q(1-w)^{m+a_1}}{}_1F_2\left(q, m+a_1+1; q+1, \frac{w}{w-1}\right), & w<-1\end{cases}$$

where $q = m+n+a_1+a_2$ and $w = 1-\frac{(b_2+V^*)(j+k-i)}{b_1+U^*}$. Notice that

$${}_2F_1\left(a, b; c, x\right) = \frac{1}{Beta\left(a, c-a\right)}\int_0^1 w^{a-1}(1-w)^{c-a-1}(1-xw)^{-b}dw, \quad |w|<1,$$

is the hypergeometric function that can be obtained using R software command $hyperg\_2F1()$ from 'gsl' package. Also, the 100(1-γ)% HPD credible interval for $R_{s,k}$ can be computed by using the method of Chen and Shao (1999) proceeding in a way similar to Section 2.2.

4. Simulation study

In this section, we perform Monte Carlo simulations to compare the performances of different estimates of $R_{s,k}$ by using the classical and Bayesian methods. In this regard, for two scenarios (η is unknown and known), we generate random samples from the stress and strength variables for different sample sizes (m, n)= (10,10), (30,30), (50,50) and (70,70). The reliability of MSS model are computed based on the point and interval estimations for two cases of 1-out-of-4: G system and 3-out-of-5: G system. It should be noted that for generating random samples of size n from the ET distribution with parameters δ and η, we used the computational pattern of Sharma *et al.* (2020) as follows

$$x_i = \frac{1}{\eta}\left[-W_{-1}\left(\frac{{u_i}^{1/\delta}-1}{e}\right)\right], \qquad i = 1, 2, ..., n,$$

where $u_i \sim U\left(0,1\right)$ and $W_{-1}\left(.\right)$ represents the negative branch of the Lambert W function. The performances of the studied methods are compared while the true value of reliability in MSS model changes from 0.05 (low extreme value) to 0.95 (high extreme value). The criteria of mean square error (MSE), average length (AL) as well as coverage probability (CP) at confidence level of 95%, are used to evaluate the simulation results. To investigate the Bayes estimates, the informative priors are considered. All of the computations are performed by using R 3.4.4 based on 10,000 replications. Furthermore, the Bayes estimate along with its credible interval, are calculated using 1,000 sampling. Table 1 represents the details of the simulations. The point estimates of $R_{s,k}$ along with MSEs are reported in Tables 2 and 3, when η is unknown and they are reported in Tables 4 and 5, when η is known. Also, the interval estimates of $R_{s,k}$ along with CPs are reported in Tables 6 and 7, when η is unknown and they are reported in Tables 8 and 9, when η is known. Clearly, lower limit of the constructed intervals smaller than 0 and upper limit of the same larger than 1 need to be replaced with 0 and 1, respectively. However, we considered the limits as they are to make a more accurate comparison between the different ACI methods. Based on these tables, the following conclusions can be deduced.

- For both scenarios and both cases, the MSEs for the estimates of $R_{s,k}$ show decrease with increases of sample size. Also, as the sample size increases, the estimates of $R_{s,k}$ become closer and closer to their true values. This reflects the fact that considered estimates are consistent.
- The MSEs for the estimates of $R_{s,k}$ and the ALs of the constructed intervals are high when $R_{s,k}$ is about 0.5 and they are low for extreme values.
- In the scenario where η is known, always the Bayes estimator has a smaller MSE than the MLE and UMVUE. Based on this criterion, there is no difference between the MLE and UMVUE except for $n = 10$. In this sample size, generally the MLE is better than UMVUE. However, for extreme values of $R_{s,k}$, the UMVUE has a smaller MSE.
- As anticipated, in the both cases, the UMVUEs have the values closest to $R_{s,k}$. This is due to its unbiasedness property.
- It is interesting to note that in the scenario where η is known, in the case of $(s, k) = (1, 4)$ the corresponding points on both sides of the point of $R_{s,k}$=0.5 have the same MSEs (for MLE and UMVUE) and the same ALs (for different ACI methods).
- In the scenario where η is unknown, always the Bayes estimator has a smaller MSE than the MLE.
- For all considered situations, always the HPD credible interval has a smaller AL than the different ACI methods as well as generally has a best CP which is mostly greater than the predefined nominal level of 95%.
- The ordering of performance in terms of ALs between the various ACI methods is AST, NT and LOGT for extreme values of $R_{s,k}$ and it is LOGT, AST and NT for moderate values of $R_{s,k}$. Also, for large sample size, the ALs of the constructed intervals are approximately the same.
- For both scenarios and both cases, the ALs of the intervals show decrease with increases of sample size.
- Comparing the different ACI methods showed that NT, AST and LOGT have the lowest to highest CPs, respectively. Additionally, the CPs of the constructed intervals in the scenario where η is known, are better than another scenario where η is unknown.
- In the scenario where η is known, the CPs based on the LOGT method reach the nominal level of 95%, but they are generally slightly less than 95%. In this regard, the performance of the other two methods is not very satisfactory.
- In the both scenarios, in the case of $(s, k) = (1, 4)$, the MSEs for the estimates of $R_{s,k}$ and the ALs of the intervals are less than the corresponding one for the case of $(s, k) = (3, 5)$.

- In the both cases, the MSEs for the estimates of $R_{s,k}$ when η is known are less than when η is unknown. Also, generally the ALs of the intervals when η is known are larger than when η is unknown.

Table 1: Different combinations of stress and strength parameters along with the true values of $R_{s,k}$ as well as different priors for Monte Carlo simulations

		η is unknown				η is known		
(s,k)	$R_{s,k}$	(δ_1,δ_2,η)	(a_1,b_1)	(a_2,b_2)	(a_3,b_3)	(δ_1,δ_2)	(a_1,b_1)	(a_2,b_2)
	0.05	(1,76,3)	(2,2)	(76,1)	(6,2)	(1,76)	(2,2)	(76,1)
	0.10	(1,36,3)	(2,2)	(36,1)	(6,2)	(1,36)	(2,2)	(36,1)
	0.20	(1,16,3)	(2,2)	(16,1)	(6,2)	(1,16)	(2,2)	(16,1)
	0.30	(1,9.3333,3)	(2,2)	(9,1)	(6,2)	(1,9.3333)	(2,2)	(9,1)
	0.40	(1,6,3)	(2,2)	(12,2)	(6,2)	(1,6)	(2,2)	(12,2)
(1,4)	0.50	(1,4,3)	(2,2)	(8,2)	(6,2)	(1,4)	(2,2)	(8,2)
	0.60	(1,2.6666,3)	(2,2)	(5,2)	(6,2)	(1,2.6666)	(2,2)	(5,2)
	0.70	(2,3.4286,3)	(4,2)	(7,2)	(6,2)	(2,3.4286)	(4,2)	(7,2)
	0.80	(2,2,3)	(4,2)	(4,2)	(6,2)	(2,2)	(4,2)	(4,2)
	0.90	(3,1.3333,3)	(6,2)	(3,2)	(6,2)	(3,1.3333)	(6,2)	(3,2)
	0.95	(5,1.0526,3)	(10,2)	(2,2)	(6,2)	(5,1.0526)	(10,2)	(2,2)
	0.05	(2,13.3159,3)	(4,2)	(13,1)	(6,2)	(2,13.3159)	(4,2)	(13,1)
	0.10	(2,8.9476,3)	(4,2)	(9,1)	(6,2)	(2,8.9476)	(4,2)	(9,1)
	0.20	(2,5.4882,3)	(4,2)	(11,2)	(6,2)	(2,5.4882)	(4,2)	(11,2)
	0.30	(2,3.8100,3)	(4,2)	(8,2)	(6,2)	(2,3.8100)	(4,2)	(8,2)
	0.40	(2,2.7520,3)	(4,2)	(6,2)	(6,2)	(2,2.7520)	(4,2)	(6,2)
(3,5)	0.50	(2,2,3)	(4,2)	(4,2)	(6,2)	(2,2)	(4,2)	(4,2)
	0.60	(2,1.4267,3)	(4,2)	(3,2)	(6,2)	(2,1.4267)	(4,2)	(3,2)
	0.70	(4,1.9388,3)	(8,2)	(4,2)	(6,2)	(4,1.9388)	(8,2)	(4,2)
	0.80	(4,1.1847,3)	(8,2)	(2,2)	(6,2)	(4,1.1847)	(8,2)	(2,2)
	0.90	(8,1.0958,3)	(8,1)	(2,2)	(6,2)	(8,1.0958)	(8,1)	(2,2)
	0.95	(16,1.0570,3)	(16,1)	(2,2)	(6,2)	(16,1.0570)	(16,1)	(2,2)

Table 2: Estimates of $R_{1,4}$ along with MSEs when η is unknown

$(\delta_1, \delta_2, \eta)$	$R_{s,k}$	(m,n)	MLE	MSE	Bayes	MSE
(1,76,3)	0.05	(10,10)	0.0463	0.0013	0.0607	0.0005
		(30,30)	0.0487	0.0005	0.0589	0.0001
		(50,50)	0.0493	0.0003	0.0561	0.0001
		(70,70)	0.0496	0.0002	0.0529	0.0001
(1,36,3)	0.1	(10,10)	0.0908	0.0036	0.1158	0.0011
		(30,30)	0.0969	0.0013	0.1084	0.0004
		(50,50)	0.0978	0.0008	0.1053	0.0002
		(70,70)	0.0990	0.0005	0.1026	0.0001
(1,16,3)	0.2	(10,10)	0.1836	0.0091	0.2150	0.0033
		(30,30)	0.1939	0.0031	0.2101	0.0013
		(50,50)	0.1962	0.0019	0.2040	0.0008
		(70,70)	0.1982	0.0013	0.2006	0.0007
(1,9.3333,3)	0.3	(10,10)	0.2772	0.0135	0.3105	0.0055
		(30,30)	0.2921	0.0044	0.3029	0.0031
		(50,50)	0.2948	0.0026	0.3005	0.0018
		(70,70)	0.2964	0.0021	0.3002	0.0014
(1,6,3)	0.4	(10,10)	0.3725	0.0166	0.3851	0.0065
		(30,30)	0.3905	0.0053	0.3911	0.0043
		(50,50)	0.3949	0.0030	0.3931	0.0026
		(70,70)	0.3964	0.0021	0.3956	0.0015
(1,4,3)	0.5	(10,10)	0.4721	0.0171	0.4865	0.0060
		(30,30)	0.4908	0.0052	0.4925	0.0040
		(50,50)	0.4948	0.0031	0.4970	0.0023
		(70,70)	0.4969	0.0023	0.4997	0.0017
(1,2.6666,3)	0.6	(10,10)	0.5749	0.0156	0.5823	0.0056
		(30,30)	0.5933	0.0044	0.5896	0.0031
		(50,50)	0.5955	0.0025	0.5956	0.0020
		(70,70)	0.5979	0.0017	0.6000	0.0015
(2,3.4286,3)	0.7	(10,10)	0.6795	0.0122	0.6850	0.0042
		(30,30)	0.6948	0.0033	0.6901	0.0022
		(50,50)	0.6959	0.0020	0.6973	0.0014
		(70,70)	0.6965	0.0014	0.6986	0.0012
(2,2,3)	0.8	(10,10)	0.7901	0.0067	0.7910	0.0034
		(30,30)	0.7962	0.0019	0.7932	0.0015
		(50,50)	0.7977	0.0011	0.7963	0.0009
		(70,70)	0.7984	0.0008	0.7974	0.0006
(3,1.3333,3)	0.9	(10,10)	0.8959	0.0023	0.8921	0.0013
		(30,30)	0.8989	0.0005	0.8958	0.0005
		(50,50)	0.8994	0.0004	0.8974	0.0003
		(70,70)	0.8997	0.0003	0.8982	0.0002
(5,1.0526,3)	0.95	(10,10)	0.9501	0.0007	0.9508	0.0004
		(30,30)	0.9502	0.0002	0.9496	0.0001
		(50,50)	0.9502	0.0001	0.9492	0.0001
		(70,70)	0.9499	0.0001	0.9494	0.0001

Table 3: Estimates of $R_{3,5}$ along with MSEs when η is unknown

$(\delta_1, \delta_2, \eta)$	$R_{s,k}$	(m, n)	MLE	MSE	Bayes	MSE
(2,13.3159,3)	0.05	(10,10)	0.0588	0.0030	0.0570	0.0010
		(30,30)	0.0516	0.0009	0.0543	0.0005
		(50,50)	0.0514	0.0005	0.0530	0.0003
		(70,70)	0.0498	0.0003	0.0531	0.0002
(2,8.9476,3)	0.1	(10,10)	0.1033	0.0066	0.1097	0.0025
		(30,30)	0.1010	0.0021	0.1066	0.0013
		(50,50)	0.1008	0.0013	0.1051	0.0010
		(70,70)	0.1004	0.0010	0.1041	0.0007
(2,5.4882,3)	0.2	(10,10)	0.2026	0.0139	0.2085	0.0046
		(30,30)	0.1989	0.0045	0.2062	0.0028
		(50,50)	0.2004	0.0027	0.2049	0.0020
		(70,70)	0.2002	0.0019	0.2044	0.0014
(2,3.8100,3)	0.3	(10,10)	0.2972	0.0185	0.3064	0.0069
		(30,30)	0.2993	0.0062	0.3047	0.0042
		(50,50)	0.2990	0.0037	0.3025	0.0029
		(70,70)	0.3005	0.0028	0.2994	0.0021
(2,2.7520,3)	0.4	(10,10)	0.3974	0.0214	0.3951	0.0080
		(30,30)	0.3980	0.0067	0.3968	0.0047
		(50,50)	0.3990	0.0041	0.3979	0.0032
		(70,70)	0.3993	0.0029	0.3994	0.0025
(2,2,3)	0.5	(10,10)	0.4955	0.0213	0.4930	0.0086
		(30,30)	0.4986	0.0063	0.4958	0.0047
		(50,50)	0.4994	0.0038	0.4963	0.0031
		(70,70)	0.4997	0.0026	0.4976	0.0023
(2,1.4267,3)	0.6	(10,10)	0.5969	0.0178	0.5818	0.0086
		(30,30)	0.5984	0.0056	0.5906	0.0042
		(50,50)	0.6005	0.0032	0.5936	0.0028
		(70,70)	0.6003	0.0024	0.5943	0.0021
(4,1.9388,3)	0.7	(10,10)	0.6991	0.0133	0.6896	0.0053
		(30,30)	0.6995	0.0041	0.6954	0.0031
		(50,50)	0.7002	0.0024	0.6975	0.0021
		(70,70)	0.7002	0.0018	0.6989	0.0012
(4,1.1847,3)	0.8	(10,10)	0.8009	0.0078	0.7903	0.0036
		(30,30)	0.8002	0.0024	0.7961	0.0018
		(50,50)	0.8003	0.0014	0.7980	0.0012
		(70,70)	0.8001	0.0009	0.7988	0.0008
(8,1.0958,3)	0.9	(10,10)	0.9012	0.0028	0.8940	0.0014
		(30,30)	0.9014	0.0008	0.8972	0.0006
		(50,50)	0.9004	0.0005	0.8979	0.0004
		(70,70)	0.9000	0.0003	0.8984	0.0003
(16,1.0570,3)	0.95	(10,10)	0.9521	0.0009	0.9444	0.0005
		(30,30)	0.9509	0.0003	0.9467	0.0002
		(50,50)	0.9506	0.0002	0.9477	0.0001
		(70,70)	0.9501	0.0001	0.9489	0.0001

Table 4: Estimates of $R_{1,4}$ along with MSEs when η is known (η=3)

(δ_1, δ_2)	$R_{s,k}$	(m, n)	MLE	MSE	UMVUE	MSE	Bayes	MSE
(1,76)	0.05	(10,10)	0.0546	0.0006	0.0501	0.0005	0.0539	0.0002
		(30,30)	0.0515	0.0002	0.0500	0.0002	0.0518	0.0001
		(50,50)	0.0509	0.0001	0.0500	0.0001	0.0511	0.0001
		(70,70)	0.0506	0.0001	0.0500	0.0001	0.0509	0.0001
(1,36)	0.1	(10,10)	0.1077	0.0020	0.1000	0.0018	0.1067	0.0007
		(30,30)	0.1024	0.0006	0.1001	0.0006	0.1032	0.0003
		(50,50)	0.1015	0.0003	0.1000	0.0003	0.1025	0.0002
		(70,70)	0.1011	0.0002	0.0999	0.0002	0.1016	0.0002
(1,16)	0.2	(10,10)	0.2103	0.0058	0.1998	0.0056	0.2100	0.0024
		(30,30)	0.2032	0.0018	0.2003	0.0017	0.2051	0.0012
		(50,50)	0.2019	0.0011	0.2001	0.0010	0.2031	0.0008
		(70,70)	0.2013	0.0007	0.1999	0.0007	0.2028	0.0006
(1,9.3333)	0.3	(10,10)	0.3079	0.0090	0.3001	0.0094	0.3139	0.0043
		(30,30)	0.3026	0.0030	0.2995	0.0030	0.3071	0.0022
		(50,50)	0.3018	0.0018	0.2998	0.0018	0.3046	0.0015
		(70,70)	0.3012	0.0013	0.3000	0.0013	0.3032	0.0011
(1,6)	0.4	(10,10)	0.4047	0.0112	0.3991	0.0122	0.4074	0.0049
		(30,30)	0.4017	0.0038	0.4000	0.0039	0.4038	0.0026
		(50,50)	0.4009	0.0023	0.3998	0.0023	0.4021	0.0018
		(70,70)	0.4006	0.0016	0.3997	0.0017	0.4017	0.0014
(1,4)	0.5	(10,10)	0.4992	0.0120	0.4999	0.0130	0.5011	0.0057
		(30,30)	0.4998	0.0040	0.4997	0.0042	0.5008	0.0030
		(50,50)	0.4998	0.0025	0.5002	0.0025	0.4999	0.0020
		(70,70)	0.5003	0.0018	0.4998	0.0018	0.5004	0.0015
(1,2.6666)	0.6	(10,10)	0.5950	0.0111	0.6007	0.0122	0.6009	0.0057
		(30,30)	0.5982	0.0038	0.6001	0.0039	0.5989	0.0030
		(50,50)	0.5988	0.0023	0.6002	0.0023	0.5997	0.0020
		(70,70)	0.5994	0.0016	0.6001	0.0017	0.5998	0.0015
(2,3.4286)	0.7	(10,10)	0.6923	0.0089	0.7002	0.0094	0.6906	0.0036
		(30,30)	0.6969	0.0030	0.7000	0.0031	0.6954	0.0021
		(50,50)	0.6983	0.0018	0.7002	0.0018	0.6970	0.0015
		(70,70)	0.6985	0.0013	0.7000	0.0013	0.6976	0.0011
(2,2)	0.8	(10,10)	0.7902	0.0057	0.8002	0.0056	0.7886	0.0028
		(30,30)	0.7969	0.0018	0.8002	0.0017	0.7952	0.0014
		(50,50)	0.7981	0.0010	0.7998	0.0010	0.7967	0.0009
		(70,70)	0.7988	0.0007	0.8000	0.0007	0.7971	0.0007
(3,1.3333)	0.9	(10,10)	0.8926	0.0020	0.9002	0.0018	0.8881	0.0011
		(30,30)	0.8975	0.0006	0.9000	0.0006	0.8951	0.0005
		(50,50)	0.8985	0.0003	0.9001	0.0003	0.8996	0.0003
		(70,70)	0.8989	0.0002	0.9001	0.0002	0.8977	0.0002
(5,1.0526)	0.95	(10,10)	0.9453	0.0006	0.9499	0.0005	0.9449	0.0003
		(30,30)	0.9485	0.0002	0.9500	0.0002	0.9478	0.0001
		(50,50)	0.9491	0.0001	0.9499	0.0001	0.9487	0.0001
		(70,70)	0.9493	0.0001	0.9499	0.0001	0.9489	0.0001

Table 5: Estimates of $R_{3,5}$ along with MSEs when η is known (η=3)

(δ_1, δ_2)	$R_{s,k}$	(m, n)	MLE	MSE	UMVUE	MSE	Bayes	MSE
(2,13.3159)	0.05	(10,10)	0.0644	0.0030	0.0501	0.0024	0.0643	0.0010
		(30,30)	0.0548	0.0007	0.0500	0.0007	0.0574	0.0005
		(50,50)	0.0527	0.0004	0.0502	0.0004	0.0552	0.0003
		(70,70)	0.0520	0.0003	0.0499	0.0003	0.0539	0.0002
(2,8.9476)	0.1	(10,10)	0.1174	0.0065	0.0997	0.0061	0.1175	0.0025
		(30,30)	0.1061	0.0019	0.1000	0.0018	0.1090	0.0013
		(50,50)	0.1034	0.0011	0.1000	0.0011	0.1055	0.0009
		(70,70)	0.1025	0.0008	0.0999	0.0008	0.1043	0.0007
(2,5.4882)	0.2	(10,10)	0.2160	0.0122	0.1998	0.0137	0.2152	0.0044
		(30,30)	0.2056	0.0041	0.1999	0.0043	0.2077	0.0027
		(50,50)	0.2029	0.0025	0.2000	0.0025	0.2047	0.0018
		(70,70)	0.2022	0.0017	0.1999	0.0018	0.2041	0.0014
(2,3.8100)	0.3	(10,10)	0.3094	0.0161	0.2997	0.0185	0.3032	0.0064
		(30,30)	0.3034	0.0057	0.2996	0.0060	0.3014	0.0039
		(50,50)	0.3032	0.0035	0.3001	0.0035	0.3008	0.0027
		(70,70)	0.3014	0.0025	0.3001	0.0025	0.3005	0.0021
(2,2.7520)	0.4	(10,10)	0.4028	0.0179	0.4015	0.0209	0.3933	0.0078
		(30,30)	0.4005	0.0064	0.3998	0.0067	0.3969	0.0046
		(50,50)	0.4004	0.0039	0.3999	0.0040	0.3981	0.0032
		(70,70)	0.3999	0.0028	0.3996	0.0029	0.3972	0.0024
(2,2)	0.5	(10,10)	0.4964	0.0178	0.4993	0.0201	0.4940	0.0083
		(30,30)	0.4982	0.0062	0.4999	0.0064	0.4971	0.0047
		(50,50)	0.4989	0.0038	0.4997	0.0038	0.4984	0.0031
		(70,70)	0.4999	0.0027	0.4997	0.0028	0.4991	0.0024
(2,1.4267)	0.6	(10,10)	0.5899	0.0156	0.6012	0.0167	0.5838	0.0085
		(30,30)	0.5964	0.0052	0.6002	0.0054	0.5930	0.0042
		(50,50)	0.5981	0.0032	0.6005	0.0032	0.5957	0.0028
		(70,70)	0.5986	0.0023	0.6001	0.0023	0.5962	0.0021
(4,1.9388)	0.7	(10,10)	0.6875	0.0118	0.7003	0.0120	0.6845	0.0046
		(30,30)	0.6958	0.0038	0.7004	0.0038	0.6925	0.0027
		(50,50)	0.6974	0.0023	0.7001	0.0022	0.6950	0.0018
		(70,70)	0.6981	0.0016	0.7000	0.0016	0.6964	0.0014
(4,1.1847)	0.8	(10,10)	0.7869	0.0070	0.8001	0.0065	0.7915	0.0030
		(30,30)	0.7958	0.0021	0.7999	0.0020	0.7954	0.0015
		(50,50)	0.7974	0.0012	0.8000	0.0012	0.7973	0.0010
		(70,70)	0.7982	0.0009	0.8000	0.0009	0.7974	0.0008
(8,1.0958)	0.9	(10,10)	0.8914	0.0023	0.9001	0.0020	0.8916	0.0011
		(30,30)	0.8972	0.0006	0.9000	0.0006	0.8960	0.0005
		(50,50)	0.8983	0.0004	0.8999	0.0004	0.8976	0.0003
		(70,70)	0.8989	0.0003	0.9000	0.0003	0.8979	0.0002
(16,1.0570)	0.95	(10,10)	0.9453	0.0006	0.9500	0.0005	0.9458	0.0003
		(30,30)	0.9485	0.0002	0.9501	0.0002	0.9478	0.0001
		(50,50)	0.9490	0.0001	0.9500	0.0001	0.9485	0.0001
		(70,70)	0.9493	0.0001	0.9500	0.0001	0.9489	0.0001

Table 6: ALs of the interval estimates of $R_{1,4}$ and their corresponding CPs when η is unknown

$(\delta_1, \delta_2, \eta)$	$R_{s,k}$	(m,n)	LOGT	CP	AST	CP	NT	CP	HPD	CP
(1,76,3)	0.05	(10,10)	0.090	0.693	0.083	0.670	0.083	0.652	0.062	0.976
		(30,30)	0.053	0.742	0.051	0.732	0.051	0.737	0.038	0.965
		(50,50)	0.041	0.754	0.040	0.751	0.040	0.745	0.032	0.961
		(70,70)	0.035	0.764	0.035	0.767	0.034	0.755	0.033	0.947
(1,36,3)	0.1	(10,10)	0.158	0.750	0.148	0.718	0.150	0.695	0.103	0.979
		(30,30)	0.097	0.793	0.094	0.784	0.095	0.777	0.076	0.968
		(50,50)	0.076	0.808	0.074	0.799	0.074	0.798	0.063	0.961
		(70,70)	0.064	0.844	0.063	0.840	0.064	0.810	0.045	0.940
(1,16,3)	0.2	(10,10)	0.262	0.808	0.255	0.779	0.262	0.751	0.197	0.985
		(30,30)	0.166	0.857	0.164	0.845	0.165	0.829	0.140	0.971
		(50,50)	0.130	0.863	0.130	0.856	0.130	0.851	0.118	0.962
		(70,70)	0.111	0.883	0.111	0.880	0.111	0.861	0.102	0.952
(1,9.3333,3)	0.3	(10,10)	0.334	0.844	0.335	0.817	0.345	0.794	0.279	0.977
		(30,30)	0.213	0.886	0.214	0.877	0.216	0.869	0.189	0.964
		(50,50)	0.168	0.899	0.169	0.891	0.170	0.886	0.153	0.958
		(70,70)	0.144	0.907	0.144	0.897	0.144	0.878	0.141	0.941
(1,6,3)	0.4	(10,10)	0.380	0.871	0.386	0.848	0.397	0.828	0.307	0.974
		(30,30)	0.242	0.912	0.244	0.903	0.246	0.886	0.212	0.961
		(50,50)	0.191	0.916	0.192	0.912	0.193	0.910	0.171	0.959
		(70,70)	0.163	0.912	0.164	0.905	0.164	0.918	0.156	0.938
(1,4,3)	0.5	(10,10)	0.399	0.896	0.408	0.877	0.422	0.865	0.336	0.973
		(30,30)	0.251	0.925	0.254	0.920	0.257	0.911	0.225	0.959
		(50,50)	0.198	0.933	0.199	0.929	0.201	0.926	0.180	0.955
		(70,70)	0.169	0.941	0.170	0.937	0.170	0.916	0.158	0.943
(1,2.6666,3)	0.6	(10,10)	0.392	0.907	0.400	0.894	0.413	0.880	0.331	0.968
		(30,30)	0.242	0.938	0.244	0.934	0.247	0.928	0.219	0.956
		(50,50)	0.190	0.943	0.191	0.941	0.192	0.941	0.177	0.952
		(70,70)	0.162	0.941	0.163	0.942	0.163	0.955	0.152	0.945
(2,3.4286,3)	0.7	(10,10)	0.352	0.919	0.355	0.908	0.366	0.893	0.282	0.971
		(30,30)	0.211	0.940	0.212	0.937	0.213	0.930	0.187	0.952
		(50,50)	0.165	0.944	0.165	0.942	0.166	0.938	0.149	0.950
		(70,70)	0.139	0.949	0.140	0.949	0.141	0.941	0.134	0.948
(2,2,3)	0.8	(10,10)	0.280	0.927	0.271	0.914	0.282	0.899	0.225	0.973
		(30,30)	0.163	0.942	0.162	0.938	0.163	0.936	0.147	0.955
		(50,50)	0.126	0.943	0.126	0.942	0.126	0.942	0.116	0.951
		(70,70)	0.106	0.946	0.106	0.942	0.107	0.934	0.099	0.942
(3,1.3333,3)	0.9	(10,10)	0.169	0.911	0.159	0.889	0.162	0.880	0.135	0.981
		(30,30)	0.095	0.938	0.092	0.932	0.093	0.922	0.085	0.960
		(50,50)	0.073	0.941	0.072	0.939	0.072	0.932	0.066	0.957
		(70,70)	0.062	0.948	0.061	0.938	0.061	0.935	0.054	0.948
(5,1.0526,3)	0.95	(10,10)	0.092	0.884	0.084	0.858	0.085	0.823	0.069	0.985
		(30,30)	0.051	0.092	0.050	0.911	0.050	0.895	0.044	0.966
		(50,50)	0.039	0.926	0.039	0.920	0.039	0.909	0.034	0.961
		(70,70)	0.033	0.912	0.033	0.910	0.033	0.920	0.029	0.943

Table 7: ALs of the interval estimates of $R_{3,5}$ and their corresponding CPs when η is unknown

$(\delta_1, \delta_2, \eta)$	$R_{s,k}$	(m, n)	LOGT	CP	AST	CP	NT	CP	HPD	CP
(2,13.3159,3)	0.05	(10,10)	0.211	0.864	0.161	0.795	0.165	0.742	0.118	0.986
		(30,30)	0.110	0.897	0.098	0.871	0.097	0.836	0.086	0.966
		(50,50)	0.082	0.907	0.076	0.890	0.076	0.877	0.068	0.963
		(70,70)	0.068	0.926	0.065	0.911	0.064	0.874	0.059	0.951
(2,8.9476,3)	0.1	(10,10)	0.297	0.891	0.252	0.831	0.260	0.780	0.197	0.987
		(30,30)	0.171	0.922	0.159	0.889	0.161	0.870	0.139	0.966
		(50,50)	0.131	0.920	0.125	0.906	0.127	0.889	0.112	0.961
		(70,70)	0.111	0.924	0.108	0.912	0.109	0.905	0.101	0.953
(2,5.4882,3)	0.2	(10,10)	0.400	0.916	0.375	0.870	0.397	0.829	0.290	0.989
		(30,30)	0.247	0.933	0.242	0.912	0.246	0.899	0.212	0.965
		(50,50)	0.194	0.941	0.192	0.930	0.194	0.914	0.173	0.960
		(70,70)	0.165	0.939	0.164	0.932	0.165	0.922	0.143	0.947
(2,3.8100,3)	0.3	(10,10)	0.453	0.925	0.449	0.885	0.473	0.857	0.355	0.975
		(30,30)	0.287	0.940	0.288	0.925	0.293	0.917	0.250	0.960
		(50,50)	0.227	0.944	0.228	0.937	0.230	0.929	0.211	0.957
		(70,70)	0.194	0.949	0.194	0.936	0.196	0.928	0.178	0.941
(2,2.7520,3)	0.4	(10,10)	0.474	0.927	0.482	0.892	0.508	0.869	0.397	0.972
		(30,30)	0.303	0.945	0.306	0.935	0.311	0.928	0.275	0.959
		(50,50)	0.239	0.946	0.241	0.941	0.244	0.934	0.221	0.952
		(70,70)	0.204	0.936	0.205	0.930	0.207	0.935	0.199	0.937
(2,2,3)	0.5	(10,10)	0.469	0.928	0.482	0.900	0.505	0.879	0.407	0.970
		(30,30)	0.297	0.945	0.301	0.936	0.307	0.936	0.274	0.953
		(50,50)	0.235	0.949	0.237	0.942	0.239	0.941	0.219	0.950
		(70,70)	0.200	0.947	0.201	0.945	0.203	0.951	0.182	0.938
(2,1.4267,3)	0.6	(10,10)	0.440	0.928	0.451	0.905	0.470	0.883	0.387	0.965
		(30,30)	0.275	0.943	0.278	0.941	0.283	0.931	0.255	0.952
		(50,50)	0.217	0.943	0.218	0.942	0.220	0.940	0.202	0.950
		(70,70)	0.184	0.955	0.185	0.953	0.187	0.940	0.176	0.940
(4,1.9388,3)	0.7	(10,10)	0.380	0.914	0.383	0.893	0.397	0.870	0.306	0.978
		(30,30)	0.234	0.936	0.235	0.932	0.237	0.919	0.211	0.955
		(50,50)	0.183	0.940	0.184	0.937	0.185	0.932	0.170	0.952
		(70,70)	0.155	0.923	0.155	0.923	0.157	0.934	0.133	0.948
(4,1.1847,3)	0.8	(10,10)	0.294	0.904	0.288	0.878	0.296	0.845	0.235	0.980
		(30,30)	0.177	0.932	0.176	0.919	0.177	0.908	0.160	0.960
		(50,50)	0.138	0.931	0.136	0.926	0.138	0.922	0.127	0.955
		(70,70)	0.117	0.949	0.116	0.939	0.117	0.931	0.107	0.949
(8,1.0958,3)	0.9	(10,10)	0.170	0.860	0.158	0.830	0.164	0.812	0.129	0.985
		(30,30)	0.099	0.900	0.097	0.884	0.097	0.880	0.087	0.964
		(50,50)	0.077	0.903	0.076	0.900	0.076	0.894	0.068	0.957
		(70,70)	0.065	0.927	0.064	0.920	0.065	0.916	0.062	0.952
(16,1.0570,3)	0.95	(10,10)	0.092	0.813	0.083	0.780	0.085	0.755	0.061	0.988
		(30,30)	0.053	0.870	0.051	0.854	0.051	0.834	0.044	0.967
		(50,50)	0.041	0.873	0.040	0.867	0.040	0.853	0.035	0.959
		(70,70)	0.034	0.874	0.034	0.869	0.034	0.875	0.033	0.947

Table 8: ALs of the interval estimates of $R_{1,4}$ and their corresponding CPs when η is known (η=3)

(δ_1, δ_2)	$R_{s,k}$	(m, n)	LOGT	CP	AST	CP	NT	CP	HPD	CP
(1,76)	0.05	(10,10)	0.097	0.943	0.089	0.936	0.089	0.919	0.058	0.966
		(30,30)	0.051	0.947	0.049	0.944	0.049	0.938	0.037	0.958
		(50,50)	0.039	0.948	0.038	0.947	0.038	0.943	0.030	0.956
		(70,70)	0.032	0.948	0.032	0.948	0.032	0.945	0.026	0.954
(1,36)	0.1	(10,10)	0.172	0.943	0.163	0.935	0.165	0.920	0.113	0.969
		(30,30)	0.094	0.947	0.092	0.944	0.092	0.939	0.074	0.960
		(50,50)	0.072	0.949	0.071	0.947	0.071	0.944	0.060	0.949
		(70,70)	0.061	0.949	0.060	0.948	0.060	0.945	0.052	0.952
(1,16)	0.2	(10,10)	0.280	0.943	0.275	0.933	0.281	0.919	0.210	0.972
		(30,30)	0.162	0.947	0.161	0.943	0.162	0.938	0.139	0.954
		(50,50)	0.126	0.947	0.125	0.945	0.126	0.942	0.112	0.945
		(70,70)	0.106	0.949	0.106	0.948	0.106	0.945	0.097	0.943
(1,9.3333)	0.3	(10,10)	0.346	0.942	0.348	0.930	0.358	0.916	0.286	0.967
		(30,30)	0.208	0.948	0.209	0.944	0.211	0.939	0.188	0.950
		(50,50)	0.163	0.947	0.163	0.945	0.164	0.942	0.150	0.940
		(70,70)	0.138	0.949	0.138	0.947	0.139	0.945	0.128	0.934
(1,6)	0.4	(10,10)	0.383	0.943	0.391	0.928	0.403	0.914	0.313	0.964
		(30,30)	0.235	0.948	0.237	0.943	0.239	0.938	0.208	0.947
		(50,50)	0.184	0.948	0.185	0.945	0.187	0.941	0.168	0.938
		(70,70)	0.157	0.949	0.157	0.947	0.158	0.944	0.145	0.937
(1,4)	0.5	(10,10)	0.395	0.944	0.405	0.928	0.417	0.915	0.335	0.964
		(30,30)	0.244	0.949	0.246	0.945	0.245	0.940	0.221	0.948
		(50,50)	0.192	0.947	0.193	0.945	0.194	0.941	0.177	0.941
		(70,70)	0.163	0.949	0.164	0.947	0.165	0.946	0.152	0.940
(1,2.6666)	0.6	(10,10)	0.384	0.944	0.391	0.930	0.403	0.917	0.333	0.961
		(30,30)	0.235	0.948	0.237	0.943	0.239	0.939	0.216	0.942
		(50,50)	0.184	0.949	0.185	0.946	0.187	0.943	0.172	0.935
		(70,70)	0.157	0.950	0.157	0.947	0.158	0.945	0.147	0.933
(2,3.4286)	0.7	(10,10)	0.346	0.944	0.348	0.931	0.358	0.916	0.280	0.974
		(30,30)	0.208	0.946	0.209	0.942	0.211	0.937	0.186	0.947
		(50,50)	0.163	0.949	0.163	0.947	0.164	0.943	0.149	0.941
		(70,70)	0.138	0.947	0.138	0.946	0.139	0.944	0.128	0.941
(2,2)	0.8	(10,10)	0.280	0.945	0.275	0.933	0.281	0.920	0.229	0.966
		(30,30)	0.162	0.947	0.161	0.943	0.162	0.938	0.146	0.948
		(50,50)	0.126	0.948	0.125	0.946	0.126	0.943	0.116	0.937
		(70,70)	0.106	0.949	0.106	0.947	0.106	0.945	0.099	0.934
(3,1.3333)	0.9	(10,10)	0.172	0.944	0.162	0.933	0.165	0.918	0.136	0.970
		(30,30)	0.094	0.947	0.092	0.945	0.093	0.939	0.084	0.944
		(50,50)	0.072	0.947	0.071	0.946	0.071	0.943	0.066	0.942
		(70,70)	0.061	0.950	0.060	0.948	0.060	0.945	0.056	0.937
(5,1.0526)	0.95	(10,10)	0.097	0.945	0.089	0.937	0.090	0.920	0.070	0.972
		(30,30)	0.051	0.950	0.049	0.947	0.049	0.941	0.044	0.949
		(50,50)	0.039	0.949	0.038	0.947	0.038	0.944	0.034	0.945
		(70,70)	0.032	0.950	0.032	0.947	0.032	0.947	0.030	0.940

Table 9: ALs of the interval estimates of $R_{3,5}$ and their corresponding CPs when η is known (η=3)

(δ_1, δ_2)	$R_{s,k}$	(m, n)	LOGT	CP	AST	CP	NT	CP	HPD	CP
(2,13.3159)	0.05	(10,10)	0.228	0.939	0.179	0.911	0.184	0.873	0.129	0.982
		(30,30)	0.110	0.945	0.099	0.933	0.100	0.915	0.085	0.954
		(50,50)	0.081	0.948	0.075	0.941	0.076	0.927	0.068	0.946
		(70,70)	0.068	0.948	0.063	0.944	0.064	0.934	0.058	0.941
(2,8.9476)	0.1	(10,10)	0.317	0.942	0.274	0.914	0.286	0.881	0.210	0.974
		(30,30)	0.172	0.946	0.161	0.935	0.163	0.920	0.141	0.952
		(50,50)	0.130	0.949	0.125	0.942	0.126	0.931	0.113	0.945
		(70,70)	0.109	0.949	0.106	0.944	0.106	0.936	0.097	0.939
(2,5.4882)	0.2	(10,10)	0.414	0.947	0.394	0.921	0.414	0.894	0.304	0.975
		(30,30)	0.247	0.949	0.243	0.939	0.247	0.928	0.210	0.950
		(50,50)	0.192	0.949	0.190	0.943	0.192	0.936	0.171	0.950
		(70,70)	0.163	0.951	0.162	0.945	0.163	0.942	0.148	0.941
(2,3.8100)	0.30	(10,10)	0.460	0.947	0.459	0.921	0.483	0.897	0.365	0.966
		(30,30)	0.286	0.949	0.287	0.940	0.292	0.929	0.252	0.945
		(50,50)	0.226	0.949	0.226	0.944	0.229	0.938	0.205	0.940
		(70,70)	0.192	0.949	0.193	0.946	0.194	0.942	0.177	0.938
(2,2.7520)	0.4	(10,10)	0.477	0.949	0.487	0.925	0.513	0.903	0.400	0.965
		(30,30)	0.302	0.949	0.305	0.940	0.310	0.931	0.273	0.945
		(50,50)	0.239	0.950	0.240	0.945	0.243	0.940	0.220	0.935
		(70,70)	0.203	0.950	0.205	0.947	0.206	0.944	0.190	0.935
(2,2)	0.5	(10,10)	0.470	0.948	0.484	0.925	0.508	0.905	0.410	0.962
		(30,30)	0.297	0.950	0.301	0.942	0.306	0.935	0.273	0.940
		(50,50)	0.234	0.948	0.236	0.944	0.239	0.939	0.219	0.937
		(70,70)	0.200	0.950	0.201	0.946	0.203	0.943	0.188	0.937
(2,1.4267)	0.6	(10,10)	0.441	0.948	0.453	0.928	0.473	0.910	0.390	0.956
		(30,30)	0.274	0.948	0.277	0.944	0.281	0.937	0.254	0.939
		(50,50)	0.215	0.948	0.217	0.944	0.219	0.940	0.202	0.934
		(70,70)	0.183	0.950	0.184	0.946	0.185	0.943	0.173	0.932
(4,1.9388)	0.7	(10,10)	0.387	0.948	0.391	0.931	0.406	0.913	0.313	0.977
		(30,30)	0.234	0.948	0.234	0.944	0.237	0.939	0.208	0.954
		(50,50)	0.182	0.949	0.182	0.946	0.184	0.943	0.167	0.944
		(70,70)	0.155	0.948	0.155	0.946	0.155	0.944	0.143	0.938
(4,1.1847)	0.8	(10,10)	0.304	0.947	0.299	0.931	0.308	0.916	0.240	0.967
		(30,30)	0.175	0.949	0.174	0.947	0.175	0.941	0.156	0.948
		(50,50)	0.135	0.948	0.135	0.945	0.135	0.941	0.124	0.938
		(70,70)	0.114	0.949	0.114	0.947	0.114	0.943	0.106	0.937
(8,1.0958)	0.9	(10,10)	0.181	0.945	0.171	0.937	0.173	0.920	0.137	0.971
		(30,30)	0.098	0.948	0.096	0.946	0.096	0.940	0.086	0.950
		(50,50)	0.075	0.950	0.074	0.948	0.074	0.944	0.068	0.945
		(70,70)	0.063	0.950	0.062	0.949	0.062	0.945	0.058	0.942
(16,1.0570)	0.95	(10,10)	0.099	0.947	0.091	0.938	0.092	0.917	0.067	0.974
		(30,30)	0.052	0.949	0.050	0.947	0.050	0.940	0.043	0.956
		(50,50)	0.039	0.949	0.039	0.948	0.039	0.944	0.034	0.951
		(70,70)	0.033	0.950	0.032	0.950	0.032	0.945	0.030	0.943

5. Real example

To demonstration the application of the various approaches developed in this paper, we present the lifetime data sets reported by Bader and Priest (1982). The data sets represent the failure stresses of single carbon fibers at four levels of gauge lengths of 1, 10, 20 and 50 mm. Here, we consider the single fibers of 20 mm (X) and 50 mm (Y) in gauge lengths. These data sets are reported as follows:

X (20 mm): 1.312, 1.314, 1.479, 1.552, 1.700, 1.803, 1.861, 1.865, 1.944, 1.958, 1.966, 1.997, 2.006, 2.021, 2.027, 2.055, 2.063, 2.098, 2.140, 2.179, 2.224, 2.240, 2.253, 2.270, 2.272, 2.274, 2.301, 2.301, 2.359, 2.382, 2.382, 2.426, 2.434, 2.435, 2.478, 2.490, 2.511, 2.514, 2.535, 2.554, 2.566, 2.570, 2.586, 2.629, 2.633, 2.642, 2.648, 2.684, 2.697, 2.726, 2.770, 2.773, 2.800, 2.809, 2.818, 2.821, 2.848, 2.880, 2.954, 3.012, 3.067, 3.084, 3.090, 3.096, 3.128, 3.233, 3.433, 3.585, 3.585.

Y (50 mm): 1.339, 1.434, 1.549, 1.574, 1.589, 1.613, 1.746, 1.753, 1.764, 1.807, 1.812, 1.840, 1.852, 1.852, 1.862, 1.864, 1.931, 1.952, 1.974, 2.019, 2.051, 2.055, 2.058, 2.088, 2.125, 2.162, 2.171, 2.172, 2.18, 2.194, 2.211, 2.270, 2.272, 2.280, 2.299, 2.308, 2.335, 2.349, 2.356, 2.386, 2.390, 2.410, 2.430, 2.431, 2.458, 2.471, 2.497, 2.514, 2.558, 2.577, 2.593, 2.601, 2.604, 2.620, 2.633, 2.670, 2.682, 2.699, 2.705, 2.735, 2.785, 3.020, 3.042, 3.116, 3.174.

The validity of the ET distribution for the considered data sets has been checked by using the Kolmogorov-Smirnov (K-S) test. Furthermore, in this example the ET distribution is compared to some well-known lifetime distributions namely Weibull (Wei), Gamma (Gam), generalized exponential (GE), inverse Weibull (IW), power Lindley (PW), generalized Lindley (GL), Burr type X (BX), exponentiated half Logistic (EHL), Gompertz (Gom), Chen and generalized half normal (GHN). The probability density function of these distributions are listed below:

$$\begin{aligned}
\text{Wei} &: \; f(x;\delta,\eta) = \delta\eta x^{\delta-1}\exp\left[-\eta x^{\delta}\right], \\
\text{Gam} &: \; f(x;\delta,\eta) = \frac{\eta^{\delta}}{\Gamma(\delta)} x^{\delta-1}\exp(-\eta x), \\
\text{GE} &: \; f(x;\delta,\eta) = \delta\eta\exp(-\eta x)\left[1-\exp(-\eta x)\right]^{\delta-1}, \\
\text{IW} &: \; f(x;\delta,\eta) = \delta\eta x^{-(\delta+1)}\exp\left(-\eta x^{-\delta}\right), \\
\text{PL} &: \; f(x;\delta,\eta) = \frac{\delta\eta^{2}}{\eta+1}\left(1+x^{\delta}\right)x^{\delta-1}\exp\left(-\eta x^{\delta}\right), \\
\text{GL} &: \; f(x;\delta,\eta) = \frac{\delta\eta^{2}}{1+\eta}(1+x)\left[1-\frac{1+\eta+\eta x}{1+\eta}\exp(-\eta x)\right]^{\delta-1}\exp(-\eta x), \\
\text{BX} &: \; f(x;\delta,\eta) = 2\delta\eta^{2}x\exp\left(-\eta^{2}x^{2}\right)\left[1-\exp\left(-\eta^{2}x^{2}\right)\right]^{\delta-1}, \\
\text{EHL} &: \; f(x;\delta,\eta) = 2\delta\eta\frac{\exp(-\eta x)}{1-\exp(-2\eta x)}\left[\frac{1-\exp(-\eta x)}{1+\exp(-\eta x)}\right]^{\delta}, \\
\text{Gom} &: \; f(x;\delta,\eta) = \delta e^{\eta x}\exp\left[-\frac{\delta}{\eta}\left(e^{\eta x}-1\right)\right], \\
\text{Chen} &: \; f(x;\delta,\eta) = \delta\eta x^{\eta-1}\exp\left[\delta\left(1-e^{x^{\eta}}\right)+x^{\eta}\right], \\
\text{GHN} &: \; f(x;\delta,\eta) = \sqrt{\frac{2}{\pi}}\left(\frac{\delta}{x}\right)\left(\frac{x}{\eta}\right)^{\delta}\exp\left[-\frac{1}{2}\left(\frac{x}{\eta}\right)^{2\delta}\right],
\end{aligned}$$

where δ, η and x are positive. For the purpose of comparison of the above distributions with the ET distribution, we use the Kolmogorov-Smirnov (K-S) distance and corresponding P-value as well as the Akaike information criterion (AIC). The MLEs of unknown parameters,

the K-S distances and the corresponding P-values as well as the AIC values are reported in Table 10 for the X and Y data sets. Based on Table 10, it is observed that the ET distribution provides the better fit compared to the other competitive models. So, it can be selected as the best model. The validity of the ET distribution is also supported by the P-P plot as shown in Figure 1. The likelihood ratio test (LRT) is employed to compare the equivalence between stress and strength scale parameters. It is observed that the LRT statistic and corresponding P-value are 1.5562 and 0.2122, respectively. So, there is no sufficient statistical evidence to reject the null hypothesis $\eta_1 = \eta_2$.
Now, we obtain the reliability of MSS model through classical and Bayesian methods for $(s, k) = (1, 4)$ and $(s, k) = (3, 5)$. First, from the above data sets, the ML estimates of δ_1, δ_2 and η are computed as $\hat{\delta_1} = 5.6323$, $\hat{\delta_2} = 3.7021$ and $\hat{\eta} = 0.6440$, respectively. To analyze the data sets from the Bayesian view, we used both informative and non-informative priors. The hyperparameters of $(a_i, b_i) = (0.0001, 0.0001)$, $i = 1, 2, 3$ are selected for the non-informative prior. Also, the hyperparameters of $(a_1, b_1) = (5.6, 1)$, $(a_2, b_2) = (3.7, 1)$ and $(a_3, b_3) = (0.6, 1)$ are selected for the informative prior based on the MLEs of unknown parameters (Kızılaslan 2017). Tables 11 and 12 give the point and interval estimates of $R_{s,k}$, respectively. Based on these results, the ML estimates of $R_{s,k}$ are somewhat greater than the Bayes estimates. In view of interval estimates, it is observed that, the HPD credible intervals based on the informative prior have the lowest ALs. Also, The ALs of the HPD credible intervals based on non-informative prior as well as different ACI methods are less or more the same.

Table 10: The MLEs of the unknown parameters, K-S statistic and P-value (P) for the real example

	X				Y			
Model	$(\hat{\delta},\hat{\eta})$	K-S (P)	AIC	Rank	$(\hat{\delta},\hat{\eta})$	K-S (P)	AIC	Rank
Wei	(5.504,0.005)	0.056(0.973)	103.2	3	(6.013,0.005)	0.056(0.979)	74.9	4
Gam	(23.38,9.537)	0.058(0.962)	104.1	4	(28.54,12.72)	0.073(0.860)	74.1	3
GE	(88.24,2.038)	0.095(0.531)	109.2	8	(176.2,2.540)	0.098(0.524)	80.7	9
IW	(4.128,23.27)	0.134(0.155)	127.2	12	(4.995,31.71)	0.125(0.241)	91.7	12
PL	(3.867,0.050)	0.044(0.998)	102.1	2	(4.221,0.052)	0.065(0.927)	74.0	2
GL	(64.09,2.312)	0.092(0.576)	112.3	9	(128.0,2.831)	0.097(0.545)	80.2	7
BX	(8.790,0.667)	0.066(0.907)	105.3	5	(12.13,0.771)	0.081(0.752)	75.5	5
EHL	(45.64,2.048)	0.093(0.558)	112.9	10	(89.74,2.546)	0.098(0.533)	80.6	8
Gom	(0.008,2.043)	0.085(0.673)	107.2	6	(0.006,2.474)	0.070(0.889)	81.8	10
Chen	(0.022,1.356)	0.096(0.523)	114.1	11	(0.022,1.493)	0.091(0.621)	85.2	11
GHN	(4.129,2.753)	0.080(0.734)	107.7	7	(4.537,2.501)	0.063(0.945)	78.9	6
ET	(4.664,0.615)	0.042(0.999)	102.0	1	(5.774,0.697)	0.064(0.938)	73.3	1

Table 11: Estimates of $R_{s,k}$ for real example

(s,k)	MLE	Bayes (Prior 1)	Bayes (Prior 2)
(1,4)	0.8589	0.8552	0.8542
(3,5)	0.6227	0.6152	0.6138

Table 12: Interval estimates of $R_{s,k}$ along with ALs for the real example

(s,k)	Method	Interval	AL
(1,4)	LOGT	(0.8104,0.8965)	0.0861
	AST	(0.8134,0.8989)	0.0855
	NT	(0.8160,0.9017)	0.0857
	HPD (Prior 1)	(0.8149,0.8992)	0.0843
	HPD (Prior 2)	(0.8186,0.8913)	0.0727
(3,5)	LOGT	(0.5228,0.7130)	0.1902
	AST	(0.5247,0.7158)	0.1911
	NT	(0.5265,0.7188)	0.1923
	HPD (Prior 1)	(0.5220,0.7056)	0.1836
	HPD (Prior 2)	(0.5314,0.6914)	0.1600

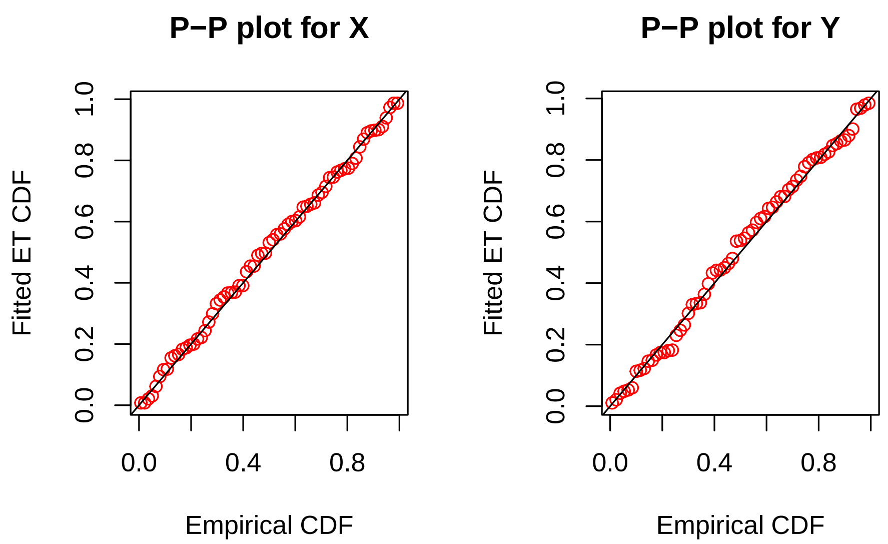

Figure 1: P-P plot for the fitted ET distribution under real example

6. Concluding remarks

In this article, we considered the reliability of multicomponent stress-strength model under the assumption that the stress and strength random variables are taken from ET distributions. The reliability of the model was obtained using the MLE and the approximate Bayes estimation, in the scenario where the common scale parameter was unknown. Also, it was derived using the MLE, UMVUE and exact Bayes estimation, in the another scenario where the common scale parameter was known. The asymptotic and HPD intervals were constructed in the both scenarios. Furthermore, two other asymptotic confidence intervals were computed based on the Logit and Arcsin transformations.

Based on simulations, for the both scenarios, the MSEs for the estimates of $R_{s,k}$ and the ALs of the intervals showed decrease with increases of sample size. Moreover, the the values of MSEs and ALs were low when $R_{s,k}$ tends to the extreme value and they were high when $R_{s,k}$

tends to moderate value. The ordering of performance in terms of MSE was Bayes estimator, UMVUE and MLE for extreme values of $R_{s,k}$ and it was Bayes estimator, MLE and UMVUE for moderate values of $R_{s,k}$. In terms of the AL, always the Bayes estimator had the best performance among the estimators. Furthermore, comparing the different ACI methods in terms of the AL indicated that the Arcsin and Logit transformations worked better respectively for the extreme and moderate values of $R_{s,k}$.

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