Cubic Transmuted Rayleigh Distribution: Theory and Application

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Abstract

In this paper, the Rayleigh distribution is generalized using the cubic transmuted (CT) family studied by Rahman, Al-Zahrani, Shahbaz, and Shahbaz (2019) to propose a cubic transmuted Rayleigh distribution. An overall description is presented here for the distributional properties, parameter estimation, inference procedure, reliability behavior, and distribution of different order statistics. A real-life data set is used to demonstrate the applicability of the proposed distribution for modeling data.

Keywords: cubic transmutation, maximum likelihood estimation, order statistics, Rayleigh distribution, reliability analysis.

1. Introduction

Modeling every phenomenon with the well-known standard probability distributions is not always straightforward. Again in statistical analysis, the feature of the procedures mostly depends on the assumed probability model. Generalizing the probability distributions is a popular and ever-existing process in statistics to capture extra variation in the data. The generalization by induction of the shape parameter(s) started at the end of the last century. The main target of the article is to generalize the Rayleigh distribution to increase the flexibility in the analysis of real-life data by induction of a shape parameter. The Rayleigh distribution is a continuous probability distribution for nonnegative-valued random variables, named after the English Lord Rayleigh (1842-1919). The distribution function of the distribution is

$$G(x; \sigma) = 1 - e^{-\frac{x^2}{2\sigma^2}}, \ x \in \mathcal{R}^{0+}, \tag{1}$$

where $\sigma \in \mathcal{R}^+$ is the scale parameter. The Weibull distribution with shape parameter 2 is nothing but the Rayleigh distribution. When the scale parameter (σ) is 1, the Rayleigh is the same as the chi-square distribution of 2 degrees of freedom. Shaw and Buckley (2009) introduced the transmuted family of distributions, which has the cumulative distribution function (CDF) as

$$F(x; \lambda) = (1 + \lambda)G(x) - \lambda G^2(x), \ x \in \mathcal{R}, \tag{2}$$

where $\lambda \in [-1, 1]$. Several standard probability models are developed and added to the literature using (2), including transmuted extreme value by Aryal and Tsokos (2009); transmuted

Weibull by Aryal and Tsokos (2011); and transmuted Pareto by Merovci and Puka (2014). Merovci (2013) introduced the transmuted Rayleigh distribution using (1) in (2), which can take a simple quadratic form as

$$F(x; \sigma, \lambda) = \left[1 - e^{-\frac{x^2}{2\sigma^2}}\right]\left[1 + \lambda e^{-\frac{x^2}{2\sigma^2}}\right], \ x \in \mathcal{R}^{0+},$$

where $\lambda \in [-1, 1]$ is the shape parameter. Rahman *et al.* (2019) introduce the cumulative distribution function of a new family of cubic transmuted distributions, which can take the simple cubic form as

$$F(x; \lambda) = (1 - \lambda)G(x) + 3\lambda G^2(x) - 2\lambda G^3(x), \ x \in \mathcal{R}, \tag{3}$$

where $\lambda \in [-1, 1]$. Observe that at $\lambda = 0$, one has the distribution of the base random variable. Akter, Khan, Rana, and Rahman (2021) develop the cubic transmuted Burr-XII distribution using (3). Researchers in this area can develop several other cubic transmuted distributions by using this family of distributions.

The layout plan of this article follows: The cubic transmuted Rayleigh distribution is introduced and described in Section 2. Section 3 describes the moments and associated results, as well as the generating functions and quantile function. In Section 4, the random number generation and parameter estimation techniques of the distribution are discussed. Section 5 provides the reliability behavior of the distribution along with the distributions of different order statistics in Section 6. Section 7 provides a real-life example for evaluating the proposed distribution's applicability. Section 8 concludes with some closing remarks.

2. Cubic transmuted Rayleigh distribution

The cumulative distribution function of the cubic transmuted Rayleigh distribution is obtained by using (1) in (3), which can be written as

$$F(x; \sigma, \lambda) = 1 - e^{-\frac{3x^2}{2\sigma^2}}\left[(1 - \lambda)e^{\frac{x^2}{\sigma^2}} + 3\lambda e^{\frac{x^2}{2\sigma^2}} - 2\lambda\right], \ x \in \mathcal{R}^{0+}, \tag{4}$$

where $\sigma \in \mathcal{R}^+$ and $\lambda \in [-1, 1]$ are the scale and shape parameters of the distribution respectively. Differentiating (4) with respect to x provides the probability density function (PDF), which is defined as follows.

Definition. A continuous random variable X is said to have a cubic transmuted Rayleigh distribution if its density function is defined by

$$f(x; \sigma, \lambda) = \frac{x}{\sigma^2}e^{-\frac{3x^2}{2\sigma^2}}\left[(1 - \lambda)e^{\frac{x^2}{\sigma^2}} + 6\lambda e^{\frac{x^2}{2\sigma^2}} - 6\lambda\right], \ x \in \mathcal{R}^{0+}, \tag{5}$$

where $\sigma \in \mathcal{R}^+$ and $\lambda \in [-1, 1]$.

Special cases

The cubic transmuted Rayleigh distribution has the following specific situations:

(i) For a value of $\lambda = 0$, the CDF of the cubic transmuted Rayleigh distribution in (4) reduces to the CDF of the Rayleigh distribution in (1).

(ii) The cubic transmuted Rayleigh distribution given in (5) for $\lambda = 0$ and $\sigma = 1$ is equivalent to the chi distribution with 2 degrees of freedom.
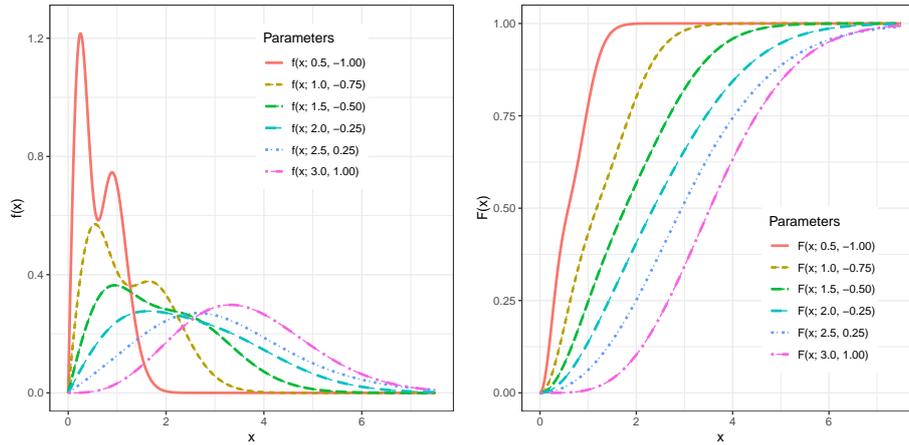
Figure 1: The PDF (left) and CDF (right) are obtained for the CT-Rayleigh distribution

(iii) The half-normal distribution is the univariate specific case of the cubic transmuted Rayleigh distribution given in (5) for $\lambda = 0$.

For selected values of the model parameters λ and σ, Figure 1 shows some potential shapes for the density function (left) and distribution function (right) of the proposed cubic transmuted Rayleigh distribution. The cubic transmuted Rayleigh distribution can capture different behavior in the data set, as shown in Figure 1.

3. Moments and related results

The moments are very much essential to know the shape characteristic of a distribution. The mean of a distribution is the first raw moment, while the variance, skewness, and kurtosis of the distribution are the second central moment, third standardized moment, and fourth standardized moment, respectively. The moments, as well as some related outcomes, are listed in the subsections below.

3.1. Moments

The following theorem describes the rth raw moment of the proposed cubic transmuted Rayleigh distribution.

Theorem 1. *Let the continuous random variable X follow a cubic transmuted Rayleigh distribution, then the rth raw moment, μ_r', is*

$$\mu_r' = \frac{r\sigma^r}{2}\Gamma\left(\frac{r}{2}\right)\left[3\lambda + (1-\lambda)2^{\frac{r}{2}} - 2\lambda\left(\frac{2}{3}\right)^{\frac{r}{2}}\right]. \tag{6}$$

Proof. The rth raw moment is defined and further proceed as

$$
\begin{aligned}
\mu_r' &= \int_0^\infty x^r f(x;\sigma,\lambda)\ \mathrm{d}x \\
&= \int_0^\infty x^r\left[\frac{x}{\sigma^2}e^{-\frac{3x^2}{2\sigma^2}}\left\{(1-\lambda)e^{\frac{x^2}{\sigma^2}} + 6\lambda e^{\frac{x^2}{2\sigma^2}} - 6\lambda\right\}\right]\ \mathrm{d}x \\
&= \frac{1-\lambda}{\sigma^2}\int_0^\infty x^{r+1}e^{-\frac{x^2}{2\sigma^2}}\ \mathrm{d}x + \frac{6\lambda}{\sigma^2}\int_0^\infty x^{r+1}e^{-\frac{x^2}{\sigma^2}}\ \mathrm{d}x - \frac{6\lambda}{\sigma^2}\int_0^\infty x^{r+1}e^{-\frac{3x^2}{2\sigma^2}}\ \mathrm{d}x \\
&= \frac{1-\lambda}{\sigma^2}\left[\sigma^{r+2}2^{\frac{r}{2}}\left(\frac{r}{2}\right)\Gamma\left(\frac{r}{2}\right)\right] + \frac{6\lambda}{\sigma^2}\left[\sigma^{r+2}\left(\frac{1}{2}\right)\left(\frac{r}{2}\right)\Gamma\left(\frac{r}{2}\right)\right]
\end{aligned}
$$

$$-\frac{6\lambda}{\sigma^2}\left[\sigma^{r+2}\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^{\frac{r}{2}}\left(\frac{r}{2}\right)\Gamma\left(\frac{r}{2}\right)\right]$$

$$=\frac{\sigma^{r+2}\left(\frac{r}{2}\right)\Gamma\left(\frac{r}{2}\right)}{\sigma^2}\left[(1-\lambda)2^{\frac{r}{2}}+\frac{6\lambda}{2}-\frac{6\lambda}{3}\left(\frac{2}{3}\right)^{\frac{r}{2}}\right]$$

$$=\frac{r\sigma^r}{2}\Gamma\left(\frac{r}{2}\right)\left[3\lambda+(1-\lambda)2^{\frac{r}{2}}-2\lambda\left(\frac{2}{3}\right)^{\frac{r}{2}}\right].$$

Hence, proved the theorem. □

The first raw moment is obtained by setting $r=1$ in (6) and given as

$$\mu_1'=\frac{\sigma}{2}\sqrt{\pi}\left[3\lambda+(1-\lambda)\sqrt{2}-2\lambda\sqrt{\frac{2}{3}}\right],$$

which is the mean of the distribution. The second raw moment can also be computed as

$$\mu_2'=\sigma^2\left(2-\frac{\lambda}{3}\right).$$

The variance of the distribution, also known as the second central moment, is obtained as

$$\mu_2=\mu_2'-\left[\mu_1'\right]^2=\sigma^2\left[\left(2-\frac{\lambda}{3}\right)-\frac{\pi}{4}\left\{3\lambda+(1-\lambda)\sqrt{2}-2\lambda\sqrt{\frac{2}{3}}\right\}^2\right].$$

Table 1: The mean and variance (in parentheses) of the CT-Rayleigh distribution

	$\lambda=-1$	$\lambda=-0.5$	$\lambda=0$	$\lambda=0.5$	$\lambda=1$
$\sigma=0.5$	0.648	0.637	0.627	0.616	0.606
	(0.164)	(0.136)	(0.107)	(0.079)	(0.050)
$\sigma=1$	1.295	1.274	1.253	1.232	1.211
	(0.656)	(0.543)	(0.429)	(0.315)	(0.199)
$\sigma=2$	2.590	2.548	2.507	2.465	2.423
	(2.624)	(2.172)	(1.717)	(1.258)	(0.796)
$\sigma=3$	3.885	3.823	3.760	3.697	3.634
	(5.903)	(4.887)	(3.863)	(2.831)	(1.791)
$\sigma=4$	5.181	5.097	5.013	4.930	4.846
	(10.495)	(8.688)	(6.867)	(5.033)	(3.184)

The mean and variance chart for the proposed cubic transmuted Rayleigh distribution is provided in Table 1 and plotted in Figure 2. When the value of σ increases, the mean and variance increase, and when the value of λ increases, the mean and variance decreases.

The normalized rth central moment of the random variable X is

$$\frac{\mu_r}{\sigma^r}=\frac{\mathrm{E}\left[(X-\mu)^r\right]}{\sigma^r}.\tag{7}$$

The first and second standardized moments are 0 and 1, respectively, as seen in (7). Also, the skewness (γ_1) and kurtosis (γ_2) of the random variable are the third and fourth standardized moments, respectively. The skewness and kurtosis of the proposed distribution are determined and expressed as follow

$$\gamma_1=\frac{\mu_3}{(\mu_2)^{\frac{3}{2}}}=\frac{2\sqrt{\pi}}{(-\pi\xi^2-\phi)^{\frac{3}{2}}}\left[\pi\xi^3+\frac{3}{2}\phi\xi-9\delta\lambda+162\sqrt{2}\right],$$
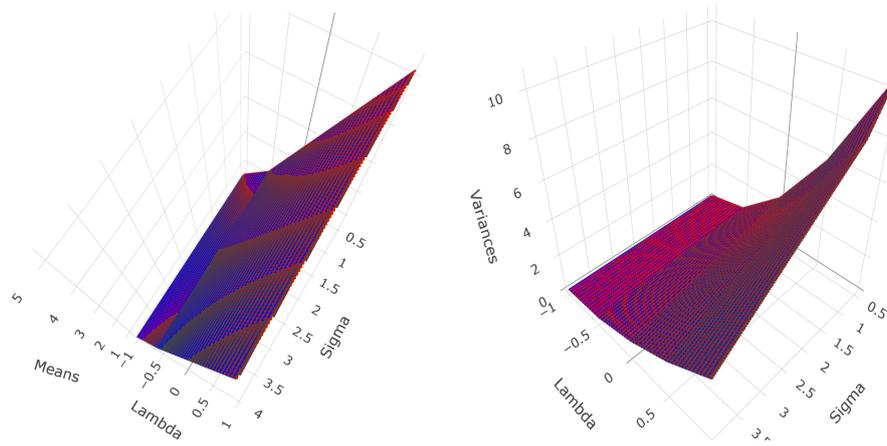
Figure 2: The mean (left) and variance (right) are plotted for the CT-Rayleigh distribution

and

$$\gamma_2 = \frac{\mu_4}{\mu_2^2} = \frac{3}{(\pi\xi^2 + \phi)^2} \left[-\pi^2\xi^4 - 2\pi\phi\xi^2 + 24\pi \left\{ \delta\lambda - 18\sqrt{2} \right\} \xi + 96(36 - 17\lambda) \right],$$

where $\xi = \left(9 - 3\sqrt{2} - 2\sqrt{6} \right) \lambda + 3\sqrt{2}$, $\phi = 12(\lambda - 6)$ and $\delta = 18\sqrt{2} + 4\sqrt{6} - 27$.

Table 2: The skewness and kurtosis of the CT-Rayleigh distribution

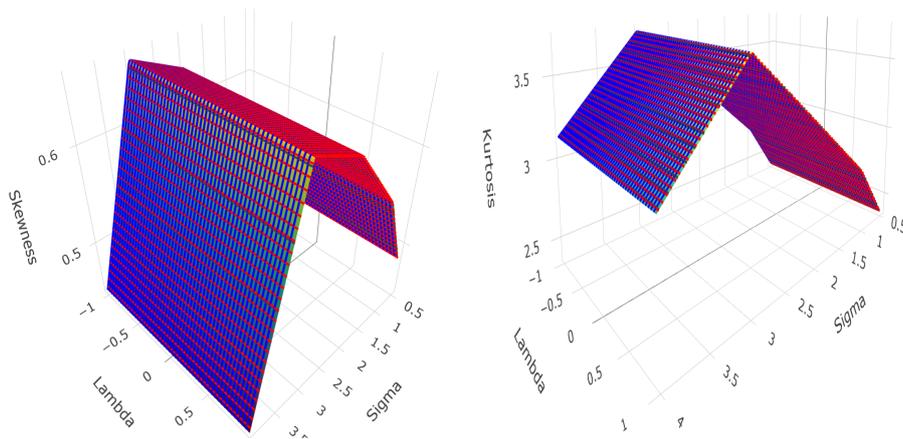|	any σ				
	$\lambda =$-1	$\lambda =$-0.5	$\lambda =$0	$\lambda =$0.5	$\lambda =$1
Skewness	0.486	0.562	0.631	0.656	0.446
Kurtosis	2.384	2.77	3.245	3.716	3.136

Figure 3: The skewness (left) and kurtosis (right) are plotted for the CT-Rayleigh distribution

Table 2 shows the skewness and kurtosis chart for the proposed cubic transmuted Rayleigh distribution. It's worth noting that skewness and kurtosis are unaffected by the scale parameter σ. Figure 3 illustrates the skewness (left) and kurtosis (right) of the proposed distribution for different combinations of the model parameters, confirming that it is positively skewed and leptokurtic.

3.2. Moment generating function

The moment generating function, $M_X(t)$, of a real-valued random variable X is an alternative specification of its probability distribution. It can be used to obtain the moments of a distribution. Specifically, the rth moment about 0 is the rth derivative of the moment generating function, evaluated at 0. The moment generating function of the proposed cubic transmuted Rayleigh distribution is described by the following theorem.

Theorem 2. *Let the continuous random variable X follow a cubic transmuted Rayleigh distribution, then the moment generating function, $M_X(t)$, is*

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} \frac{r\sigma^r}{2} \Gamma\left(\frac{r}{2}\right) \left[3\lambda + (1-\lambda)2^{\frac{r}{2}} - 2\lambda\left(\frac{2}{3}\right)^{\frac{r}{2}}\right], \tag{8}$$

where $t \in \mathcal{R}$.

Proof. The moment generating function is defined as

$$M_X(t) = E\left[e^{tX}\right] = \int_0^{\infty} e^{tx} f(x)\mathrm{d}x,$$

where $f(x)$ is given in (5). Using the series representation of e^{tx} given by Gradshteyn and Ryzhik (2014), above equation can be further expressed as

$$M_x(t) = \int_0^{\infty} \sum_{r=0}^{\infty} \frac{t^r}{r!} x^r f(x)\mathrm{d}x = \sum_{r=0}^{\infty} \frac{t^r}{r!} E(X^r). \tag{9}$$

Using $E(X^r)$ from (6) in (9), we have (8). □

3.3. Characteristic generating function

The characteristic function of a real-valued distribution is always existed, unlike the moment generating function. The following theorem describes the characteristic function of the proposed cubic transmuted Rayleigh distribution.

Theorem 3. *Let the continuous random variable X follow a cubic transmuted Rayleigh distribution, then the characteristic generating function, $\phi_X(t)$, is*

$$\phi_X(t) = \sum_{r=0}^{\infty} \frac{(\mathrm{i}t)^r}{r!} \frac{r\sigma^r}{2} \Gamma\left(\frac{r}{2}\right) \left[3\lambda + (1-\lambda)2^{\frac{r}{2}} - 2\lambda\left(\frac{2}{3}\right)^{\frac{r}{2}}\right],$$

where $\mathrm{i} = \sqrt{-1}$ is the imaginary unit and $t \in \mathcal{R}$.

Proof. The proof is simple. □

3.4. Quantile function and median

The quantile function of a distribution specifies the value of a random variable. It is also called the inverse cumulative distribution function. The proposed cubic transmuted Rayleigh distribution's qth quantile x_q is obtained by using (4) as follow

$$1 - e^{-\frac{3x_q^2}{2\sigma^2}} \left[(1-\lambda)e^{\frac{x_q^2}{\sigma^2}} + 3\lambda e^{\frac{x_q^2}{2\sigma^2}} - 2\lambda\right] = q,$$

which can be further presented as

$$x_q = \sigma \left[2 \log \left\{ \frac{\sqrt[3]{\delta_1}}{3\sqrt[3]{2}(q-1)} - \frac{\sqrt[3]{2}\xi}{3(q-1)\sqrt[3]{\delta_1}} - \frac{1-\lambda}{3(q-1)} \right\} \right]^{\frac{1}{2}},$$

(10)

where

$$\begin{aligned}
\xi &= -\lambda^2 - 7\lambda + 9\lambda q - 1, \\
\delta_1 &= \delta_2 + \sqrt{(\delta_2 + \delta_3)^2 + 4\xi^3} + \delta_3, \\
\delta_2 &= 2\lambda^3 + 21\lambda^2 + 33\lambda, \text{ and} \\
\delta_3 &= 54\lambda q^2 - 27\lambda^2 q - 81\lambda q - 2.
\end{aligned} \right\}$$

(11)

By setting $q = \frac{1}{4}$, $\frac{1}{2}$ and $\frac{3}{4}$ in (10), respectively, will give the first quartile, median (second quartile), and third quartile.

4. Random number generation and parameter estimation

An inversion method is applied to generate random numbers from the proposed cubic transmuted Rayleigh distribution with parameters σ and λ. According to the inverse transform sampling method, this can proceed as

$$1 - e^{-\frac{3x^2}{2\sigma^2}} \left[(1-\lambda)e^{\frac{x^2}{\sigma^2}} + 3\lambda e^{\frac{x^2}{2\sigma^2}} - 2\lambda \right] = u,$$

where $u \sim U(0,1)$ and which can be further expressed as

$$x = \sigma \left[2 \log \left\{ \frac{\sqrt[3]{\delta_1}}{3\sqrt[3]{2}(u-1)} - \frac{\sqrt[3]{2}\xi}{3(u-1)\sqrt[3]{\delta_1}} - \frac{1-\lambda}{3(u-1)} \right\} \right]^{\frac{1}{2}},$$

(12)

where ξ, δ_1, δ_2 and δ_3 are obtained from (11) by setting u in place of q. One can use (12) to generate random numbers from the cubic transmuted Rayleigh distribution when the model parameters σ and λ are known.

Let $x_1, x_2, \cdots, x_n$ be a random sample of size n drawn from a cubic transmuted Rayleigh distribution. The likelihood function of the distribution is

$$\begin{aligned}
\mathcal{L}(x; \sigma, \lambda) &= \prod_{i=1}^{n} f(x; \sigma, \lambda) \\
&= \frac{\prod_{i=1}^{n} x_i}{\sigma^{2n}} \cdot e^{-\sum_{i=1}^{n} \frac{3x^2}{2\sigma^2}} \cdot \prod_{i=1}^{n} \left[(1-\lambda)e^{\frac{x^2}{\sigma^2}} + 6\lambda e^{\frac{x^2}{2\sigma^2}} - 6\lambda \right],
\end{aligned}$$

with respective sample log-likelihood function as

$$\ell(x; \sigma, \lambda) = \sum_{i=1}^{n} \log(x_i) - 2n \log(\sigma) - \sum_{i=1}^{n} \frac{3x_i^2}{2\sigma^2} + \sum_{i=1}^{n} \log \left[(1-\lambda)e^{\frac{x_i^2}{\sigma^2}} + 6\lambda e^{\frac{x_i^2}{2\sigma^2}} - 6\lambda \right]. \quad (13)$$

The estimators of the model parameters σ, and λ are obtained by maximizing (13). The derivatives of (13) with respect to the unknown parameters are

$$\frac{\delta\ell}{\delta\sigma} = \sum_{i=1}^{n} \frac{3x_i^2}{\sigma^3} - \frac{2n}{\sigma} - \sum_{i=1}^{n} \frac{\frac{2(1-\lambda)}{\sigma^3}x_i^2 e^{\frac{x_i^2}{\sigma^2}} + \frac{6\lambda}{\sigma^3}x_i^2 e^{\frac{x_i^2}{2\sigma^2}}}{(1-\lambda)e^{\frac{x_i^2}{\sigma^2}} + 6\lambda e^{\frac{x_i^2}{2\sigma^2}} - 6\lambda}, \quad (14)$$

and

$$\frac{\delta\ell}{\delta\lambda} = \sum_{i=1}^{n} \frac{6e^{\frac{x_i^2}{2\sigma^2}} - e^{\frac{x_i^2}{\sigma^2}} - 6}{(1-\lambda)e^{\frac{x_i^2}{\sigma^2}} + 6\lambda e^{\frac{x_i^2}{2\sigma^2}} - 6\lambda}. \tag{15}$$

The maximum likelihood estimator $\hat{\Theta} = \left(\hat{\sigma}, \hat{\lambda}\right)'$ of $\Theta = (\sigma, \lambda)'$ is obtained by solving the resulting nonlinear system of equations, setting $\frac{\delta\ell}{\delta\sigma} = 0$, and $\frac{\delta\ell}{\delta\lambda} = 0$. It is usually more convenient to use nonlinear optimization algorithms such as quasi-Newton algorithm to numerically maximize the log-likelihood function given in (13). Applying the usual large sample approximation, the maximum likelihood estimator of $\hat{\Theta}$ can be treated as being approximately bivariate normal with mean Θ and variance-covariance matrix equal to the inverse of the expected information matrix. That is,

$$\sqrt{n}\left(\hat{\Theta} - \Theta\right) \sim N_2\left[0, I^{-1}(\Theta)\right],$$

where $I^{-1}(\Theta)$ is the limiting variance-covariance matrix of $\hat{\Theta}$. The elements of the 2×2 matrix $I(\Theta)$ can be estimated by

$$I_{ij}\left(\hat{\Theta}\right) = -\ell_{\Theta_i\Theta_j|\Theta=\hat{\Theta}}, \ i, j \in \{1, 2\}.$$

The second partial derivatives of the log-likelihood function (13) are obtained from the equations (14) and (15), and further expressed as

$$
\begin{aligned}
I_{11} &= -\frac{\delta^2\ell}{\delta\sigma^2} \\
&= \sum_{i=1}^{n} \frac{9x_i^2}{\sigma^4} - \frac{2n}{\sigma^2} + \sum_{i=1}^{n}\left[\frac{\left\{-\frac{2(1-\lambda)}{\sigma^3}x_i^2 e^{\frac{x_i^2}{\sigma^2}} - \frac{6\lambda}{\sigma^3}x_i^2 e^{\frac{x_i^2}{2\sigma^2}}\right\}^2}{\left\{(1-\lambda)e^{\frac{x_i^2}{\sigma^2}} + 6\lambda e^{\frac{x_i^2}{2\sigma^2}} - 6\lambda\right\}^2}\right. \\
&\quad \left. -\frac{\left\{\frac{4(1-\lambda)}{\sigma^6}x_i^4 e^{\frac{x_i^2}{\sigma^2}} + \frac{6\lambda}{\sigma^6}x_i^4 e^{\frac{x_i^2}{2\sigma^2}} + \frac{6(1-\lambda)}{\sigma^4}x_i^2 e^{\frac{x_i^2}{\sigma^2}} + \frac{18\lambda}{\sigma^4}x_i^2 e^{\frac{x_i^2}{2\sigma^2}}\right\}}{\left\{(1-\lambda)e^{\frac{x_i^2}{\sigma^2}} + 6\lambda e^{\frac{x_i^2}{2\sigma^2}} - 6\lambda\right\}}\right],
\end{aligned}
$$

$$
\begin{aligned}
I_{12} &= -\frac{\delta^2\ell}{\delta\sigma \cdot \delta\lambda} \\
&= \sum_{i=1}^{n}\left[\frac{\frac{6}{\sigma^3}x_i^2 e^{\frac{x_i^2}{2\sigma^2}} - \frac{2}{\sigma^3}x_i^2 e^{\frac{x_i^2}{\sigma^2}}}{(1-\lambda)e^{\frac{x_i^2}{\sigma^2}} + 6\lambda e^{\frac{x_i^2}{2\sigma^2}} - 6\lambda} - \frac{\left(6e^{\frac{x_i^2}{2\sigma^2}} - e^{\frac{x_i^2}{\sigma^2}} - 6\right)\left\{\frac{2(1-\lambda)}{\sigma^3}x_i^2 e^{\frac{x_i^2}{\sigma^2}} + \frac{6\lambda}{\sigma^3}x_i^2 e^{\frac{x_i^2}{2\sigma^2}}\right\}}{\left\{(1-\lambda)e^{\frac{x_i^2}{\sigma^2}} + 6\lambda e^{\frac{x_i^2}{2\sigma^2}} - 6\lambda\right\}^2}\right],
\end{aligned}
$$

and

$$I_{22} = -\frac{\delta^2\ell}{\delta\lambda^2} = \sum_{i=1}^{n} \frac{\left(6e^{\frac{x_i^2}{2\sigma^2}} - e^{\frac{x_i^2}{\sigma^2}} - 6\right)^2}{\left\{(1-\lambda)e^{\frac{x_i^2}{\sigma^2}} + 6\lambda e^{\frac{x_i^2}{2\sigma^2}} - 6\lambda\right\}^2}.$$

Approximate two sided $100(1 - \alpha)\%$ confidence intervals for the model parameters σ and λ are respectively given by

$$\hat{\sigma} \pm z_{\alpha/2}\sqrt{I_{11}^{-1}\left(\hat{\Theta}\right)} \quad \text{and} \quad \hat{\lambda} \pm z_{\alpha/2}\sqrt{I_{22}^{-1}\left(\hat{\Theta}\right)},$$

where z_α is the upper αth percentile of the standard normal distribution. R coding is used to obtain the Hessian matrix and its inverse, as well as the standard errors and asymptotic confidence intervals.

In order to check the superiority of the cubic transmuted Rayleigh distribution as compared with transmuted and base Rayleigh distributions, use the LR test statistic. Actually a LR test is used to tests the hypothesis $H_0 : \Theta = \Theta_0$ versus $H_1 : \Theta \neq \Theta_0$. Hence, the LR test statistic for testing H_0 versus H_1 is $\lambda_{LR} = -2\left[\ell\left(\hat{\Theta}_0\right) - \ell\left(\hat{\Theta}\right)\right]$, where $\hat{\Theta}_0$ and $\hat{\Theta}$ are the maximum likelihood estimates (MLEs) under H_0 and H_1 respectively. The test statistic λ_{LR} asymptotically (as $n \to \infty$) distributed as χ_k^2, where k is the difference in dimensionality of Θ and Θ_0. The LR test reject H_0 if $\lambda_{LR} > \chi_{k;\alpha}^2$, where $\chi_{k;\alpha}^2$ denotes the upper $100\alpha\%$ quantile of the χ_k^2 distribution.

5. Reliability analysis

The reliability function (RF) is used recurrently in life data analysis and reliability engineering. This function returns the probability of an item operating for a certain amount of time t without failure. This function is defined by $R(t) = 1 - F(t)$ and obtained for a cubic transmuted Rayleigh distribution as

$$R(t) = e^{-\frac{3t^2}{2\sigma^2}}\left[(1 - \lambda)e^{\frac{t^2}{\sigma^2}} + 3\lambda e^{\frac{t^2}{2\sigma^2}} - 2\lambda\right], \quad t \in \mathcal{R}^+.$$

In reliability analysis, the hazard function (HF) is a way to model data distribution. This function may be interpreted as the frequency of failure within a very narrow time frame. Let $f(t)$ be the density function of the time-to-failure of a random variable T, and let $R(t)$ be its reliability function. Then the hazard rate function, $h(t)$, is defined as

$$h(t) = \frac{f(t)}{R(t)} = \frac{f(t)}{1 - F(t)},$$

and further obtained for the cubic transmuted Rayleigh distribution as

$$h(t) = \frac{\frac{t}{\sigma^2}\left[(1 - \lambda)e^{\frac{t^2}{\sigma^2}} + 6\lambda e^{\frac{t^2}{2\sigma^2}} - 6\lambda\right]}{(1 - \lambda)e^{\frac{t^2}{\sigma^2}} + 3\lambda e^{\frac{t^2}{2\sigma^2}} - 2\lambda}, \quad t \in \mathcal{R}^+.$$

Figure 4 illustrates the reliability function (left) and hazard rate function (right) of the proposed cubic transmuted Rayleigh distribution for various combinations of the model parameters σ and λ. The increasing hazard rate, followed by an increasing then bathtub hazard rate are observed from Figure 4.

6. Order statistics

The rth order statistic of a sample, as described by David and Nagaraja (2003), is equal to its rth smallest value. The first order statistic (or smallest order statistic) is always the minimum of the sample, that is, $X_{1:n} = \min\{X_1, X_2, \cdots, X_n\}$, following a common convention, here use upper-case letters to refer to random variables. Similarly, for a sample of size n, the nth order statistic (or largest order statistic) is the maximum, that is, $X_{n:n} = \max\{X_1, X_2, \cdots, X_n\}$.
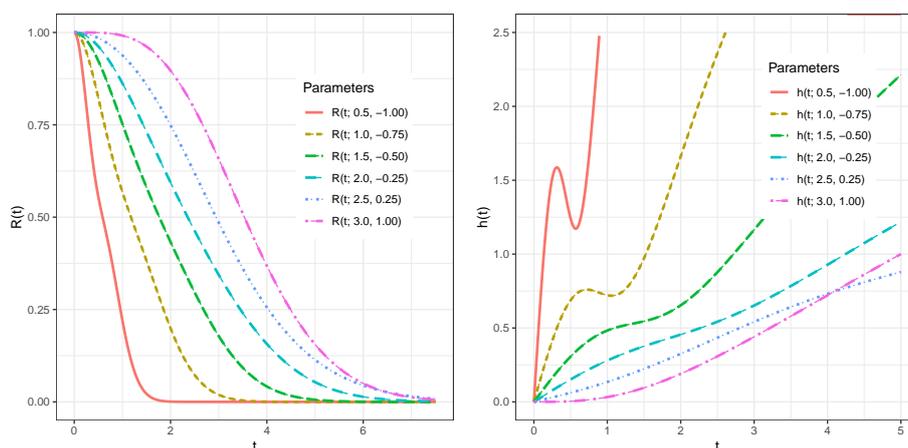
Figure 4: The RF (left) and HF (right) are obtained for the CT-Rayleigh distribution

Let $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ denotes the order statistic of a random sample $X_1, X_2, \cdots, X_n$ from a continuous population with distribution function $F_X(x)$ and density function $f_X(x)$ then the density function of $X_{r:n}$ is given by

$$f_{X_{r:n}}(x) = \frac{n!}{(r-1)!(n-r)!}[F(x)]^{r-1}[1-F(x)]^{n-r}f(x).$$ (16)

By using (16), the probability density function of the rth order statistic for the cubic transmuted Rayleigh distribution is

$$
\begin{aligned}
f_{X_{r:n}}(x) &= \frac{n!}{(r-1)!(n-r)!}\left[\frac{x}{\sigma^2}e^{-\frac{3x^2}{2\sigma^2}}\left\{(1-\lambda)e^{\frac{x^2}{\sigma^2}} + 6\lambda e^{\frac{x^2}{2\sigma^2}} - 6\lambda\right\}\right] \\
&\times \left[1 - e^{-\frac{3x^2}{2\sigma^2}}\left\{(1-\lambda)e^{\frac{x^2}{\sigma^2}} + 3\lambda e^{\frac{x^2}{2\sigma^2}} - 2\lambda\right\}\right]^{r-1} \\
&\times \left[e^{-\frac{3x^2}{2\sigma^2}}\left\{(1-\lambda)e^{\frac{x^2}{\sigma^2}} + 3\lambda e^{\frac{x^2}{2\sigma^2}} - 2\lambda\right\}\right]^{n-r},
\end{aligned}
$$ (17)

where $r = 1, 2, \cdots, n$. Using $r = 1$ in (17), obtain the density function of the lowest order statistic $X_{1:n}$, and is given as

$$
\begin{aligned}
f_{X_{1:n}}(x) &= \frac{nx}{\sigma^2}e^{-\frac{3x^2}{2\sigma^2}}\left\{(1-\lambda)e^{\frac{x^2}{\sigma^2}} + 6\lambda e^{\frac{x^2}{2\sigma^2}} - 6\lambda\right\} \\
&\times \left[e^{-\frac{3x^2}{2\sigma^2}}\left\{(1-\lambda)e^{\frac{x^2}{\sigma^2}} + 3\lambda e^{\frac{x^2}{2\sigma^2}} - 2\lambda\right\}\right]^{n-1},
\end{aligned}
$$

and for using $r = n$ in (17), the density function of the highest order statistic $X_{n:n}$, is

$$
\begin{aligned}
f_{X_{n:n}}(x) &= \frac{nx}{\sigma^2}e^{-\frac{3x^2}{2\sigma^2}}\left\{(1-\lambda)e^{\frac{x^2}{\sigma^2}} + 6\lambda e^{\frac{x^2}{2\sigma^2}} - 6\lambda\right\} \\
&\times \left[1 - e^{-\frac{3x^2}{2\sigma^2}}\left\{(1-\lambda)e^{\frac{x^2}{\sigma^2}} + 3\lambda e^{\frac{x^2}{2\sigma^2}} - 2\lambda\right\}\right]^{n-1}.
\end{aligned}
$$

Note that for $\lambda = 0$, one has the density function of the rth order statistic for the Rayleigh distribution, expressed as

$$g_{X_{r:n}}(x) = \frac{n!}{(r-1)!(n-r)!}\frac{x}{\sigma^2}e^{-\frac{x^2}{2\sigma^2}}\left[1 - e^{-\frac{x^2}{2\sigma^2}}\right]^{r-1}\left[e^{-\frac{x^2}{2\sigma^2}}\right]^{n-r}, \quad r = 1, 2, \cdots, n.$$

The kth order moment of $X_{r:n}$ for the cubic transmuted Rayleigh distribution is obtained by using the following formula

$$E(X_{r:n}^k) = \int_0^\infty x_r^k \cdot f_{X_{r:n}}(x) \cdot dx,$$

where $f_{X_{r:n}}(x)$ is given in (17).

7. Application

To investigate the applicability of the proposed cubic transmuted Rayleigh distribution considers the transmuted and base Rayleigh distributions. A real-life data set is used here, which has previously been used by Choulakian and Stephens (2001); Merovci and Puka (2014). Table 3 shows some descriptive statistics based on the data. The obtained skewness value is greater than 0, indicating that the distribution is skewed to the right (positive skewness). Again the obtained kurtosis value confirms that the distribution is platykurtic.

Table 3: Summary statistics for the Wheaton River flood data

Min.	1st Qu.	Median	Mean	3rd Qu.	Max.	Variance	Skew.	Kurt.
0.100	2.125	9.500	12.204	20.125	64.000	151.222	1.442	2.727

Along with the selected models, the maximum likelihood estimation method is used to estimate the model parameters of the proposed distribution. The majority of the numerical solutions are based on the R package "bbmle" created by Bolker and Team (2016). The numerical estimate of the model parameters, standard error of the estimates, 95% confidence interval, and log-likelihoods are given in Table 4. The log-likelihood obtained for the proposed distribution holds maximum when compared to other selected models, as shown in Table 4.

Table 4: MLEs of the parameters for the Rayleigh and transmuted Rayleigh of first and second orders distributions

Distribution	Parameter estimate	Standard error	95% Confidence interval	$\ell(x; \cdot)$
Cubic transmuted Rayleigh	$\hat{\sigma} = 11.952$	0.717	[10.547, 13.357]	-285.439
	$\hat{\lambda} = -1$	0.230	[-1, -0.549]	
Transmuted Rayleigh	$\hat{\sigma} = 13.896$	1.169	[11.605, 16.187]	-296.565
	$\hat{\lambda} = 0.634$	0.172	[0.297, 0.971]	
Rayleigh	$\hat{\sigma} = 12.209$	0.719	[10.8, 13.618]	-302.838

The likelihood ratio test statistic to test the hypothesis $H_0 : \lambda = 0$ versus $H_1 : \lambda \neq 0$ is $\lambda_{LR} = 34.79857 > 3.841 = \chi^2_{1,0.05}$, hence reject the null hypothesis.

The TTT plot (total time on test) introduced by Barlow and Campo (1975) is used here to observe the behavior of the empirical hazard function. Figure 5 shows a TTT plot (left) and an approximate hazard curve plotted over an empirical plot (right) for the cubic transmuted Rayleigh distribution. According to the TTT plot, there is an indication that the hazard function has an increasing shape, which is observed by the estimated hazard plot as presented in the right of Figure 5.

The variance covariance matrix of the maximum likelihood estimates under the cubic transmuted Rayleigh distribution is computed as

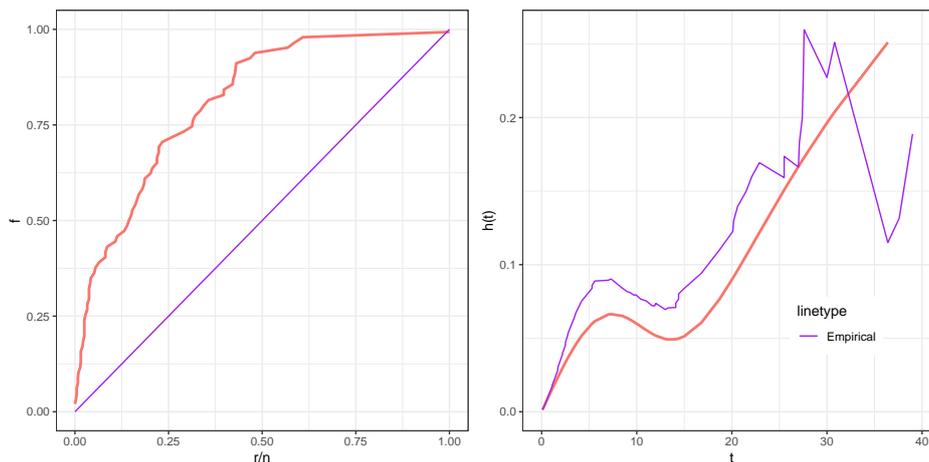$$I(\hat{\Theta})^{-1} = \begin{bmatrix} 0.514 & -0.002 \\ -0.002 & 0.053 \end{bmatrix},$$

Figure 5: TTT plot (left) and hazard curve (right) empirical and estimated by CT-Rayleigh distribution are obtained for the Wheaton River flood data

thus, the variances of the maximum likelihood estimates of σ and λ are $\text{Var}(\hat{\sigma}) = 0.514$ and $\text{Var}(\hat{\lambda}) = 0.053$. The 95% confidence intervals for the proposed model parameters σ and λ are given in Table 4.
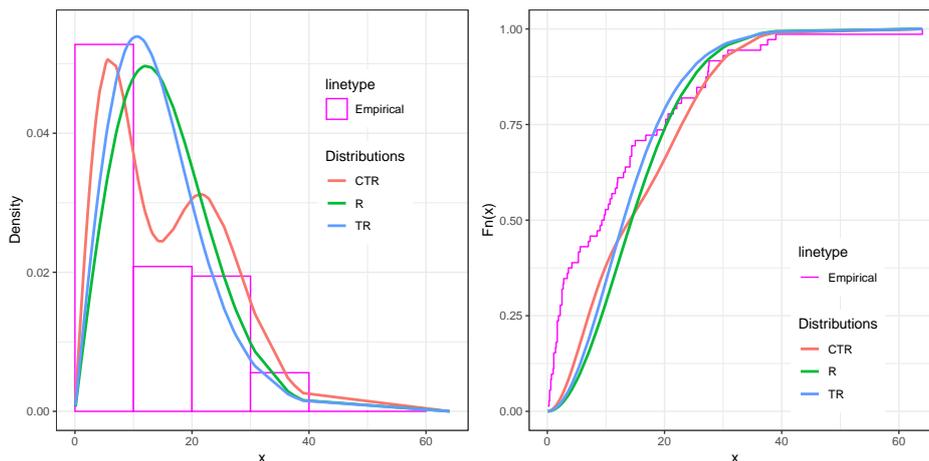

Figure 6: The PDF (left) and CDF (right) of the empirical, fitted Rayleigh, first and second order transmuted Rayleigh distributions are plotted for the Wheaton River flood data

The estimated density and distribution functions of the cubic transmuted Rayleigh distribution along with other selected models are plotted over empirical density and distribution functions and presented in Figure 6. The proposed model, as seen in Figure 6, provides a better fit than the other models.

Several model selection criteria including -2ℓ, AIC (Akaike information criterion), AICc (corrected Akaike information criterion), BIC (Bayesian information criterion), KS (Kolmogorow-Smirnow statistic), AD (Anderson-Darling statistic), and C-vM (Cramér-von Mises statistic), are considered to make a comparative study among the proposed and selected models. The better distribution corresponds to the smaller -2ℓ, AIC, AICc, BIC, KS, AD, and C-vM values:

$$\text{AIC} = 2k - 2\ln(\hat{\mathcal{L}}),$$

$$\text{AICc} = \text{AIC} + \frac{2k^2 + 2k}{n - k - 1},$$

$$\text{BIC} = k \ln(n) - 2 \ln(\hat{\mathcal{L}}),$$

$$KS = \sup_x |F_n(x) - F(x)|,$$

$$AD = n \int_{-\infty}^{\infty} [F_n(x) - F(x)]^2 \, w(x) dF(x),$$

and

$$CvM = \int_{-\infty}^{\infty} [F_n(x) - F(x)]^2 \, dF(x),$$

where k is the number of parameters, $\hat{\mathcal{L}}$ is the maximized value of the likelihood function, n is the sample size and ℓ is the maximized value of the log-likelihood function, as well as $\sup_x$ is the supremum of the set of distances, $F_n(x)$ is the empirical distribution function, $F(x)$ is the cumulative distribution function and $w(x) = [F(x) \{1 - F(x)\}]^{-1}$ is the weight function of the considered model.

Table 5: Several model selection criteria estimated for the Rayleigh and transmuted Rayleigh of first and second orders distributions

Distribution	$-2\ell(x; \cdot)$	AIC	AICc	BIC	KS	AD	C-vM
Cubic transmuted Rayleigh	570.878	574.877	575.051	579.430	0.295	9.270	0.250
Transmuted Rayleigh	593.130	597.130	597.304	601.683	0.322	10.119	0.259
Rayleigh	605.676	607.676	607.733	609.952	0.332	10.637	0.284

The obtained values of -2ℓ, AIC, AICc, BIC, KS, AD, and C-vM are presented in Table 5. It has been investigated from Table 5 that the proposed cubic transmuted Rayleigh distribution provides quite better fits as compare with the transmuted Rayleigh distribution and base Rayleigh distribution.

8. Concluding remarks

The cubic transmuted Rayleigh distribution, which extends the Rayleigh distribution, to capture complex behavior in the analysis of real-life data, is proposed in this study. Modeling real-life data is flexible enough with the proposed distribution. Moments, moment generating function, characteristic function, and quantile function are some of the distributional properties described here. The maximum likelihood estimation technique is applied to estimate the model parameters, also obtain its information matrix and confidence interval. The reliability behavior, as well as the distributions of various order statistics, are also discussed. The likelihood ratio statistic is applied to compare the proposed model with its base model. The cubic transmuted Rayleigh distribution performs significantly better in real-life application than any other models considered in this study.

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