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Analysis of Survival Data by a Weibull-generalized Sibuya Distribution

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Abstract

In this paper, we consider a survival model of a series system with random sample size, Z. Such a situation arises in competing risk analysis where the number of causes of failure is random and only the minimum of the survival times due to various causes is observed. Considering the distribution of Z as generalized Sibuya and the baseline distribution as Weibull, a Weibull-generalized Sibuya distribution is derived. The structural properties of the proposed model are studied along with the maximum likelihood estimation of the parameters. Extensive simulation studies are carried out to study the performance of the estimators. For illustration, two real data sets are analyzed and it is shown that the proposed model fits better than some of the existing models.

Keywords: generalized Sibuya distribution, Weibull-Bessel distribution, Weibull-generalized Poisson distribution, maximum likelihood estimation, Rao's score test.

1. Introduction

In survival analysis , many times the complete survival times $T_1, T_2, ..., T_Z$ are not available. However, $X = \min(T_1, T_2, ..., T_Z)$ or $Y = \max(T_1, T_2, ...T_Z)$ is observed. The first situation arises in competing risk theory where $T_1, T_2, ..., T_Z$ are the survival times due to different causes of failure and only X is observed along with the cause of failure. The second situation arises in the industrial set up where $T_1, T_2, ..., T_Z$ are the number of defects in a product and only Y is observed. In both the situations, the number of observations is unknown and so Z is considered as a random variable.

In this paper, we consider a random sample where only X is observed and Z is considered as random. More specifically, let $T_1, T_2, ..., T_Z$ be a random sample of size Z with the survival function (s.f.) $S_0(t)$. Then the conditional survival function of X given $Z = z$ is given by

$$S(x|Z = z) = [S_0(x)]^z \,. \tag{1}$$

The unconditional survival function of X is given by

$$\begin{aligned} S(x) &= \sum_{z=1}^{\infty} [S_0(x)]^z \; \mathbb{P}(Z = z) \\ &= G_Z(S_0(x)), \end{aligned} \tag{2}$$

where $G_Z(s) = \sum_{z=1}^{\infty} s^z \; \mathbb{P}(Z = z), \; 0 < s < 1$, is the probability generating function (p.g.f.) of Z.

The model above can be interpreted as a proportional hazard model whose hazard rate is given by

$$\lambda(x|Z) = Z \; \lambda_0(x),$$

where $\lambda_0(x) = -\frac{d}{dx} \ln S_0(x)$ is the baseline hazard rate function and Z is the proportionality factor, whence in this case is random and has a discrete distribution. This is analogous to the proportional hazard continuous frailty model, see Gupta and Gupta (2009).

Let $f(x|z), S(x|z), \lambda(x|z)$ be the conditional p.d.f, s.f and h.r.f. of X given $Z = z$. The unconditional p.d.f, s.f and h.r.f. of X, respectively, are given by $f(x), S(x), \lambda(x)$. It follows that

$$\begin{aligned} \frac{\lambda(x)}{\lambda_0(x)} &= \frac{f(x)}{S(x) \; \lambda_0(x)} \\ &= \frac{1}{S(x) \; \lambda_0(x)} \sum_{z=1}^{\infty} f(x|z) \; \mathbb{P}(Z = z) \\ &= \frac{1}{S(x) \; \lambda_0(x)} \sum_{z=1}^{\infty} \lambda(x|z) \; S(x|z) \; \mathbb{P}(Z = z) \\ &= \frac{S_0(x)}{S(x)} \sum_{z=1}^{\infty} z \; [S_0(x)]^{z-1} \; \mathbb{P}(Z = z) \\ &= S_0(x) \; \frac{G'_Z(S_0(x))}{G_Z(S_0(x))}, \end{aligned}$$

where $G'_Z(s) = \frac{d}{ds} G_Z(s)$.

For some theoretical properties of the above model, in the continuous case, the reader is referred to Gupta and Gupta (2009) and the references therein. For the proposed model, the expression for the hazard rate will be derived in Section3.

Various distributions of T and Z have been considered in the literature. For example, Morais and Barreto-Souza (2011), Gupta and Huang (2014)and Gupta and Waleed (2018) have considered the baseline distribution as Weibull and the distribution of Z as power series, generalized Poisson, Conway-Maxwell Poisson and Bessel, respectively. In addition, Cooner, Banerjee, Carlin, and Sinha (2007), Chen, Ibrahim, and Sinha (1999), Kus (2007) and Karlis (2009) have considered the baseline distribution as exponential and the distribution of Z as Poisson. Cordeiro, Rodrigues, and de Castro (2012) and Rodrigues, de Castro, Cancho, and Balakrishnan (2009) studied the resulting model with Conway-Maxwell Poisson as the distribution of Z.

In the present paper, we propose a model with baseline distribution as Weibull and the distribution of Z as generalized Sibuya giving rise to a four parameter model which has decreasing, increasing, bathtub and upside-down bathtub failure rate. The maximum likelihood estimation of the parameters is studied and Rao's score test is developed for various parameters. Examples are presented to illustrate the validity of the proposed model. The fitting of the proposed model, for these examples, has been compared with several of the existing models. Simulation studies have been carried out to examine the performance of the estimators.

The contents of this paper are organized as follows. In Section 2, a brief background of the generalized Sibuya distribution and its properties are given. Section 3 contains the development of the Weibull-generalized Sibuya and its structural properties. The estimation of the parameters along with the score tests for various parameters is presented in Section 4. Simulation studies are carried out in Section 5 to examine the performance of the estimators. Some applications are presented in Section 6. Finally, some conclusions and comments are provided in Section 7.

2. Generalized Sibuya distribution

A discrete random variable Z is said to have a generalized Sibuya distribution with parameters α and ν, denoted by $GS(\alpha,\nu)$, if its probability mass function (p.m.f.) is given by

$$p_Z(z) = \mathbb{P}(Z=z) = \frac{\alpha}{\nu+z}\prod_{i=1}^{z-1}\left(1-\frac{\alpha}{\nu+i}\right) = \frac{\alpha}{\nu+z}\,\frac{(\nu+1-\alpha)_{z-1}}{(\nu+1)_{z-1}}, \qquad z=1,2,\ldots, \quad (3)$$

where $\nu \geq 0,\ 0<\alpha<\nu+1$, and

$$(a)_0 = 1, \qquad (a)_k = a(a+1)(a+2)\ldots(a+k-1), \qquad k=1,2,\ldots,$$

is the (rising) Pochhammer symbol. Note that $(a)_k = \Gamma(a+k)/\Gamma(a)$, for $a>0$, where $\Gamma(\cdot)$ is the gamma function. The special case $\nu=0$ and $0<\alpha<1$ implies the Sibuya (1979) distribution with p.m.f.

$$p_Z(z) = \mathbb{P}(Z=z) = \alpha\,\frac{(1-\alpha)_{z-1}}{z!}, \qquad z=1,2,\ldots,, \quad 0<\alpha<1.$$

Figure 1 shows the p.m.f. of $GS(\alpha,\nu)$ for selected values of α and ν.

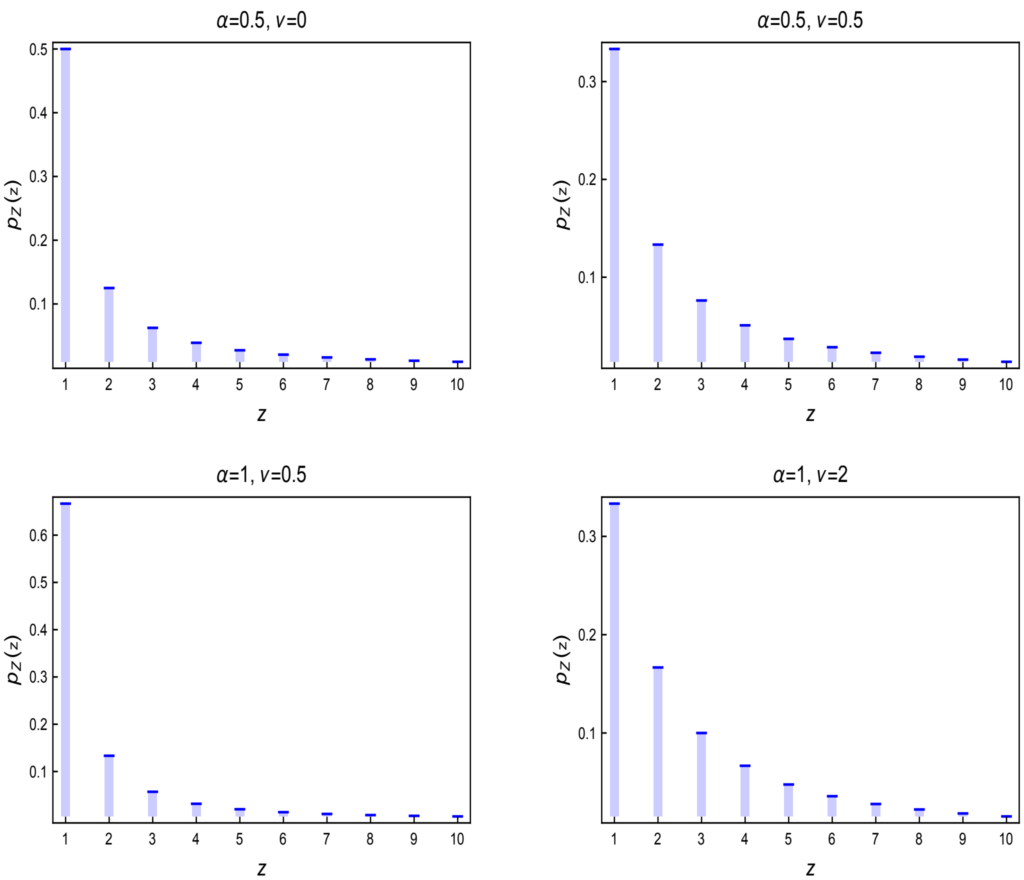

Figure 1: The p.m.f. of $GS(\alpha,\nu)$ distribution

For $\nu \geq 0, 0 < \alpha < \nu + 1$, the survival function (s.f.) and hazard rate function (h.r.f.), respectively, are given by

$$S_Z(z) = \mathbb{P}(Z > z) = \frac{(\nu + 1 - \alpha)_z}{(\nu + 1)_z}, \qquad z = 1, 2, \ldots, \tag{4}$$

$$h_Z(z) = \mathbb{P}(Z = z | Z \geq z) = \frac{\alpha}{\nu + z}, \qquad z = 1, 2, \ldots. \tag{5}$$

For $\nu > 0, 1 < \alpha < \nu + 1$, the mean residual life function (m.r.l.f.) is given by

$$\mu_Z(z) = \mathbb{E}(Z - z | Z \geq z) = \frac{\nu + z - 1}{\alpha - 1} - 1, \qquad z = 1, 2, \ldots. \tag{6}$$

The GS distribution has the following important properties:

(i) $p(z)$ is decreasing in z and hence GS distribution has mode at 1.

(ii) $p(z)$ is log-convex and hence GS distribution has decreasing hazard rate (DHR) $h_Z(z)$ and hence increasing mean residual life (IMRL) $\mu_Z(z)$. Also, log-convexity implies that GS distribution is infinitely divisible, see Hansen (1988). Kozubowski and Podgorski (2018) showed infinite divisibility via the representation as mixture of geometric distribution.

The mean and variance of the GS distribution, respectively, are given by

$$\begin{aligned} \mu_Z &= \frac{\nu}{\alpha - 1}, \qquad \nu > 0, \quad 1 < \alpha < \nu + 1, \\ \sigma_Z^2 &= \frac{\alpha\nu(\nu + 1 - \alpha)}{(\alpha - 1)^2 \ (\alpha - 2)}, \qquad \nu > 1, \quad 2 < \alpha < \nu + 1. \end{aligned}$$

Note that Sibuya distribution, i.e. $\nu = 0$ and $0 < \alpha < 1$, does not have mean and variance. The index of dispersion of the GS distribution is given by

$$\gamma_Z = \frac{\sigma_Z^2}{\mu_Z} = \frac{\alpha(\nu + 1 - \alpha)}{(\alpha - 1)(\alpha - 2)}, \qquad \nu > 1, \quad 2 < \alpha < \nu + 1.$$

The GS distribution is over-dispersed, i.e. $\gamma_Z > 1$, (under-dispersed, i.e. $\gamma_Z < 1$,) if $2 < \alpha < \nu_1$ ($\nu_1 < \alpha < \nu + 1$), where $\nu_1 = \frac{\nu+4+\sqrt{\nu(\nu+8)}}{4} > 2$.

For details about the above properties of GS model, see Kozubowski and Podgorski (2018). The p.g.f. of GS distribution is given by

$$\begin{aligned} G_Z(s) &= \mathbb{E}(s^Z) \qquad (7) \\ &= \sum_{z=1}^{\infty} s^z \ \frac{\alpha}{\nu + z} \ \frac{(\nu + 1 - \alpha)_{z-1}}{(\nu + 1)_{z-1}} \\ &= \sum_{k=0}^{\infty} s^{k+1} \ \frac{\alpha}{\nu + k + 1} \ \frac{(\nu + 1 - \alpha)_k}{(\nu + 1)_k} \\ &= \frac{\alpha}{\nu + 1} \ s \sum_{k=0}^{\infty} \frac{(1)_k \ (\nu + 1 - \alpha)_k}{(\nu + 2)_k} \ \frac{s^k}{k!} \\ &= \frac{\alpha}{\nu + 1} \ s \ {}_2F_1(1, \nu + 1 - \alpha; \nu + 2; s), \quad 0 < s < 1, \qquad (8) \end{aligned}$$

where

$${}_2F_1(a, b; c; s) = \sum_{k=0}^{\infty} \frac{(a)_k \ (b)_k}{(c)_k} \ \frac{s^k}{k!}, \qquad |s| < 1, \quad c \neq 0, -1, -2, \ldots,$$

is the Gauss hypergeometric function. Note that our expression of the p.g.f. (8) is different than the one given by Kozubowski and Podgorski (2018).

Note that the Gauss hypergeometric function has the integral representation:

$$ {}_2F_1(a,b;c;s) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-st)^{-a}\, dt, \qquad c > b > 0. $$

For $\nu = 0$ and $0 < \alpha < 1$, i.e. Sibuya distribution, we have

$$ \begin{aligned} G_Z(s) &= \alpha\ s\ {}_2F_1(1, 1-\alpha; 2; s) \\ &= \alpha\ s\ {}_2F_1(1-\alpha, 1; 2; s) \\ &= \alpha\ s \int_0^1 (1-st)^{-(1-\alpha)}\, dt \\ &= \alpha \int_{1-s}^1 y^{\alpha-1}\, dy \\ &= 1-(1-s)^\alpha, \qquad 0 < s < 1, \quad 0 < \alpha < 1. \end{aligned} $$

3. Weibull-generalized Sibuya distribution

Suppose that $T_1, T_2, \ldots, T_Z$ are independent and identically distributed random variables having the Weibull distribution with s.f.

$$ S_0(t) = e^{-(\beta t)^\gamma}, \qquad t > 0, \quad \beta, \gamma > 0, \tag{9} $$

and Z is a discrete random variable having $GS(\alpha, \nu)$ distribution with p.g.f. (8).

Consider $X = \min(T_1, T_2, \ldots, T_Z)$ and assume that the random variables T_i, $i = 1, 2, ..., Z$, and Z are independent. It follows that the survival function of X is

$$ S(x; \boldsymbol{\theta}) = G_Z\left(S_T(x)\right) = \frac{\alpha}{\nu+1}\ e^{-(\beta x)^\gamma}\ {}_2F_1(1, \nu+1-\alpha; \nu+2; e^{-(\beta x)^\gamma}), \qquad x > 0, \tag{10} $$

where $\boldsymbol{\theta} = (\beta, \gamma, \alpha, \nu)$ and $\beta, \gamma > 0$ and $0 < \alpha < \nu+1, \nu \geq 0$. See the `R` function `pweibullGS1` in Appendix B for calculating the cumulative distribution function (c.d.f.) $F(x; \boldsymbol{\theta}) = 1 - S(x; \boldsymbol{\theta})$. Also, in Appendix B, the `R` function `qweibullGS1` can be used for calculating the quantile function $F^{-1}(p; \boldsymbol{\theta})$ where $0 < p < 1$.

The continuous distribution with s.f. (10) will be called Weibull-generalized Sibuya distribution and will be denoted by $WGS(\boldsymbol{\theta})$.

Special submodels:

(i) $\gamma = 1$: Exponential-GS (EGS)distribution,

(ii) $\nu = 0$: Weibull-Sibuya (WS) distribution,

(iii) $\gamma = 1, \nu = 0$: Exponential-Sibuya (ES) distribution.

Since

$$ \frac{d}{dx}\ {}_2F_1(a, b; c; w(x)) = \frac{ab}{c}\ {}_2F_1(a+1, b+1; c+1; w(x))\ \frac{dw(x)}{dx}, $$

(see Andrews (1998), p. 362), it follows that the probability density function (p.d.f.) of WGS distribution is

$$ \begin{aligned} f(x; \boldsymbol{\theta}) &= -\frac{d}{dx} S(x; \boldsymbol{\theta}) \\ &= \frac{\alpha}{\nu+1}\ \beta\gamma(\beta x)^{\gamma-1}\ e^{-(\beta x)^\gamma}\ \{\ {}_2F_1(1, \nu+1-\alpha; \nu+2; e^{-(\beta x)^\gamma}) \\ &\qquad\qquad + \frac{\nu+1-\alpha}{\nu+2}\ e^{-(\beta x)^\gamma}\ {}_2F_1(2, \nu+2-\alpha; \nu+3; e^{-(\beta x)^\gamma})\}. \end{aligned} $$

Now, since

$$a \;\; {}_2F_1(a,b;c;s) + \frac{ab}{c} \; s \;\; {}_2F_1(a+1,b+1;c+1;s) = \; {}_2F_1(a+1,b;c;s),$$

(see Andrews (1998), p. 363), we have

$$f(x;\boldsymbol{\theta}) = \frac{\alpha}{\nu+1} \; \beta\gamma(\beta x)^{\gamma-1} \; e^{-(\beta x)^\gamma} \;\; {}_2F_1(2,\nu+1-\alpha;\nu+2;e^{-(\beta x)^\gamma}). \tag{11}$$

See the R function `dweibullGS1` in Appendix B for calculating the p.d.f. (11).

For $\nu = 0$ and $0 < \alpha < 1$, i.e. Sibuya distribution, we have ${}_2F_1(2,1-\alpha;2;s) = (1-s)^{\alpha-1}$ and hence

$$f(x;\boldsymbol{\theta}) = \alpha\beta\gamma(\beta x)^{\gamma-1} \; e^{-(\beta x)^\gamma} \; \left[1 - e^{-(\beta x)^\gamma}\right]^{\alpha-1}, \qquad x > 0, \quad \beta,\gamma > 0, 0 < \alpha < 1.$$

The last expression is the p.d.f. of the exponentiated Weibull distribution with resilience parameter $\alpha \in (0,1)$, see Mudholkar and Srivastava (1993) and (Mudholkar, Srivastava, and Freimer 1995).

Figure 1 shows the p.d.f. (11) of the WGS distribution for selected values of the parameters. This figure shows that such p.d.f. is strictly decreasing for $0 < \gamma \leq 1$, and is unimodal for $\gamma > 1$.

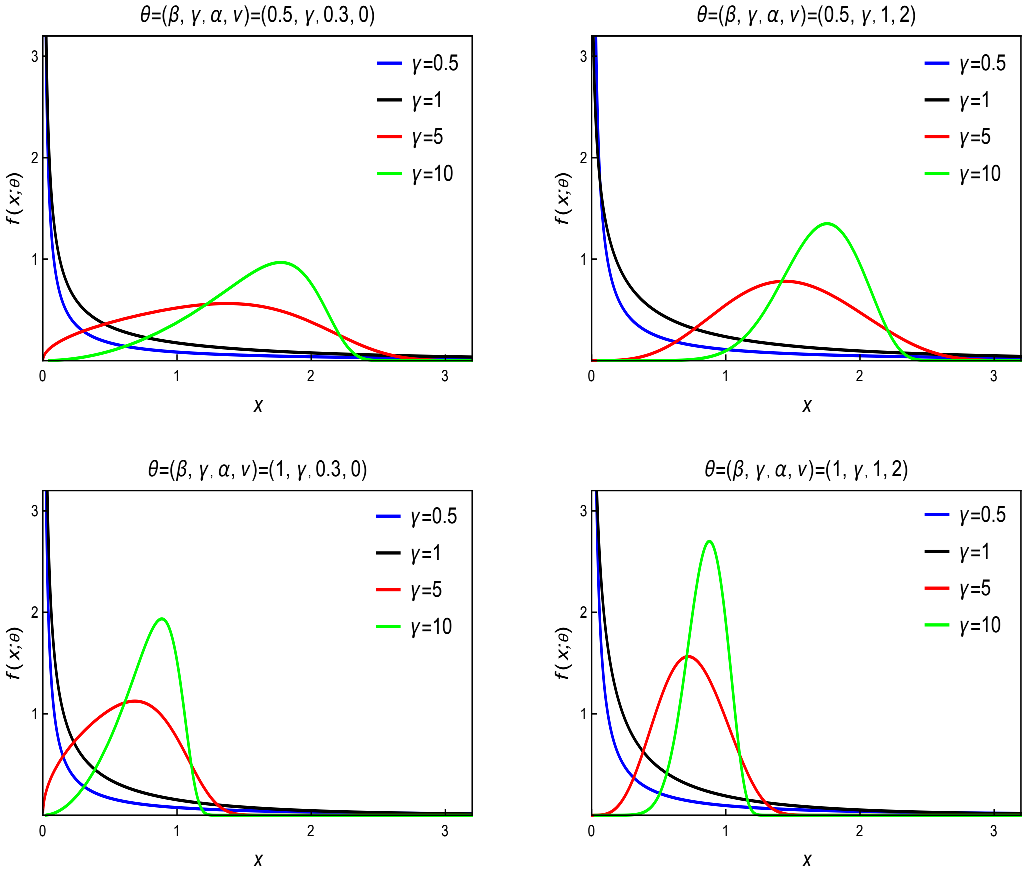

Figure 2: The p.d.f. of WGS distribution for selected values of the parameters

The h.r.f. of WGS distribution is

$$\begin{aligned} h(x;\boldsymbol{\theta}) &= \frac{f(x;\boldsymbol{\theta})}{S(x;\boldsymbol{\theta})} \\ &= \beta\gamma(\beta x)^{\gamma-1} \; \frac{{}_2F_1(2,\nu+1-\alpha;\nu+2;e^{-(\beta x)^\gamma})}{{}_2F_1(1,\nu+1-\alpha;\nu+2;e^{-(\beta x)^\gamma})}. \end{aligned} \tag{12}$$

Note that the expression (12) is a weighted function of $h_0(x) = \beta\gamma(\beta x)^{\gamma-1}$, the h.r.f. of Weibull distribution.

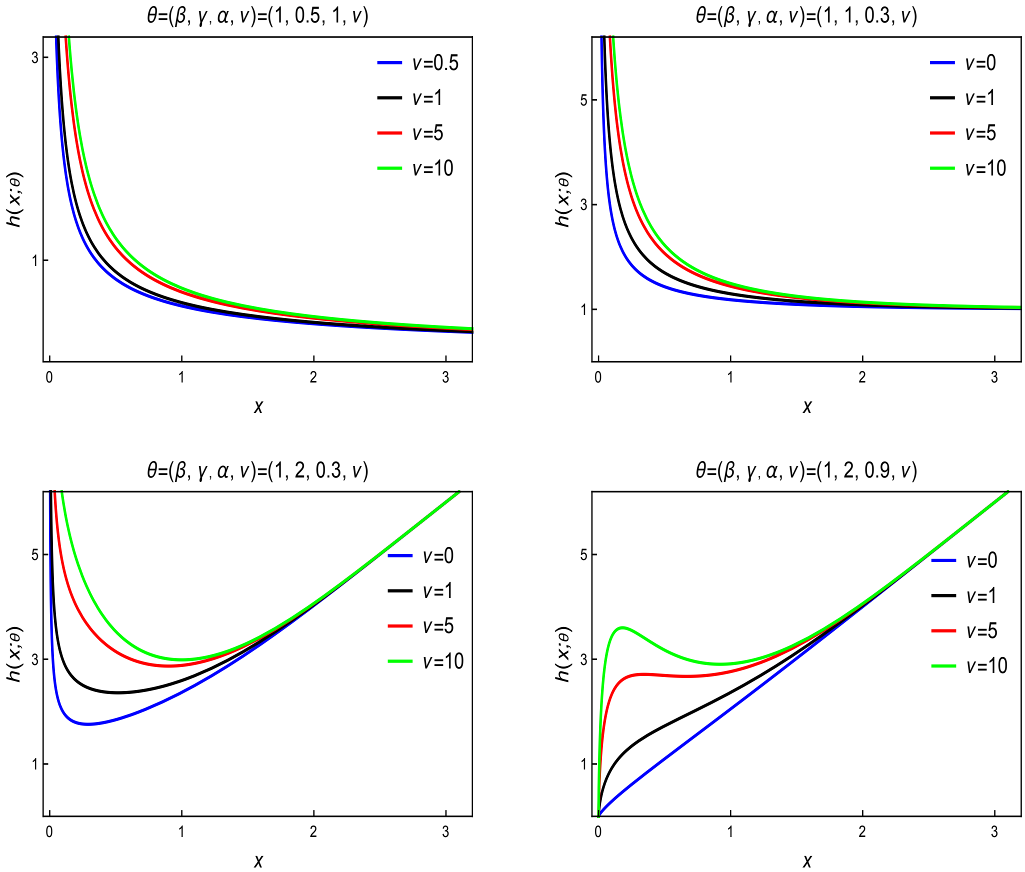

Figure 3: The h.r.f. of WGS distribution for selected values of the parameters

Figure (3) shows the h.r.f. (12) of the WGS distribution for selected values of the parameters.

The kth moment of the WGS distribution is given by

$$\mu'_k = \mathbb{E}(X^k) = \frac{\Gamma(1+k/\gamma)}{\beta^k}\,\mathbb{E}(Z^{-k/\gamma}) = \frac{\Gamma(1+k/\gamma)}{\beta^k}\sum_{z=1}^{\infty} z^{-k/\gamma}\,p_Z(z). \tag{13}$$

The above moments can be computed numerically using computer software language such as Mathematica, R, MATLAB, etc.

It can be verified that the mean and variance of the WGS distribution are, respectively, given by the following expressions

$$\begin{aligned}\mathbb{E}(X) &= \frac{\Gamma(1+1/\gamma)}{\beta}\,\mathbb{E}(Z^{-1/\gamma}),\\ \mathbb{V}ar(X) &= \frac{\Gamma(1+2/\gamma)}{\beta^2}\,\mathbb{E}(Z^{-2/\gamma}) - \left[\frac{\Gamma(1+1/\gamma)}{\beta}\,\mathbb{E}(Z^{-1/\gamma})\right]^2.\end{aligned}$$

Table 1: Mean and variance of WGS for selected values of the parameters

β	γ	α	ν	$E(X)$	$Var(X)$
0.5	0.5	0.3	0	1.354	27.689
	1			0.816	2.043
	5			1.259	0.393
	10			1.507	0.248
0.5	0.5	1	2	1.580	30.660
	1			1.000	2.159
	5			1.486	0.228
	10			1.700	0.083
1	0.5	0.3	0	0.677	6.922
	1			0.408	0.511
	5			0.630	0.098
	10			0.754	0.062
1	0.5	1	2	0.790	7.665
	1			0.500	0.540
	5			0.743	0.057
	10			0.850	0.021
2	0.5	0.3	0	0.339	1.731
	1			0.204	0.128
	5			0.315	0.025
	10			0.377	0.015
2	0.5	1	2	0.394	1.916
	1			0.250	0.135
	5			0.371	0.014
	10			0.425	0.005

4. Maximum likelihood estimation

Let $x_1, x_2, \ldots, x_n$ be a random sample of size n from $WGS(\beta, \gamma, \alpha, \nu)$ distribution. The log-likelihood function is given by

$$\begin{aligned}\ell_n(\boldsymbol{\theta}) &= \sum_{i=1}^{n} \ln f(x_i; \boldsymbol{\theta}) \\ &= n[\ln(\alpha) + \gamma \ln(\beta) + \ln(\gamma) - \ln(\nu+1)] + (\gamma - 1)\sum_{i=1}^{n} \ln(x_i) - \sum_{i=1}^{n} (\beta x_i)^{\gamma} \\ &\quad + \sum_{i=1}^{n} \ln[\,{}_2F_1(2, \nu+1-\alpha; \nu+2; e^{-(\beta x_i)^{\gamma}})]. \qquad (14)\end{aligned}$$

The score vector

$$\mathbf{U}_n(\boldsymbol{\theta}) = (U_1(\boldsymbol{\theta}), U_2(\boldsymbol{\theta}), U_3(\boldsymbol{\theta}), U_4(\boldsymbol{\theta}))^{\top} = \left(\frac{\partial \ell_n(\boldsymbol{\theta})}{\partial \beta}, \frac{\partial \ell_n(\boldsymbol{\theta})}{\partial \gamma}, \frac{\partial \ell_n(\boldsymbol{\theta})}{\partial \alpha}, \frac{\partial \ell_n(\boldsymbol{\theta})}{\partial \nu}\right)^{\top}$$

has elements

$$U_1(\boldsymbol{\theta}) = \frac{n\gamma}{\beta} - \gamma\beta^{\gamma-1}\sum_{i=1}^{n} x_i^{\gamma} + \sum_{i=1}^{n} \frac{D_1(x_i,\boldsymbol{\theta})}{{}_2F_1(2,\nu+1-\alpha;\nu+2;e^{-(\beta x_i)^{\gamma}})}, \quad (15)$$

$$U_2(\boldsymbol{\theta}) = n\left(\ln(\beta)+\frac{1}{\gamma}\right) + \sum_{i=1}^{n}\ln(x_i) - \sum_{i=1}^{n}(\beta x_i)^{\gamma}\ \ln(\beta x_i) + \sum_{i=1}^{n} \frac{D_2(x_i,\boldsymbol{\theta})}{{}_2F_1(2,\nu+1-\alpha;\nu+2;e^{-(\beta x_i)^{\gamma}})}, \quad (16)$$

$$U_3(\boldsymbol{\theta}) = \frac{n}{\alpha} + \sum_{i=1}^{n} \frac{D_3(x_i,\boldsymbol{\theta})}{{}_2F_1(2,\nu+1-\alpha;\nu+2;e^{-(\beta x_i)^{\gamma}})}, \quad (17)$$

$$U_4(\boldsymbol{\theta}) = -\frac{n}{\nu+1} + \sum_{i=1}^{n} \frac{D_4(x_i,\boldsymbol{\theta})}{{}_2F_1(2,\nu+1-\alpha;\nu+2;e^{-(\beta x_i)^{\gamma}})}, \quad (18)$$

where

$$D_1(x_i,\boldsymbol{\theta}) = -\frac{2(\nu+1-\alpha)\gamma}{(\nu+2)\beta}\ {}_2F_1(3,\nu+2-\alpha;\nu+3;e^{-(\beta x_i)^{\gamma}})\ (\beta x_i)^{\gamma}\ e^{-(\beta x_i)^{\gamma}}, \quad (19)$$

$$D_2(x_i,\boldsymbol{\theta}) = \frac{\beta}{\gamma}\ \ln(\beta x_i)\ D_1(x_i,\boldsymbol{\theta}), \quad (20)$$

$$D_3(x_i,\boldsymbol{\theta}) = {}_2F_1(2,\nu+1-\alpha;\nu+2;s)\ \psi_0(\nu+1-\alpha) - \sum_{k=0}^{\infty}\frac{(2)_k\ (\nu+1-\alpha)_k}{(\nu+2)_k}\ \psi_0(\nu+k+1-\alpha)\ \frac{s^k}{k!}, \quad (21)$$

$$D_4(x_i,\boldsymbol{\theta}) = {}_2F_1(2,\nu+1-\alpha;\nu+2;s)\ \psi_0(\nu+2) - \sum_{k=0}^{\infty}\frac{(2)_k\ (\nu+1-\alpha)_k}{(\nu+2)_k}\ \psi_0(\nu+2+k)\ \frac{s^k}{k!} - D_3(x_i,\boldsymbol{\theta}), \quad (22)$$

with $\psi_0(z) = \frac{d}{dz}\ln\Gamma(z)$ as the polygamma function of order 0, see Appendix A for details.

The MLE $\widehat{\boldsymbol{\theta}}$ of $\boldsymbol{\theta}$ can be obtained by solving the system of equations $\mathbf{U}_n(\boldsymbol{\theta}) = \mathbf{0}$ numerically. For interval estimation and tests of hypotheses on $\boldsymbol{\theta}$, we require the observed information matrix of a random sample of size n from the WGS distribution, given by

$$\mathbf{J}_n(\boldsymbol{\theta}) = -\begin{bmatrix} \frac{\partial^2\ell_n(\boldsymbol{\theta})}{\partial\beta^2} & \frac{\partial^2\ell_n(\boldsymbol{\theta})}{\partial\beta\partial\gamma} & \frac{\partial^2\ell_n(\boldsymbol{\theta})}{\partial\beta\partial\alpha} & \frac{\partial^2\ell_n(\boldsymbol{\theta})}{\partial\beta\partial\nu} \\ \frac{\partial^2\ell_n(\boldsymbol{\theta})}{\partial\beta\partial\gamma} & \frac{\partial^2\ell_n(\boldsymbol{\theta})}{\partial\gamma^2} & \frac{\partial^2\ell_n(\boldsymbol{\theta})}{\partial\gamma\partial\alpha} & \frac{\partial^2\ell_n(\boldsymbol{\theta})}{\partial\gamma\partial\nu} \\ \frac{\partial^2\ell_n(\boldsymbol{\theta})}{\partial\beta\partial\alpha} & \frac{\partial^2\ell_n(\boldsymbol{\theta})}{\partial\gamma\partial\alpha} & \frac{\partial^2\ell_n(\boldsymbol{\theta})}{\partial\alpha^2} & \frac{\partial^2\ell_n(\boldsymbol{\theta})}{\partial\alpha\partial\nu} \\ \frac{\partial^2\ell_n(\boldsymbol{\theta})}{\partial\beta\partial\nu} & \frac{\partial^2\ell_n(\boldsymbol{\theta})}{\partial\gamma\partial\nu} & \frac{\partial^2\ell_n(\boldsymbol{\theta})}{\partial\alpha\partial\nu} & \frac{\partial^2\ell_n(\boldsymbol{\theta})}{\partial\nu^2} \end{bmatrix}.$$

Under mild regularity conditions (see Lehmann and Casella (1998), pp. 461-463), the asymptotic distribution of the MLE $\widehat{\boldsymbol{\theta}}$ is multivariate normal distribution with mean $\boldsymbol{\theta}$ and variance-covariance matrix $\mathbf{J}_n(\boldsymbol{\theta})$.

This estimated multivariate normal distribution can be used to construct approximate confidence intervals for the parameters and to test hypotheses about these parameters.

It is of interest to test whether the baseline distribution is exponential versus Weibull. That is, for our proposed model, we need to test $H_0 : \gamma = 1$ versus $H_1 : \gamma \neq 1$. For this purpose, we use the Rao's score and the likelihood ratio tests.

The Rao's score test statistic is given by

$$R = \mathbf{U}_n^\top(\widetilde{\boldsymbol{\theta}}_1)\ \mathbf{I}_n^{-1}(\widetilde{\boldsymbol{\theta}}_1)\ \mathbf{U}_n(\widetilde{\boldsymbol{\theta}}_1),$$

where $\mathbf{U}_n(\boldsymbol{\theta})$ is the score vector, $\mathbf{I}_n(\boldsymbol{\theta})$ is the expected information matrix and $\widetilde{\boldsymbol{\theta}}_1$ is the restricted MLE under $H_0 : \gamma = 1$.

For large n, the expected information matrix matrix $\mathbf{I}_n(\boldsymbol{\theta})$ is approximated by $\mathbf{J}(\boldsymbol{\theta})$.

The likelihood ratio test statistic is

$$\Lambda = -2[\ell_n(\widetilde{\boldsymbol{\theta}}_1) - \ell_n(\widehat{\boldsymbol{\theta}})].$$

Each of the above test statistics has a chi-square distribution with one degree of freedom.

For more details about the Rao score and likelihood ratio tests and their asymptotic distribution, see Rao (2001).

5. Simulation study

To generate a variate x from the $WGS(\beta, \gamma, \alpha, \nu)$ distribution, we (i) simulate a value z of the random variable $Z \sim GS(\alpha, \nu)$, (ii) simulate z values $t_1, \ldots, t_z$ from the random variable $T \sim Weibull(\beta, \gamma)$ and (iii) take $x = \min(t_1, \ldots, t_z)$. See the `R` function `rweibullGS1` in Appendix B for generating random data from WGS distribution.

In this section, we present the results of the simulation studies which were carried out to examine the performance of the MLEs of the WGS distribution. To suffice our purpose, we generated 10000 samples of size 50, 100, 150, 200, 250 and 300 considering the following four scenarios.

Table 2: Scenarios of the true values of the parameters used in the simulations

		β	γ	α	ν
Scenario 1	case 1	0.5	0.5	0.3	0
	case 2	0.5	1	0.3	0
	case 3	0.5	5	0.3	0
	case 4	0.5	10	0.3	0
Scenario 2	case 1	1	0.5	0.3	0
	case 2	1	1	0.3	0
	case 3	1	5	0.3	0
	case 4	1	10	0.3	0
Scenario 3	case 1	0.5	0.5	1	2
	case 2	0.5	1	1	2
	case 3	0.5	5	1	2
	case 4	0.5	10	1	2
Scenario 4	case 1	1	0.5	1	2
	case 2	1	1	1	2
	case 3	1	5	1	2
	case 4	1	10	1	2

All simulations were performed in *Ox* Console, version 8.02 (Doornik 2007).

Figure 4 shows the average bias of the MLEs which are seen to be quite small. This figure also shows that the bias of $\widehat{\beta}, \widehat{\gamma}, \widehat{\alpha}$ may be positive or negative, while the bias of $\widehat{\nu}$ is positive. Figure 5 shows the MSEs of the estimates which decrease as the sample size increases.

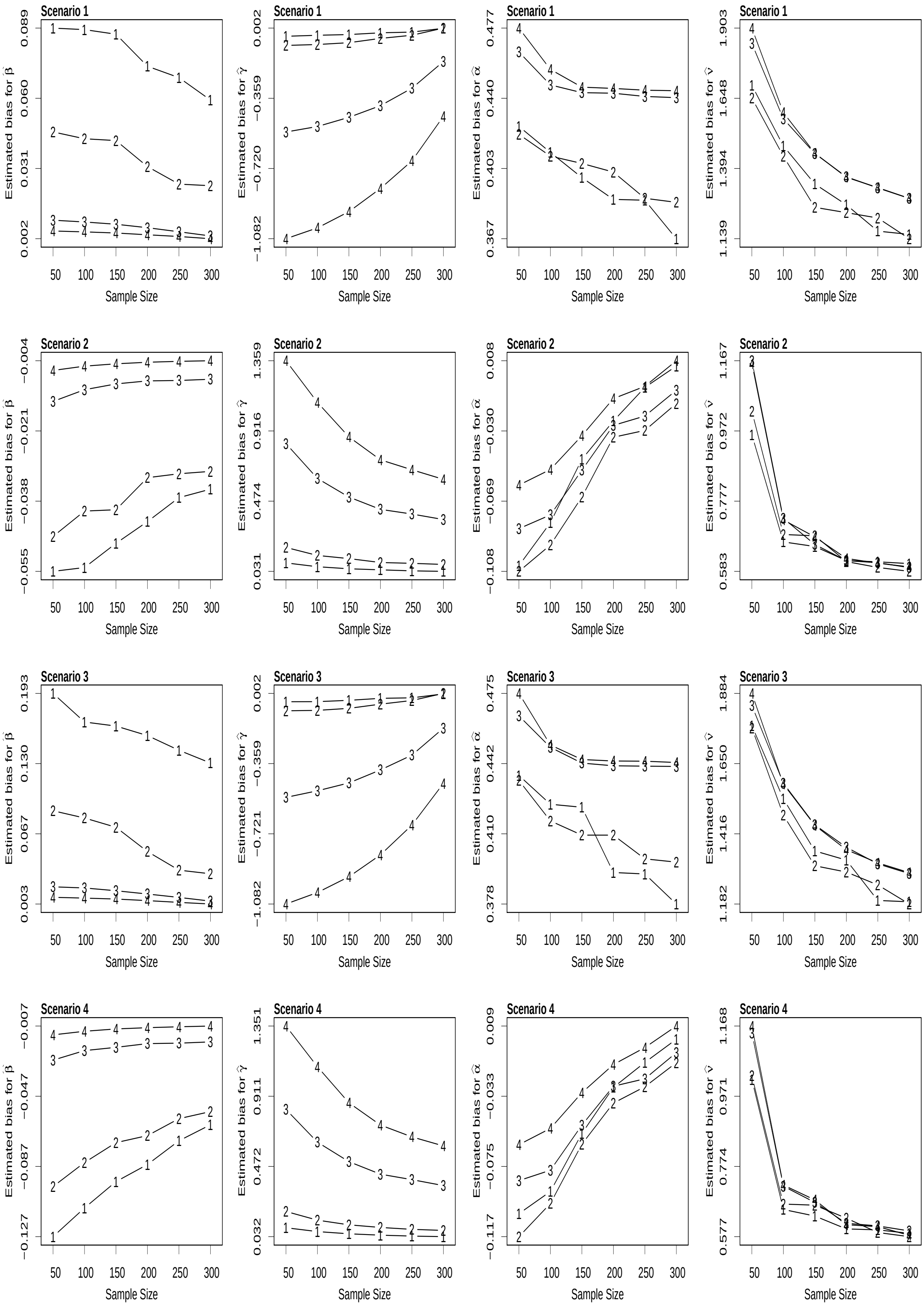

Figure 4: Estimated bias of the MLEs under the considered four scenarios

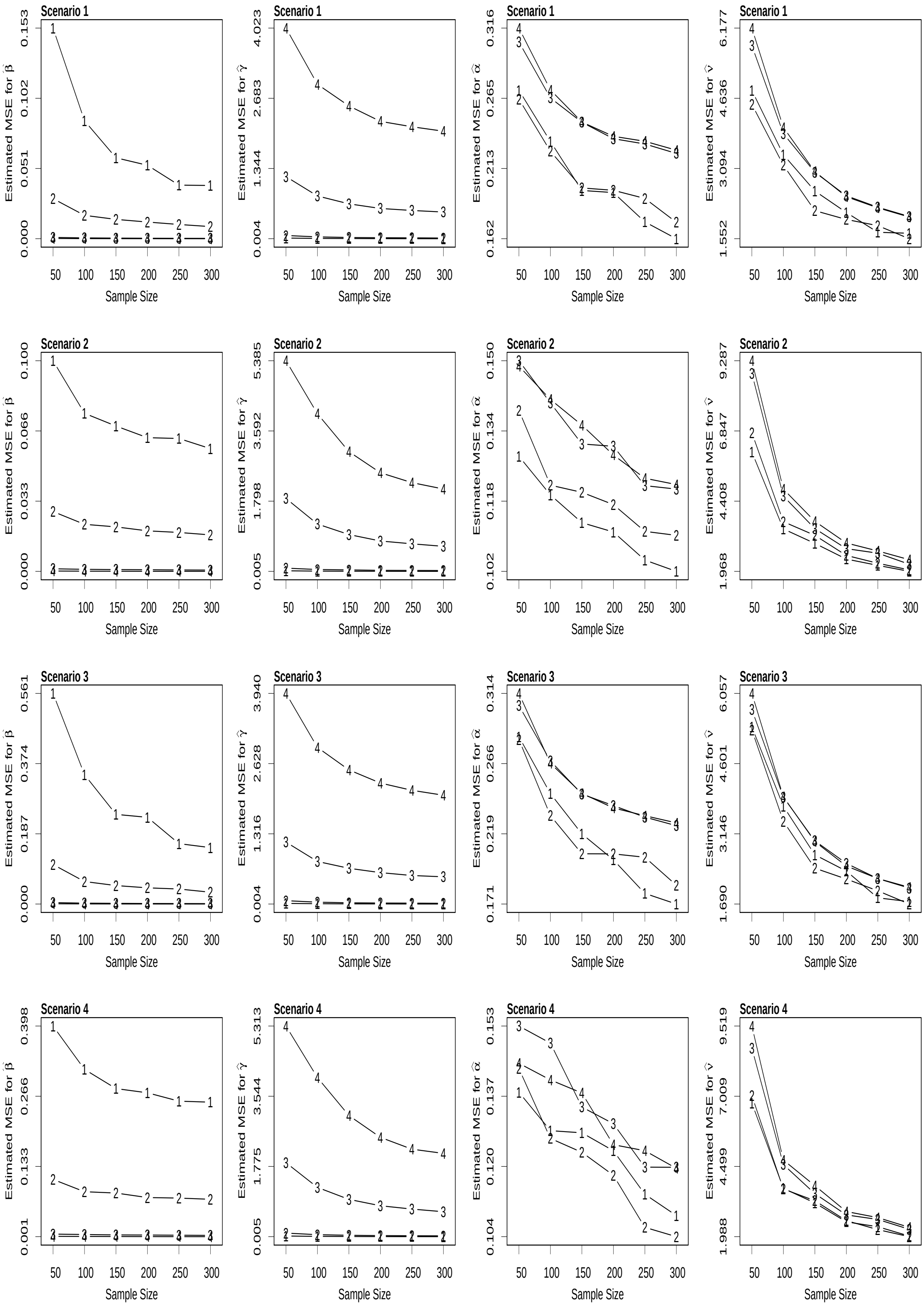

Figure 5: Estimated MSE of the MLEs under the considered four scenarios

6. Applications

We compare our proposed Weibull-generalized sibuya (WGS) model with other four-parameter models, namely, Weibull-Bessel (WB), see (Gupta and Waleed 2018), and Weibull-generalized Poisson (WGP), see (Gupta and Huang 2014). We also consider the three-parameter versions when $\gamma = 1$, when the base distribution is exponential.

The p.d.f. of the WB model is given by

$$f_{WB}(x;\boldsymbol{\psi}) = \frac{\gamma\beta^{\gamma}\omega^{1/2}\,\Gamma(\tau+1)}{\Gamma(\tau+1)I_{\nu}(2\sqrt{\omega}) - \omega^{\nu/2}}\,\frac{x^{\gamma-1}\,I_{\tau+1}\left(2\sqrt{e^{-(\beta x)^{\gamma}}\omega}\right)}{\sqrt{e^{(1-\tau)(\beta x)^{\gamma}}}}, \qquad x > 0,$$

where $\boldsymbol{\psi} = (\beta, \gamma, \omega, \tau)$, $\beta, \gamma, \omega > 0$, $\tau > -1$, and

$$I_{\tau}(a) = \sum_{k=0}^{\infty} \frac{1}{k!\;\Gamma(k+\tau+1)}\,(a/2)^{2k+\tau}, \qquad a > 0, \tau > -1,$$

is the modified Bessel function of the first kind.

The p.d.f. of the WGP model is given by

$$f_{WGP}(x;\boldsymbol{\phi}) = \frac{\lambda\gamma\beta^{\gamma}x^{\gamma-1}e^{-\frac{\lambda}{\rho}W(g)-\lambda}\,(-W(g))}{\rho(1-e^{-\lambda})(1+W(g))}, \qquad x > 0,$$

where $\boldsymbol{\phi} = (\beta, \gamma, \lambda, \rho)$, $\beta, \gamma, \lambda > 0$, $|\rho| < 1$, and $W(g)$ is the Lambert function, i.e. $W(g)e^{W(g)} = g$ and $g = -\rho e^{-\alpha-(\beta x)^{\gamma}}$.

For model selection, we use Akaiake information criterion (AIC) and Bayesian information criterion (BIC) where $AIC = 2k - 2\widehat{\ell_n}$ and $BIC = 2\log(n) - 2\widehat{\ell_n}$ where k is the number of parameters in the model and $\widehat{\ell_n}$ is the estimated log-likelihood function of this model.

Data set 1 This data set represents the number of successive failures for the air conditioning system of each member in a fleet of 7 Boeing 720 airplanes. The data consisting of 125 observations pertain to aircraft numbers 7910, 7911, 7912, 7913, 7914, 7915, 7916, see Gupta and Waleed (2018).

74, 57, 48, 29, 502, 12, 70, 21, 29, 386, 59, 27, 153, 26, 326, 55, 320, 56, 104, 220, 239, 47, 246, 176, 182, 33, 15, 104, 35, 23, 261, 87, 7, 120, 14, 62, 47, 225, 71, 246, 21, 42, 20, 5, 12, 120, 11, 3, 14, 71, 11, 14, 11, 16, 90, 1, 16, 52, 95, 97, 51, 11, 4, 141, 18, 142, 68, 77, 80, 1, 16, 106, 206, 82, 54, 31, 216, 46, 111, 39, 63, 18, 191, 18, 163, 24, 50, 44, 102, 72, 22, 39, 3, 15, 197, 188, 79, 88, 46, 5, 5, 36, 22, 139, 210, 97, 30, 23, 13, 14, 359, 9, 12, 270, 603, 3, 104, 2, 438, 50, 254, 5, 283, 35, 12

Table 3 shows the MLEs of the parameters of the various competing models, their estimated log-likelihood, AIC and BIC. This table also shows that the EGS model has the smallest AIC and BIC. This selection of EGS model is also consistent with Rao's and likelihood ratio tests presented in Table 4. In addition, several goodness-of-fit plots for the EGS model for data set 1 presented in Figure 6 support this selection.

Table 3: MLEs, estimated log-likelihood, AIC, and BIC for data set 1 (n=125)

Model	MLE	log-likelihood	AIC	BIC
WGS	$\widehat{\boldsymbol{\theta}} = (\widehat{\beta}, \widehat{\gamma}, \widehat{\alpha}, \widehat{\nu})$ $= (0.0056, 1.2516, 1.0368, 2.0028)$	-686.29	1380.6	1391.9
EGS	$\widetilde{\boldsymbol{\theta}}_1 = (\widetilde{\beta}_1, \widetilde{\alpha}_1, \widetilde{\nu}_1)$ $= (0.0071, 11.7252, 25.0701)$	-686.33	1378.7	1387.2
WB	$\widehat{\boldsymbol{\psi}} = (\widehat{\beta}, \widehat{\gamma}, \widehat{\omega}, \widehat{\tau})$ $= (0.0122, 0.8249, 0.0099, 7.5434)$	- 686.60	1381.2	1392.4
EB	$\widetilde{\boldsymbol{\psi}}_1 = (\widetilde{\beta}_1, \widetilde{\omega}_1, \widetilde{\tau}_1)$ $= (0.0023, 31.8331, -0.6675)$	-687.70	1381.4	1389.7
WGP	$\widehat{\boldsymbol{\phi}} = (\widehat{\beta}, \widehat{\gamma}, \widehat{\lambda}, \widehat{\rho})$ $= (0.0046, 1.5395, 1.5490, 0.8440)$	-1064.70	2140.8	2149.3
EGP	$\widetilde{\boldsymbol{\phi}}_1 = (\widetilde{\beta}_1, \widetilde{\lambda}_1, \widetilde{\rho}_1)$ $= (0.0094, 2.3689, -0.9083)$	-1067.40	2139.0	2144.7

Table 4: Likelihood ratio and Rao's score tests for $H_0 : \gamma = 1$ (EGS) for data set 1

Test	Statistic	p-value	Decision
Rao's score	0.818	0.366	Accept H_0
LR	166.180	0.077	Accept H_0

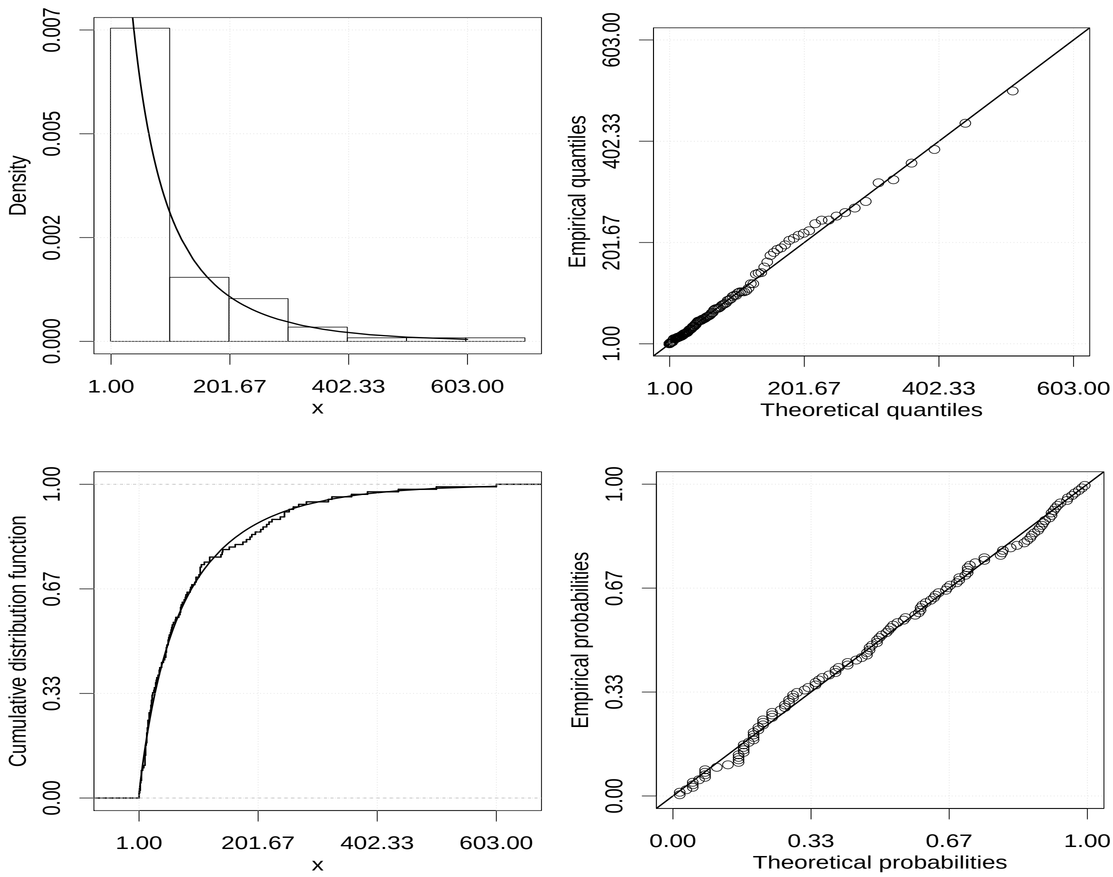

Figure 6: Goodness-of-fit plots for EGS model for data set 1

Data set 2 This data set represents the failure stresses (in GPa) of 65 single carbon fibers of length 50 mm (see Bader and Priest (1982)).

1.339, 1.434, 1.549, 1.574, 1.589, 1.613, 1.746, 1.753, 1.764, 1.807, 1.812, 1.840, 1.852, 1.852, 1.862, 1.864, 1.931, 1.952, 1.974, 2.019, 2.051, 2.055, 2.058, 2.088, 2.125, 2.162, 2.171, 2.172, 2.180, 2.194, 2.211, 2.270, 2.272, 2.280, 2.299, 2.308, 2.335, 2.349, 2.356, 2.386, 2.390, 2.410, 2.430, 2.431, 2.458, 2.471, 2.497, 2.514, 2.558, 2.577, 2.593, 2.601, 2.604, 2.620, 2.633, 2.670, 2.682, 2.699, 2.705, 2.735, 2.785, 3.020, 3.042, 3.116, 3.174

Table 5 shows the MLEs of the parameters of the various competing models, their estimated log-likelihood, AIC and BIC. This table also shows that the WGS model has the smallest AIC and BIC. This selection of WGS model is also consistent with Rao's score and likelihood ratio tests presented in Table 6. In addition, several goodness-of-fit plots for the EGS model for data set 2 presented in Figure 7 support this selection.

Table 5: MLEs, estimated log-likelihood, AIC, and BIC for data set 2

Model	MLE	log-likelihood	AIC	BIC
WGS	$\widehat{\boldsymbol{\theta}} = (\widehat{\beta}, \widehat{\gamma}, \widehat{\alpha}, \widehat{\nu})$ $= (0.3457, 10.2519, 1.0565, 8.1575)$	-34.45	76.9	85.6
EGS	$\widetilde{\boldsymbol{\theta}}_1 = (\widetilde{\beta}_1, \widetilde{\alpha}_1, \widetilde{\nu}_1)$ $= (0.4456, 1.9008, 0.9009)$	-117.54	241.1	247.6
WB	$\widehat{\boldsymbol{\psi}} = (\widehat{\beta}, \widehat{\gamma}, \widehat{\omega}, \widehat{\tau})$ $= (0.3730, 7.0299, 359.1241, 148.9401)$	-34.80	77.6	86.3
EB	$\widetilde{\boldsymbol{\psi}}_1 = (\widetilde{\beta}_1, \widetilde{\omega}_1, \widetilde{\tau}_1)$ $= (0.4457, 4.98942 \times 10^{-6}, 1.5812)$	-117.54	241.1	247.6
WGP	$\widehat{\boldsymbol{\phi}} = (\widehat{\beta}, \widehat{\gamma}, \widehat{\lambda}, \widehat{\rho})$ $= (0.3666, 8.4830, 0.9161, 0.7723)$	-34.55	77.1	85.8
EGP	$\widetilde{\boldsymbol{\phi}}_1 = (\widetilde{\beta}_1, \widetilde{\lambda}_1, \widetilde{\rho}_1)$ $= (0.7791, 0.0011, -0.9902)$	-106.50	219	225.5

Table 6: Likelihood ratio and Rao's score tests for $H_0 : \gamma = 1$ (EGS) for data set 2

Test	Statistic	p-value	Decision
Rao's score	58.616	<0.001	Reject H_0
LR	166.180	<0.001	Reject H_0

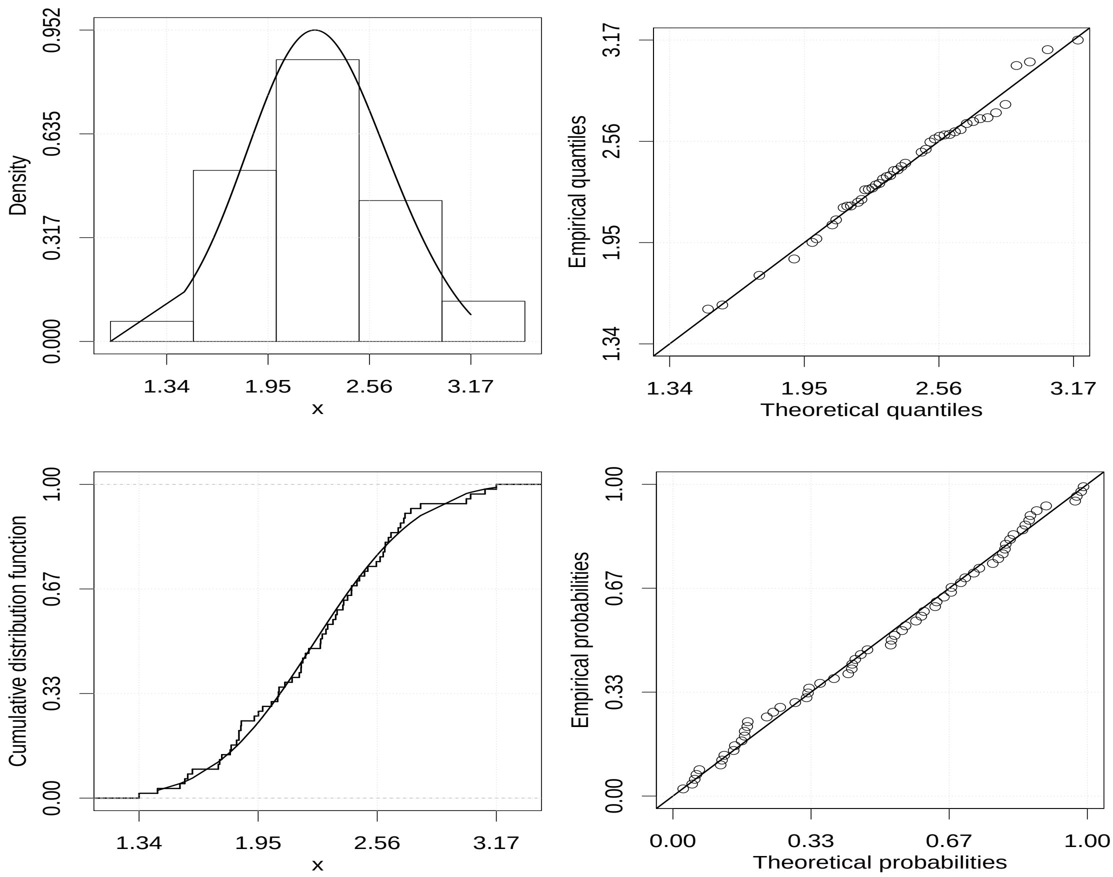

Figure 7: Goodness-of-fit plots for WGS model for data set 2

7. Conclusion and comments

In this paper, we have proposed a Weibull-generalized Sibuya distribution to analyze survival data where only the minimum of the sample is observed and the sample size is random. The resulting model is very flexible with decreasing, increasing, bathtub and upside-down bathtub shaped failure rate. The generalized Sibuya distribution, which is used in proposing the present model, is relatively new and has not received much attention in the literature. We hope that our proposed model will be a viable alternative to the existing models available in the literature and will be useful to the data analysts.

Acknowledgment

We are grateful to the editor and anonymous referees for their valuable comments which greatly improved the presentation of the paper.

Appendix A

The expressions (19) and (20) follow directly

$$\begin{aligned} D_1(x,\boldsymbol{\theta}) &= \frac{\partial}{\partial\beta}\ {}_2F_1(a,b;c;e^{-(\beta x)^\gamma}) \\ &= \frac{ab}{c}\ {}_2F_1(a+1,b+1;c+1;e^{-(\beta x)^\gamma})\ e^{-(\beta x)^\gamma}\ \left(-\frac{\gamma}{\beta}\ (\beta x)^\gamma\right), \\ D_2(x,\boldsymbol{\theta}) &= \frac{\partial}{\partial\gamma}\ {}_2F_1(a,b;c;e^{-(\beta x)^\gamma}) \\ &= \frac{ab}{c}\ {}_2F_1(a+1,b+1;c+1;e^{-(\beta x)^\gamma})\ e^{-(\beta x)^\gamma}\ (-\ (\beta x)^\gamma\ \ln(\beta x)) \\ &= \frac{\beta}{\gamma}\ \ \ln(\beta x)\ D_1(x,\boldsymbol{\theta}), \end{aligned}$$

where $a=1,\ b=\nu+1-\alpha,\ c=\nu+2$.

To prove expression (21), let

$$\psi_0(z)=\frac{d}{dz}\ln\Gamma(z)=\frac{\frac{d}{dz}\Gamma(z)}{\Gamma(z)},$$

be the polygamma function of order 0. Since

$$\begin{aligned} \frac{\partial}{\partial\alpha}(\nu+1-\alpha)_k &= \frac{\partial}{\partial\alpha}\ \frac{\Gamma(\nu+1-\alpha+k)}{\Gamma(\nu+1-\alpha)} \\ &= (\nu+1-\alpha)_k\ [\psi_0(\nu+1-\alpha)-\psi_0(\nu+1-\alpha+k)], \end{aligned}$$

it follows that

$$\begin{aligned} D_3(x,\boldsymbol{\theta}) &= \frac{\partial}{\partial\alpha}\ {}_2F_1(2,\nu+1-\alpha;\nu+2;e^{-(\beta x)^\gamma}) \\ &= \sum_{k=0}^{\infty}\frac{(2)_k}{(\nu+2)_k}\ \left[\frac{\partial}{\partial\alpha}(\nu+1-\alpha)_k\right]\ \frac{e^{-k(\beta x)^\gamma}}{k!} \\ &= \sum_{k=0}^{\infty}\frac{(2)_k\ (\nu+1-\alpha)_k}{(\nu+2)_k}\ [\psi_0(\nu+1-\alpha)-\psi_0(\nu+k+1-\alpha)]\ \frac{e^{-k(\beta x)^\gamma}}{k!} \\ &= {}_2F_1(2,\nu+1-\alpha;\nu+2;e^{-(\beta x)^\gamma})\ \psi_0(\nu+1-\alpha) \\ &\qquad -\sum_{k=0}^{\infty}\frac{(2)_k\ (\nu+1-\alpha)_k}{(\nu+2)_k}\ \psi_0(\nu+k+1-\alpha)\ \frac{e^{-k(\beta x)^\gamma}}{k!}. \end{aligned}$$

To prove expression (22), we proceed as follows. Since

$$\frac{\partial}{\partial\nu}\frac{(\nu+1-\alpha)_k}{(\nu+2)_k}=\frac{(\nu+1-\alpha)_k}{(\nu+2)_k}\ [\psi_0(\nu+2)-\psi_0(\nu+1-\alpha)-\psi_0(\nu+2+k)+\psi_0(\nu+1-\alpha+k)].$$

we have

$$\begin{aligned} D_4(x,\boldsymbol{\theta}) &= \frac{\partial}{\partial\nu}\ {}_2F_1(2,\nu+1-\alpha;\nu+2;e^{-(\beta x)^\gamma}) \\ &= \sum_{k=0}^{\infty}(2)_k\ \left[\frac{\partial}{\partial\nu}\frac{(\nu+1-\alpha)_k}{(\nu+2)_k}\right]\ \frac{e^{-k(\beta x)^\gamma}}{k!} \\ &= \sum_{k=0}^{\infty}\frac{(2)_k\ (\nu+1-\alpha)_k}{(\nu+2)_k}\ \big[\psi_0(\nu+2)-\psi_0(\nu+1-\alpha) \\ &\qquad -\psi_0(\nu+2+k)+\psi_0(\nu+1-\alpha+k)\big]\ \frac{e^{-k(\beta x)^\gamma}}{k!} \\ &= {}_2F_1(2,\nu+1-\alpha;\nu+2;e^{-(\beta x)^\gamma})\ \psi_0(\nu+2) \\ &\qquad -\sum_{k=0}^{\infty}\frac{(2)_k\ (\nu+1-\alpha)_k}{(\nu+2)_k}\ \psi_0(\nu+2+k)\ \frac{e^{-k(\beta x)^\gamma}}{k!}-D_3(x,\boldsymbol{\theta}). \end{aligned}$$

Appendix B

This appendix presents the R codes used to calculate the density function (dweibullGS1), distribution function (pweibullGS1), quantile function (qweibullGS1) and random number generation (rweibullGS1) for the Weibull-generalized Sibuya distribution. Thus, the fitdistrplus package (Delignette-Muller and Dutang 2015) can be used to obtain the MLEs and associated goodness-of-fit plots. The package gsl (Hankin 2006) is required.

```
##  Weibull-GS1 pdf ---  Equation 11
dweibullGS1 <- function(x, beta, gamma, alpha, nu)
{
    if(alpha < 0 || alpha > nu +1 ) return(NaN)
    aux0 <- beta * x;
    aux1 <- aux0 ^ gamma;
    eaux <- exp(-aux1);
    F21 <- sapply(x, function(xx) hyperg_2F1(2, nu + 1 - alpha,
                    nu + 2, exp(-(beta * xx) ^ gamma)));
    alpha / (nu + 1) * beta * gamma * aux1 / aux0 * eaux * F21;
}

## Weibull-GS1 cdf --- Equation 10
pweibullGS1 <- function(q, beta, gamma, alpha, nu)
{
    F21 <- sapply(q, function(x) hyperg_2F1(1, nu + 1 - alpha,
                    nu + 2, exp(-(beta * x) ^ gamma)))
    return(1 - alpha / (nu + 1) * exp(-(beta * q) ^ gamma) * F21);
}

## Weibull-GS1 qtf
## Calculated by solving pweibullGS1(q, beta, gamma, alpha, nu) - p = 0
qweibullGS1 <- function (p, beta, gamma, alpha, nu, L = 1e-04, U = 50)
{
    fx <- function(p)
    {
        tryCatch(uniroot(function(q) pweibullGS1(q, beta, gamma, alpha, nu) - p,
                    lower = L, upper = U)$root, error = function(e) NaN)
    }
        qtf <- sapply(p, fx)
    return(qtf)
}

## Weibull-GS1 random deviates (see Section 6)
rweibullGS1 <- function(n, beta, gamma, alpha, nu, tol = 1000)
{
    i <- 0; X <- c();
    while(i <= n)
    {
      Z <- rGSibuya(1, alpha, nu);
      if(Z < 1 || Z >= tol) next;
      Y <- rweibull(Z, scale = 1 / beta, shape = gamma);
      X[i] <- min(Y);
      i <- i + 1;
    }
    return(X);
}
```

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