Austrian Journal of Statistics
January 2022, Volume 51, 124–143.
http://www.ajs.or.at/
doi:10.17713/ajs.v51i2.1253

Applications of HLMOL-X Family of Distributions to Time Series, Acceptance Sampling and Stress-strength Parameter

Lishamol Tomy
Department of Statistics,
Deva Matha College

Meenu Jose
Department of Statistics,
Carmel College, Mala

Abstract

In this paper, the applications of the half logistic-Marshall Olkin X family of distributions are investigated with special emphasis to the half logistic-Marshall Olkin Lomax distribution. The specific areas we concentrated are time series modeling, acceptance sampling plan and stress- strength analysis. Different autoregressive minification structures of order one are introduced. The acceptance sampling plan is detailed by considering life time of products following the half logistic-Marshall Olkin Lomax distribution. The stress-strength reliability of the half logistic-Marshall Olkin Lomax distribution is derived and estimated. A simulation study is carried out to examine the bias, mean square error, average confidence length and coverage probability of the maximum likelihood estimator of the stress-strength reliability. Finally a real-life data analysis has also been presented.

Keywords: half logistic-Marshall Olkin X family of distribution, half logistic-Marshall Olkin Lomax distribution, autoregressive minification process, acceptance sampling plan, stress-strength analysis.

1. Introduction

Many real life phenomena are well described by statistical distributions. Although in case where existing distributions are found inadequate for a phenomenon, new generated classes of distributions are defined to meet the requirements. Such extended distributions are proved to be extremely useful for modeling real life situations by many authors. Several generators are existing in statistical literature. Two well known generators are, the Marshall-Olkin generator (MO-G) by Marshall and Olkin (1997) and the transformer (T-X) by Alzaatreh, Lee, and Famoye (2013).

For the last decades, there has been increasing interest in developing time series models for real valued observations using non-Gaussian distributions and the reason behind this is that many naturally occurring time series are non-Gaussian with Markovian structure. The pioneering work of autoregressive models with minification structure were proposed by Tavares (1980), and it followed by Sim (1986), Yeh, Arnold, and Robertson (1988), Arnold and Robertson (1989), Pillai (1991), Jose, Naik, and Ristić (2010) and others .

The acceptance sampling plan is an important tool in statistical quality control because it helps manufactures to minimize variability and protect the outgoing quality of their products. It is a sampling inspection procedure for determining the acceptability of the product. The acceptance sampling plans have been investigated in the past few decades by many authors, for instance Rosaiah, Gadde, Kalyani, and Kumar (2018), Rosaiah and Kantam (2005), Gillariose and Tomy (2018), Jose and Joseph (2018), Jose and Sebastian (2011) and Jose, Tomy, and Thomas (2018).

When assessing system reliability, a satisfactory performance is done when the strength applied to the component exceeds stress. Suppose that X represents the strength of a component with a stress Y, then R=$P(X > Y)$ can be considered as a measure of reliability of system. The system becomes out of control if the system stress exceeds its strength. Since R represents a relation between the stress and strength of a system. The estimation of the stress-strength reliability R has received considerable attention in the statistical literature. The pioneering work is given by Birnbaum *et al.* (1956) and Birnbaum, McCarty *et al.* (1958) .

Tomy and Jose (2020) introduced a new family of distributions called T-Marshall Olkin X family of distributions, it having the properties contained in both Marshall-Olkin and T-X family of distributions. They showed that several families of distributions can be derived from T-Marshall Olkin X family for different choices of variable T. In this article as a special case, the half logistic-Marshall Olkin X (HLMO-X) family of distributions is investigated. The cumulative density function (CDF) of the HLMO-X family of distributions by Tomy and Jose (2020) is given by

$$R(x) = \frac{1 - \left\{\frac{c(1-F(x))}{c+(1-c)F(x)}\right\}^{\lambda}}{1 + \left\{\frac{c(1-F(x))}{c+(1-c)F(x)}\right\}^{\lambda}} \tag{1}$$

where F(x) is the CDF of a random variable X. For convenience one special model of this family, the half logistic-Marshall Olkin Lomax (HLMOL) distribution, is studied in detail.

The Lomax distribution is one of the most commonly used distributions to model lifetime data and it has applications in several fields such as lifetime and reliability modeling, biological sciences and actuarial sciences. The CDF of the Lomax distribution is given by.

$$F(x) = 1 - [1 + \frac{x}{\theta}]^{-\alpha};\ x > 0,\ \alpha, \theta > 0$$

The HLMOL distribution has CDF given by

$$R(x) = \frac{\left[(1 + \frac{x}{\theta})^{\alpha} + c - 1\right]^{\lambda} - c^{\lambda}}{\left[(1 + \frac{x}{\theta})^{\alpha} + c - 1\right]^{\lambda} + c^{\lambda}} \tag{2}$$

The corresponding probability density function (PDF) is

$$r(x; c, \lambda, \alpha, \theta) = \frac{2\lambda\alpha c^{\lambda}}{\theta} \frac{\left[(1 + \frac{x}{\theta})^{\alpha} + c - 1\right]^{\lambda-1} [1 + \frac{x}{\theta}]^{\alpha-1}}{\left[\left[(1 + \frac{x}{\theta})^{\alpha} + c - 1\right]^{\lambda} + c^{\lambda}\right]^{2}};\ x > 0,\ c, \lambda, \alpha, \theta > 0 \tag{3}$$

With this context the main motivation behind this study is to investigate the diverse applications of the HLMO-X family of distributions in various fields like time series, quality control and reliability.

The paper unfolds as follows: In Section 2, we consider some applications of the HLMOL distribution in time series modeling. Section 3 presents the acceptance sampling plan of HLMOL distribution. In Section 4, The derivation and estimation of stress-strength reliability parameter R are given. The conclusion of the paper appears in Section 5.

2. Autoregressive time series modeling

Here, we develop different autoregressive minification processes of order one with HLMOL distribution as marginal distribution. We call the processes as HLMOL AR(1) Processes. Now we have the following theorem.

Theorem 2.1. *Consider an AR(1) structure given below*

$$X_n = \begin{cases} \varepsilon_n & \textit{with probability c} \\ min(X_{n-1}, \varepsilon_n) & \textit{with probability 1-c, } 0 \leq c \leq 1,\ n \geq 1 \end{cases} \tag{4}$$

where $\{\varepsilon_n\}$ is a sequence of independent and identically distributed (iid) random variables and is independent of $\{X_n\}$. Then the process is stationary AR(1) minification process with HLMOL(1,α,θ,c) as marginals if and only if ε_n is distributed as half logistic Lomax(α, θ) and $X_0 \underline{\underline{d}}$ HLMOL(1,α,θ,c).

Proof. From (4) it follows that

$$P(X_n > x) = cP(\varepsilon_n > x) + (1-c)P(X_{n-1} > x)P(\varepsilon_n > x)$$

That is,

$$\bar{R}_{X_n}(x) = \bar{R}_{\varepsilon_n}(x)[c + (1-c)\bar{R}_{X_{n-1}}(x)] \tag{5}$$

If the process is stationary with HLMOL(1,α,θ,c) marginals, then

$$\begin{aligned} \bar{R}_{\varepsilon_n}(x) &= \frac{\bar{R}_X(x)}{c + (1-c)\bar{R}_X(x)} \\ &= \frac{\frac{2c}{(1+\frac{x}{\theta})^\alpha + 2c - 1}}{c + (1-c)\frac{2c}{(1+\frac{x}{\theta})^\alpha + 2c - 1}} \\ &= \frac{2}{(1+\frac{x}{\theta})^\alpha + 1} \end{aligned} \tag{6}$$

Which is the survival function of half logistic Lomax(α, θ) distribution.

Coversely, If $\varepsilon_n(x)'s$ are iid random variables follows half logistic Lomax(α, θ) distribution with $X_0 \underline{\underline{d}}$ HLMOL(1,α,θ,c), then from (5), we have

$$\begin{aligned} \bar{R}_{X_1}(x) &= \bar{R}_{\varepsilon_1}(x)[c + (1-c)\bar{R}_{X_0}(x)] \\ &= \frac{2}{(1+\frac{x}{\theta})^\alpha + 1}\left\{c + (1-c)\frac{2c}{(1+\frac{x}{\theta})^\alpha + 2c - 1}\right\} \\ &= \frac{2}{(1+\frac{x}{\theta})^\alpha + 1}\left\{\frac{c(1+\frac{x}{\theta})^\alpha - c + 2c}{(1+\frac{x}{\theta})^\alpha + 2c - 1}\right\} \\ &= \frac{2c}{(1+\frac{x}{\theta})^\alpha + 2c - 1} \end{aligned} \tag{7}$$

That is X_1 has HLMOL(1,α,θ,c) distribution.
Similarly if X_{n-1} has HLMOL(1,α,θ,c) distribution, we get X_n also has HLMOL(1,α,θ,c) distribution. Hence the process $\{X_n\}$ is stationary with HLMOL marginals. □

The corresponding sample paths are given in Figure 1. The sample path behaviour of the process seems to be distinct and is adjustable through the parameters c,α and θ.

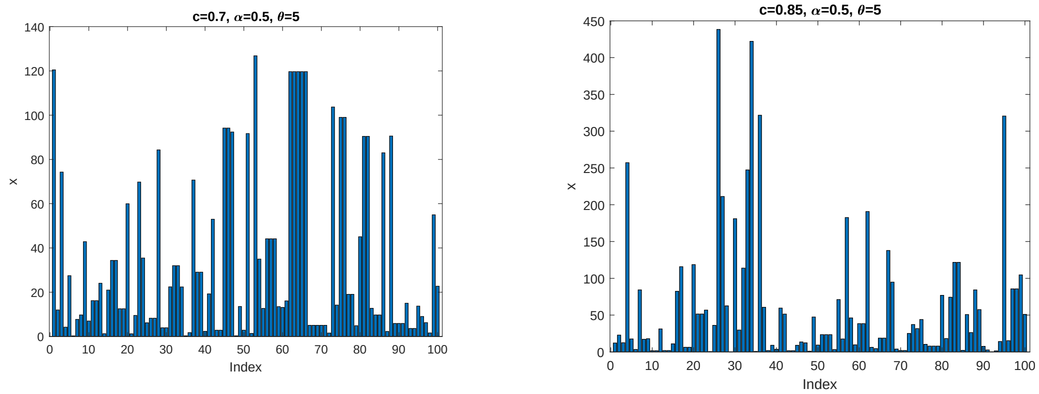

Figure 1: Sample path behaviour of HLMOL minification process given in Theorem 2.1

The following theorem gives a first order autoregressive minification process with marginals following HLMOL(λ, α, θ, β) distribution.

Theorem 2.2. *Consider an AR(1) structure given below*

$$X_n = \begin{cases} \varepsilon_n & \textit{with probability } c \\ min(X_{n-1}, \varepsilon_n) & \textit{with probability 1-c, } 0 \leq c \leq 1, \ n \geq 1 \end{cases} \tag{8}$$

where $\{\varepsilon_n\}$ *is a sequence of iid distributed random variables and is independent of* $\{X_n\}$. *Then the process is stationary AR(1) minification process with HLMOL(*λ, α, θ, β*) as marginals if and only if* $\varepsilon_n(x)'s$ *having the survival function,*

$$\bar{R}_{\varepsilon_n}(x) = \frac{2\beta^\lambda}{c[(1+\frac{x}{\theta})^\alpha + \beta - 1]^\lambda + (2-c)\beta^\lambda} \tag{9}$$

and $X_0 \stackrel{d}{=}$ *HLMOL (*λ, α, θ, β*).*

Proof. From (8) it follows that

$$P(X_n > x) = cP(\varepsilon_n > x) + (1-c)P(X_{n-1} > x)P(\varepsilon_n > x)$$

That is,

$$\bar{R}_{X_n}(x) = \bar{R}_{\varepsilon_n}(x)[c + (1-c)\bar{R}_{X_{n-1}}(x)] \tag{10}$$

If the process is stationary with HLMOL(λ, α, θ, β) marginals, then

$$\begin{aligned} \bar{R}_{\varepsilon_n}(x) &= \frac{\bar{R}_X(x)}{c + (1-c)\bar{R}_X(x)} \\ &= \frac{\frac{2\beta^\lambda}{[(1+\frac{x}{\theta})^\alpha+\beta-1]^\lambda+\beta^\lambda}}{c + (1-c)\frac{2\beta^\lambda}{[(1+\frac{x}{\theta})^\alpha+\beta-1]^\lambda+\beta^\lambda}} \\ &= \frac{2\beta^\lambda}{c[(1+\frac{x}{\theta})^\alpha + \beta - 1]^\lambda + (2-c)\beta^\lambda} \end{aligned} \tag{11}$$

That is, $\varepsilon_n(x)$'s are iid random variables having survival function given in (11).

Coversely, If $\varepsilon_n(x)'s$ are iid random variables having survival functions given in (11) with $X_0 \stackrel{d}{=}$ HLMOL(λ, α, θ, β), then from (10), we have

$$\bar{R}_{X_1}(x) = \bar{R}_{\varepsilon_1}(x)[c + (1-c)\bar{R}_{X_0}(x)] \tag{12}$$

$$= \frac{2\beta^\lambda}{c[(1+\frac{x}{\theta})^\alpha + \beta - 1]^\lambda + (2-c)\beta^\lambda}\left\{c + (1-c)\frac{2\beta^\lambda}{[(1+\frac{x}{\theta})^\alpha + \beta - 1]^\lambda + \beta^\lambda}\right\} \tag{13}$$

$$= \frac{2\beta^\lambda}{[(1+\frac{x}{\theta})^\alpha + \beta - 1]^\lambda + \beta^\lambda} \tag{14}$$

That is X_1 has HLMOL$(\lambda, \alpha, \theta, \beta)$ distribution.
Similarly if X_{n-1} has HLMOL$(\lambda, \alpha, \theta, \beta)$ distribution, we get X_n also has HLMOL$(\lambda,\ \alpha,\ \theta,$ $\beta)$ distribution. Hence the process $\{X_n\}$ is stationary with HLMOL marginals. □

The corresponding sample paths are given in Figure 2. The sample path behaviour of the process seems to be distinct and is adjustable through the parameters c, λ, α, θ and β.

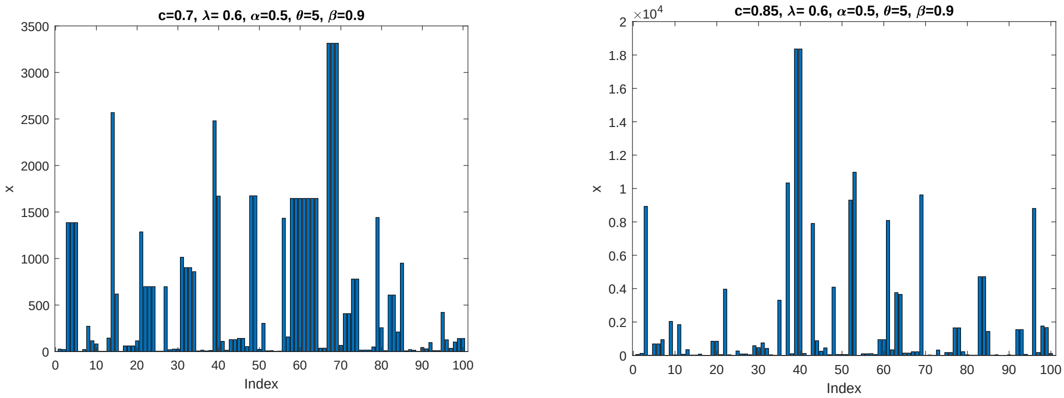

Figure 2: Sample path behaviour of HLMOL minification process given in Theorem 2.2

Krishnarani and Jayakumar (2008) gives a general class of autoregressive process with the following monotone transformation $\phi(x) = log\frac{R(x)}{\bar{R}(x)}$ where R(x) is a nondegenerate CDF, $\phi(-\infty) = -\infty$, $\phi(\infty) = \infty$ and $\bar{R}(x) = 1 - R(x)$. The corresponding markov process is

$$X_n = \begin{cases} \phi^{-1}[\phi(X_{n-1}) - log(c)] & \text{with probability c} \\ min[\phi^{-1}(\phi(X_{n-1}) - log(c)), \varepsilon_n] & \text{with probability 1-c, } 0 < c < 1, \end{cases} \tag{15}$$

where $\{\varepsilon_n\}$ is a sequences of iid random variables with CDF R , ε_n independent of X_i's, i=0, 1, 2,...n-1 with X_0 having distribution function R.
We use this concept in HLMOL(λ, α, θ, c) distribution. Then we get Theorem 2.3.

Theorem 2.3. *Consider an AR(1) structure given below*

$$X_n = \begin{cases} \theta\Big\{\Big[\big[\frac{[(1+\frac{X_{n-1}}{\theta})^{\alpha}+c-1]^{\lambda}+c^{\lambda}(c-1)}{c}\big]^{1/\lambda}+1-c\Big]^{1/\alpha}-1\Big\} & \textit{with probability c} \\ min\Big\{\theta\Big\{\Big[\big[\frac{[(1+\frac{X_{n-1}}{\theta})^{\alpha}+c-1]^{\lambda}+c^{\lambda}(c-1)}{c}\big]^{1/\lambda}+1-c\Big]^{1/\alpha}-1\Big\}, \varepsilon_n\Big\} & \textit{with probability 1-c} \end{cases} \tag{16}$$

where $0 < c < 1$, $\{\varepsilon_n\}$ is a sequence of iid distributed random variables and $\varepsilon_n$is independent of X_i, i=0, 1, 2,..n-1. Then the process is stationary AR(1) minification process with HLMOL(λ,α,θ,c) as marginals if and only if $\{\varepsilon_n\}$ is distributed as HLMOL(λ,α,θ,c) and X_0 $\underline{\underline{d}}\ \varepsilon_1$.

Proof. From (16) it follows that

$$\begin{aligned} P(X_n > x) = cP\Big\{X_{n-1} > \theta\big\{\big[[c((1+\frac{x_{n-1}}{\theta})^{\alpha}+c-1)^{\lambda}-c^{\lambda}(c-1)]^{1/\lambda}+1-c\big]^{1/\alpha}-1\big\}\Big\} \\ +(1-c)P\Big\{X_{n-1} > \theta\big\{\big[[c((1+\frac{x_{n-1}}{\theta})^{\alpha}+c-1)^{\lambda}-c^{\lambda}(c-1)]^{1/\lambda}+1-c\big]^{1/\alpha}-1\big\}\Big\} \\ P(\varepsilon_n > x) \end{aligned} \tag{17}$$

That is,

$$\bar{R}_{X_n}(x) = \bar{R}_{X_{n-1}}\Big\{\theta\big\{\big[[c((1+\frac{x_{n-1}}{\theta})^{\alpha}+c-1)^{\lambda}-c^{\lambda}(c-1)]^{1/\lambda}+1-c\big]^{1/\alpha}-1\big\}\Big\}\Big[c+(1-c)\bar{R}_{\varepsilon_n}(x)\Big] \tag{18}$$

If the process is stationary with HLMOL(λ,α,θ,c) marginals, then

$$\begin{aligned}
\left[c+(1-c)\bar{R}_\varepsilon(x)\right] &= \frac{\bar{R}_X(x)}{\bar{R}_X\left\{\theta\left\{\left[[c((1+\frac{x}{\theta})^\alpha+c-1)^\lambda - c^\lambda(c-1)]^{1/\lambda}+1-c\right]^{1/\alpha}-1\right\}\right\}} \\
&= \frac{\frac{2c^\lambda}{[(1+\frac{x}{\theta})^\alpha+c-1]^\lambda+c^\lambda}}{\frac{2c^\lambda}{c[(1+\frac{x}{\theta})^\alpha+c-1]^\lambda-c^\lambda(c-2)}} \\
&= \frac{c[(1+\frac{x}{\theta})^\alpha+c-1]^\lambda-c^\lambda(c-2)}{[(1+\frac{x}{\theta})^\alpha+c-1]^\lambda+c^\lambda} \qquad (19)
\end{aligned}$$

That is, $\bar{R}_\varepsilon(x) = \frac{2c^\lambda}{[(1+\frac{x}{\theta})^\alpha+c-1]^\lambda+c^\lambda}$, which is the survival function of HLMOL$(\lambda,\alpha,\theta,c)$ distribution.

Conversely, If $\varepsilon_n(x)'s$ are iid random variables following HLMOL$(\lambda,\alpha,\theta,c)$ distribution with $X_0 \stackrel{d}{=}$ HLMOL$(\lambda,\alpha,\theta,c)$, then from (18), we have

$$\begin{aligned}
\bar{R}_{X_1}(x) &= \bar{R}_{X_0}\left\{\theta\left\{\left[[c((1+\frac{x_{n-1}}{\theta})^\alpha+c-1)^\lambda - c^\lambda(c-1)]^{1/\lambda}+1-c\right]^{1/\alpha}-1\right\}\right\}\left[c+(1-c)\bar{R}_{\varepsilon_n}(x)\right] \\
&= \frac{2c^\lambda}{c[(1+\frac{x}{\theta})^\alpha+c-1]^\lambda-c^\lambda(c-2)}\left\{c+(1-c)\frac{2c^\lambda}{[(1+\frac{x}{\theta})^\alpha+c-1]^\lambda+c^\lambda}\right\} \\
&= \frac{2c^\lambda}{c[(1+\frac{x}{\theta})^\alpha+c-1]^\lambda-c^\lambda(c-2)}\left\{\frac{c[(1+\frac{x}{\theta})^\alpha+c-1]^\lambda+c^{\lambda+1}+2c^\lambda-2c^{\lambda+1}}{[(1+\frac{x}{\theta})^\alpha+c-1]^\lambda+c^\lambda}\right\} \\
&= \frac{2c^\lambda}{[(1+\frac{x}{\theta})^\alpha+c-1]^\lambda+c^\lambda} \qquad (20)
\end{aligned}$$

That is X_1 has HLMOL$(\lambda,\alpha,\theta,c)$ distribution.
Similarly if X_{n-1} has HLMOL$(\lambda,\alpha,\theta,c)$ distribution, we get X_n also has HLMOL$(\lambda,\alpha,\theta,c)$ distribution. Hence the process $\{X_n\}$ is stationary with HLMOL marginals. □

The corresponding sample paths are given in Figure 3. The sample path behaviour of the process seems to be distinct and is adjustable through the parameters c, λ, α and θ.

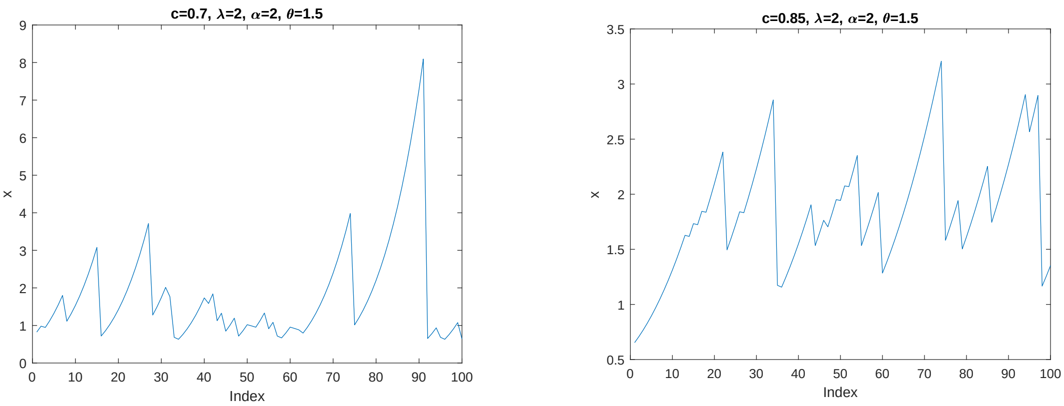

Figure 3: Sample path behaviour of HLMOL minification process given in Theorem 2.3

Theorem 2.4 stated below is a particaular case of Theorem 2.3, that is it gives a first order autoregressive minification process with marginals following HLMOL(1, α, θ, β) distribution.

Theorem 2.4. *Consider an AR(1) structure given below*

$$X_n = \begin{cases} \theta\left\{\frac{\left[(1+\frac{X_{n-1}}{\theta})^\alpha+c-1\right]^{1/\alpha}-c^{1/\alpha}}{c^{1/\alpha}}\right\} & \textit{with probability c} \\ min\left\{\theta\left\{\frac{\left[(1+\frac{X_{n-1}}{\theta})^\alpha+c-1\right]^{1/\alpha}-c^{1/\alpha}}{c^{1/\alpha}}\right\},\varepsilon_n\right\} & \textit{with probability 1-c} \end{cases} \qquad (21)$$

where 0 < c < 1, $\{\varepsilon_n\}$ is a sequence of iid distributed random variables and $\varepsilon_n$is independent of X_i, i=0, 1, 2,..n-1. Then the process is stationary AR(1) minification process with HLMOL(1,α,θ,c) as marginalas if and only if $\{\varepsilon_n\}$ is distributed as HLMOL(1,α,θ,c) and $X_0 \stackrel{d}{=} \varepsilon_1$.

Proof. The proof is similar to that of Theorem 2.3. □

The corresponding sample paths are given in Figure 4. The sample path behaviour of the process seems to be distinct and is adjustable through the parameters c, α and θ.

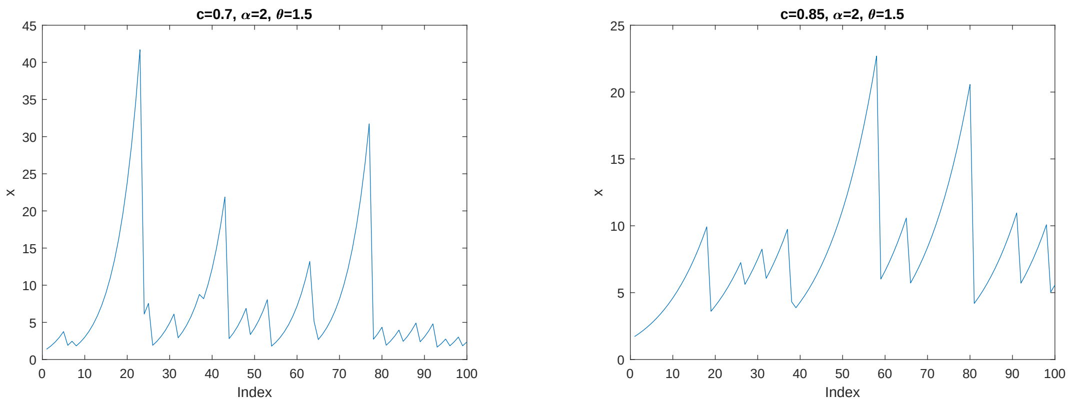

Figure 4: Sample path behaviour of HLMOL minification process given in Theorem 2.4

3. Acceptance sampling plan

In this section, we provide the acceptance sampling plan (ASP) under the assumption that lifetime of items follows a four parameter HLMOL($\lambda, \alpha, \theta, c$) distribution. The ASP involves determining the number of items to be inspected (n) and the maximum allowable number of defective items among the inspected items for acceptance of the item, that is the acceptance number (C). The test is terminated at a pre-specified time t and note the number of defective items(D). The decision procedure is to accept the lot if and only if at the end of the fixed time t, D does not exceed C, with a given probability $p^\star$. The test may get terminated before the time t is reached when D exceed C in which case we reject the lot. Here we are interested in obtaining the minimum sample size required to reach at the decision. It is assumed that the distribution parameters λ, α, c are known, while θ is unknown. In this case the average lifetime depends only on θ. Let θ_0 be the required minimum average lifetime, then the following holds:

$$R(t, \lambda, \alpha, \theta, c) \leq R(t, \lambda, \alpha, \theta_0, c) \iff \theta \geq \theta_0$$

An ASP consists of the following quantities

- The number of units 'n' on test
- The acceptance number ' C'
- The ratio $\frac{t}{\theta_0}$, where θ_0 is the specified average life and t is the maximum test duration

The probability of accepting a bad lot, that is consumers risk is not to exceed $1 - p^\star$, so that $p^\star$ is a minimum confidence level with which a lot of true average life θ below θ_0 is rejected, by the sampling plan. Therefore, for fixed $p^\star$, the ASP can be characterized by the triplet $(n, C, \frac{t}{\theta_0})$. Here we consider sufficiently large lots so that the binomial distribution can be applied. Then our aim is to find the minimum positive integer n for given values of $p^\star$ $(0 < p^\star < 1)$, θ_0 and C such that.

$$\sum_{i=0}^{C} \binom{n}{i} p_0^i (1 - p_0)^{n-i} \leq 1 - p^\star \tag{22}$$

where $p_0 = R(t, \lambda, \alpha, \theta_0, c)$, indicates the failure probabilities before time 't' which depends only on the ratio $\frac{t}{\theta_0}$. The minimum values of n satisfying the inequality (22) are obtained and displayed in Table 1 for $p^\star$=0.75, 0.90, 0.95, 0.99 and $\frac{t}{\theta_0}$ =0.4, 0.6, 0.8, 1.0, 1.2, 1.4, 1.6, 1.8, 2.0, C=0, 1, 2,...,10, $\lambda = 2, \alpha = 2$ and c=2.

If $p_0 = R(t, \lambda, \alpha, \theta_0, c)$ is small and n is large, the binomial probability may be approximated by Poisson probability with parameter $\beta = np$ so that (22) can be written as

$$\sum_{i=0}^{C} \frac{e^{-\beta}\beta^i}{i!} \leq 1 - p^\star \tag{23}$$

where $\beta = nR(t; \theta_0)$. The minimum values of 'n' satisfying (23) are obtained for the same combination of $p^\star$, $\frac{t}{\theta_0}$, C, λ, α and c values as those used for (22). The results are given in Table 2.

The operating characteristic (OC) function is the probability of accepting the lot with:

$$L(p) = \sum_{i=0}^{C} \binom{n}{i} p^i (1-p)^{n-i}$$

where $p = R(t, \lambda, \alpha, \theta, c)$, is considered as a function of θ, that is, the true average life of the lot. For given $p^\star$, $\frac{t}{\theta_0}$, the choice of C and n will be made on the basis of OC. Values of OC as a function of $d = \frac{\theta}{\theta_0}$ for the sampling plan $(n, 2, \frac{t}{\theta_0})$ with $\lambda = 2, \alpha = 2$ and c=2 are given in Table 3. Figure 5 shows the OC curves for the sampling plan $(n, C, 0.6)$ with $p^\star = 0.75$ for $\lambda = 2, \alpha = 2$ and c=2.

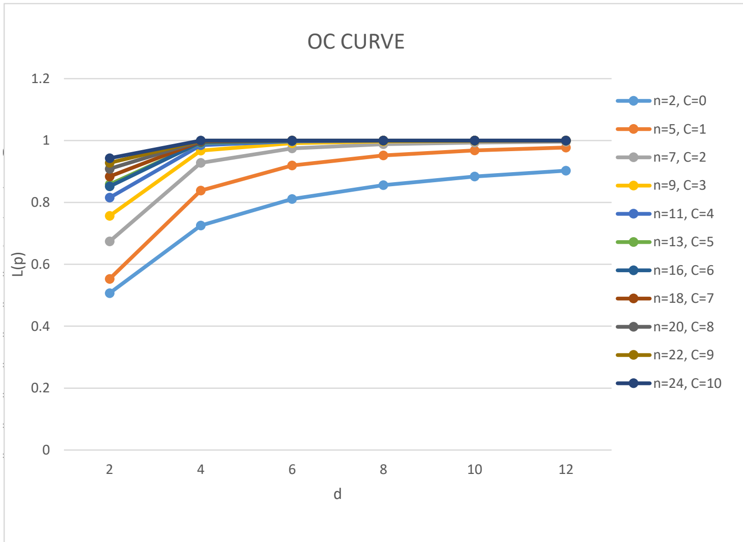

Figure 5: OC curves for C = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, respectively under $p^\star = 0.75$, $\frac{t}{\theta_0}$ = 0.6, $\lambda = 2$, $\alpha = 2$ and c = 2, of ASP for HLMOL distribution

The producer's risk is the probability of rejecting a good lot. For a specified value of the producer's risk, say 0.05, one may be interested in knowing what value of the ratio d will ensure a producer's risk less than or equal to 0.05 for a given sampling plan. Hence, the value of d, is the smallest positive number for which $p = R(\frac{t}{\theta_0}\frac{\theta_0}{\theta})$ holds the following inequality

$$\sum_{i=C+1}^{n} \binom{n}{i} p^i (1-p)^{n-i} \leq 0.05$$

Table 1: Minimum sample size for the specified ratio $\frac{t}{\theta_0}$, confidence level $p^\star$, acceptance number C, $\lambda = 2$, $\alpha = 2$ c $= 2$ using the binomial approximation

$p^\star$	C	$\frac{t}{\theta_0}$								
		0.4	0.6	0.8	1	1.2	1.4	1.6	1.8	2
0.75	0	3	2	2	2	1	1	1	1	1
	1	7	5	4	3	3	3	2	2	2
	2	10	7	6	5	4	4	4	4	3
	3	13	9	7	6	6	5	5	5	5
	4	16	11	9	8	7	6	6	6	6
	5	19	13	11	9	8	8	7	7	7
	6	22	16	12	11	10	9	9	8	8
	7	25	18	14	12	11	10	10	9	9
	8	28	20	16	14	12	12	11	11	10
	9	31	22	18	15	14	13	12	12	11
	10	34	24	19	17	15	14	13	13	12
0.9	0	5	4	3	2	2	2	2	1	1
	1	9	6	5	4	4	3	3	3	3
	2	13	9	7	6	5	5	4	4	4
	3	16	11	9	7	7	6	6	5	5
	4	20	14	11	9	8	7	7	7	6
	5	23	16	13	11	9	9	8	8	7
	6	26	18	14	12	11	10	9	9	9
	7	29	20	16	14	12	11	11	10	10
	8	33	23	18	15	14	13	12	11	11
	9	36	25	20	17	15	14	13	13	12
	10	39	27	21	18	16	15	14	14	13
0.95	0	7	5	3	3	2	2	2	2	2
	1	11	8	6	5	4	4	3	3	3
	2	15	10	8	7	6	5	5	4	4
	3	19	13	10	8	7	7	6	6	6
	4	22	15	12	10	9	8	7	7	7
	5	26	18	14	12	10	9	9	8	8
	6	29	20	16	13	12	11	10	9	9
	7	32	22	17	15	13	12	11	11	10
	8	36	25	19	16	15	13	13	12	11
	9	39	27	21	18	16	15	14	13	13
	10	42	29	23	20	17	16	15	14	14
0.99	0	10	7	5	4	3	3	3	2	2
	1	15	10	8	6	5	5	4	4	4
	2	20	13	10	8	7	6	6	5	5
	3	24	16	12	10	9	8	7	7	6
	4	27	18	14	12	10	9	9	8	8
	5	31	21	16	14	12	11	10	9	9
	6	35	24	18	15	13	12	11	11	10
	7	38	26	20	17	15	14	13	12	11
	8	42	29	22	19	16	15	14	13	13
	9	45	31	24	20	18	16	15	14	14
	10	49	33	26	22	19	18	16	16	15

Table 2: Minimum sample size for the specified ratio $\frac{t}{\theta_0}$, confidence level $p^\star$, acceptance number C, $\lambda = 2$, $\alpha = 2$ c $= 2$ using the Poisson approximation

$p^\star$	C	$\frac{t}{\theta_0}$								
		0.4	0.6	0.8	1	1.2	1.4	1.6	1.8	2
0.75	0	4	3	3	2	2	2	2	2	2
	1	8	6	5	4	4	4	4	3	3
	2	11	8	7	6	5	5	5	5	5
	3	14	10	9	8	7	7	6	6	6
	4	17	13	10	9	8	8	8	7	7
	5	20	15	12	11	10	9	9	9	9
	6	23	17	14	12	11v	11	10	10	10
	7	26	19	16	14	13	12	12	11	11
	8	29	21	17	15	14	13	13	12	12
	9	32	23	19	17	16	15	14	14	13
	10	35	26	21	18	17	16	15	15	15
0.9	0	7	5	4	4	3	3	3	3	3
	1	11	8	7	6	5	5	5	5	5
	2	15	11	9	8	7	7	7	6	6
	3	18	13	11	10	9	8	8	8	8
	4	22	13	13	12	11	10	10	9	9
	5	25	18	15	13	12	12	11	11	11
	6	29	21	17	15	14	13	13	12	12
	7	32	23	19	17	15	15	14	14	13
	8	35	25	21	18	17	16	15	15	15
	9	39	28	23	20	18	17	17	16	16
	10	42	30	25	22	20	19	18	18	17
0.95	0	9	6	5	5	4	4	4	4	3
	1	13	10	8	7	7	6	6	6	6
	2	17	13	10	9	8	8	8	7	7
	3	21	15	13	11	10	10	9	9	9
	4	25	18	15	13	12	11	11	11	10
	5	29	21	17	15	14	13	13	12	12
	6	32	23	19	17	15	15	14	14	13
	7	37	26	21	19	17	16	16	15	15
	8	39	28	23	20	19	18	17	16	16
	9	43	31	25	22	20	19	18	18	18
	10	46	33	27	24	22	21	20	19	19
0.99	0	13	9	8	7	6	6	6	6	5
	1	18	13	11	10	9	8	8	8	8
	2	23	17	14	12	11	11	10	10	10
	3	27	20	16	14	13	12	12	12	11
	4	32	23	19	17	15	14	14	13	13
	5	36	26	21	19	17	16	15	15	15
	6	40	29	23	21	19	18	17	17	16
	7	43	31	26	23	21	20	19	18	18
	8	47	34	28	25	23	21	20	20	19
	9	51	37	30	26	24	23	22	21	21
	10	54	39	32	28	26	25	24	23	22

Table 3: OC values for the ASP $(n, C, \frac{t}{\theta_0})$ for given confidence level $p^\star$, acceptance number C=2, $\lambda = 2$, $\alpha = 2$ c = 2

$p^\star$	n	$\frac{t}{\theta_0}$	d					
			2	4	6	8	10	12
0.75	10	0.4	0.6890	0.9306	0.9752	0.9885	0.9938	0.9963
	7	0.6	0.6741	0.9282	0.9746	0.9883	0.9937	0.9963
	6	0.8	0.5999	0.9056	0.9661	0.9844	0.9916	0.9949
	5	1	0.5919	0.9038	0.9657	0.9843	0.9915	0.9950
	4	1.2	0.6566	0.9251	0.9742	0.9884	0.9938	0.9964
	4	1.4	0.5559	0.8905	0.96093	0.9821	0.9904	0.9943
	4	1.6	0.4618	0.8504	0.9445	0.9742	0.9861	0.9917
	4	1.8	0.3780	0.8057	0.92518	0.9645	0.9807	0.9884
	3	2	0.6203	0.9085	0.9681	0.9857	0.9924	0.9956
0.9	13	0.4	0.5155	0.8676	0.9491	0.9756	0.9865	0.9918
	9	0.6	0.4952	0.8626	0.9477	0.9751	0.9863	0.9917
	7	0.8	0.4796	0.8583	0.9465	0.9746	0.9861	0.9916
	6	1	0.4402	0.8417	0.9398	0.9714	0.9844	0.9905
	5	1.2	0.4622	0.8524	0.9449	0.9742	0.9859	0.9916
	5	1.4	0.3492	0.7929	0.9188	0.9611	0.9786	0.9870
	4	1.6	0.4618	0.8504	0.9445	0.9742	0.9861	0.9917
	4	1.8	0.3780	0.8057	0.9251	0.9645	0.9807	0.9884
	4	2	0.3060	0.7577	0.9027	0.9531	0.9742	0.9844
0.95	15	0.4	0.4121	0.8178	0.9266	0.9639	0.9797	0.9875
	10	0.6	0.4154	0.8244	0.9306	0.9663	0.9813	0.9885
	8	0.8	0.3740	0.8051	0.9226	0.9624	0.9791	0.9872
	7	1	0.3152	0.7710	0.9073	0.9546	0.9746	0.9845
	6	1.2	0.3067	0.7662	0.9056	0.9540	0.9744	0.9844
	5	1.4	0.3492	0.7929	0.9188	0.9611	0.9786	0.9870
	5	1.6	0.2573	0.7279	0.8878	0.9449	0.9693	0.9812
	4	1.8	0.3780	0.8057	0.9251	0.9645	0.9807	0.9884
	4	2	0.3061	0.7577	0.9027	0.9531	0.9742	0.9844
0.99	20	0.4	0.2178	0.6797	0.8556	0.9247	0.9562	0.9724
	13	0.6	0.2298	0.6981	0.8676	0.9321	0.9610	0.9756
	10	0.8	0.2146	0.6890	0.8641	0.9306	0.9602	0.9752
	8	1	0.2190	0.6962	0.8692	0.9339	0.9624	0.9767
	7	1.2	0.1948	0.6741	0.8583	0.9282	0.9591	0.9746
	6	1.4	0.2052	0.6841	0.8645	0.9320	0.9615	0.9762
	6	1.6	0.1332	0.5999	0.8175	0.9056	0.9457	0.9661
	5	1.8	0.1860	0.6601	0.8524	0.9258	0.9581	0.9742
	5	2	0.1326	0.5919	0.8135	0.9038	0.9449	0.9657

that is,

$$\sum_{i=0}^{C} \binom{n}{i} p^i (1-p)^{n-i} \geq 0.95 \tag{24}$$

For some sampling plan (n, C, $\frac{t}{\theta_0}$) and values of $p^\star$, minimum values of $\frac{\theta}{\theta_0}$ satisfying (24) are given in Table 4.

3.1. Description of the tables and example

Assume that the lifetime distribution is HLMOL distribution with $\lambda = 2$, $\alpha = 2$ c = 2. Suppose that the experimenter is wants to establishing that the true unknown average life is at least 1000 hours with confidence $p^\star = 0.75$. It is desired to stop the experiment at t = 600 hours. Then, for an acceptance number c = 2, the required n in Table 1 corresponding to the values of $p^\star = 0.75$, $\frac{t}{\theta_0}$=0.6 and C=2 is 7. If, during 600 hours, no more than 2 failures out of 7 are observed, then the experimenter can assert, with a confidence level of 0.75 that the average life is at least 1000 hours. If the Poisson approximation to binomial probability is used, the value of n = 8 is obtained from Table 2 for the same situation.
Figure 6 shows that all the values of n tabulated by us corresponding to the sampling plan (n,C,1) with the confidence $p^\star = 0.75$ are found to be less than the corresponding values of n tabulated in Rosaiah *et al.* (2018) for Odds Exponential Log-Logistic (OELL) Distribution, Rosaiah and Kantam (2005)) for Inverse Rayleigh (IR) distribution, Jose and Sebastian (2011) for Marshall–Olkin Gumbel-maximum (MOGM) distribution, and Jose *et al.* (2018) for Harris extended Weibull (HEW) distribution. This improvement makes the new ASP more advantageous and helps in making optimal decisions.

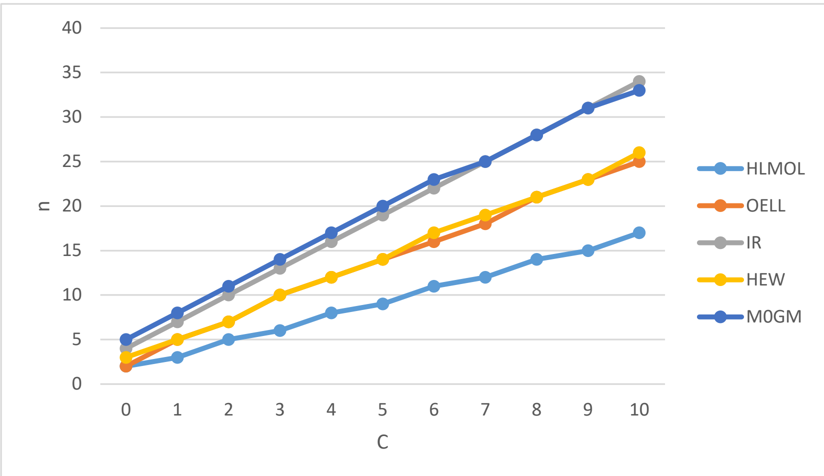

Figure 6: Comparisons between the sample sizes obtained by ASP for different distributions

For the sampling plan (n = 7, C = 2, $\frac{t}{\theta_0}$= 0.6) and confidence level $p^\star = 0.75$ under HLMOL distribution with $\lambda = 2$, $\alpha = 2$, c = 2, the values of the OC function from Table 3 are as given in Table 5.

Table 4: Minimum ratio of true average life to specified average life for the acceptability of a lot with producer's risk of 0.05

$p^\star$	C	$\frac{t}{\theta_0}$								
		0.4	0.6	0.8	1	1.2	1.4	1.6	1.8	2
0.75	0	28.569	28.565	38.087	47.609	27.247	31.788	36.329	40.870	45.411
	1	7.990	8.430	8.755	7.812	9.374	10.937	7.457	8.389	9.321
	2	4.689	4.965	5.461	5.629	5.084	5.931	6.779	7.626	5.555
	3	3.685	3.657	3.638	3.706	4.447	4.020	4.594	5.168	5.742
	4	3.125	3.108	3.171	3.436	3.445	3.182	3.637	4.092	4.546
	5	2.789	2.715	2.940	2.824	2.851	3.326	3.079	3.463	3.848
	6	2.550	2.663	2.505	2.777	2.920	2.883	3.295	3.001	3.335
	7	2.392	2.452	2.418	2.448	2.591	2.568	2.935	2.749	3.054
	8	2.265	2.328	2.336	2.448	2.309	2.694	2.668	3.001	2.766
	9	2.187	2.187	2.297	2.241	2.404	2.489	2.474	2.783	2.601
	10	2.080	2.102	2.090	2.241	2.231	2.305	2.296	2.583	2.415
0.9	0	45.453	54.542	55.167	45.444	54.532	63.621	72.710	40.870	45.411
	1	10.519	10.328	11.401	11.067	13.280	10.937	12.499	14.062	15.624
	2	6.545	6.640	6.620	6.923	6.755	7.881	6.779	7.626	8.473
	3	4.581	4.577	4.875	4.525	5.430	5.188	5.929	5.168	5.742
	4	3.980	4.039	4.137	3.963	4.124	4.020	4.594	5.168	4.546
	5	3.456	3.480	3.620	3.675	3.388	3.953	3.801	4.276	3.848
	6	3.051	3.020	3.039	3.132	3.333	3.407	3.295	3.706	4.118
	7	2.830	2.796	2.859	3.023	2.938	3.023	3.454	3.302	3.670
	8	2.730	2.728	2.697	2.688	2.938	3.048	3.078	3.001	3.335
	9	2.550	2.542	2.598	2.646	2.689	2.805	2.849	3.200	3.092
	10	2.422	2.430	2.377	2.433	2.437	2.602	2.634	2.964	2.870
0.95	0	63.491	68.179	55.167	68.020	54.532	63.621	72.710	81.798	90.887
	1	12.897	13.941	13.770	14.251	13.280	15.494	12.499	14.062	15.624
	2	7.537	7.293	7.727	8.275	8.368	7.881	9.007	7.626	8.473
	3	5.541	5.524	5.461	5.316	5.457	6.366	5.929	6.670	7.411
	4	4.427	4.339	4.503	4.547	4.756	4.811	4.594	5.168	5.742
	5	3.901	3.956	3.922	4.142	3.970	3.953	4.518	4.276	4.751
	6	3.456	3.418	3.551	3.436	3.758	3.888	3.893	3.706	4.118
	7	3.051	3.103	3.103	3.308	3.2787	3.427	3.454	3.886	3.669
	8	2.981	3.020	2.882	2.920	3.226	3.048	3.484	3.463	3.335
	9	2.789	2.796	2.739	2.871	2.920	3.137	3.206	3.200	3.556
	10	2.637	2.601	2.647	2.77	2.650	2.843	2.974	2.964	3.293
0.99	0	90.908	92.306	88.885	90.904	79.991	93.323	106.655	81.798	90.887
	1	18.177	17.637	18.588	17.213	16.859	19.395	17.707	19.920	22.134
	2	10.249	9.818	9.968	9.754	9.930	9.7624	11.157	10.133	11.259
	3	7.006	6.951	6.733	6.826	7.407	7.442	7.275	8.185	7.411
	4	5.465	5.324	5.423	5.629	5.457	2 5.549	6.341	6.186	6.873
	5	4.689	4.650	4.640	4.903	4.970	5.166	5.196	5.082	5.647
	6	4.192	4.244	4.092	4.142	4.124	4.384	4.445	4.999	4.867
	7	3.789	3.703	3.728	3.798	3.970	4.159	4.371	4.406	4.318
	8	3.518	3.544	3.421	3.602	3.504	3.764	3.917	3.919	4.355
	9	3.255	3.244	3.212	3.248	3.445	3.407	3.585	3.606	4.007
	10	3.125	3.020	3.103	3.132	3.125	3.376	3.250	3.656	3.718

Table 5: OC values for the ASP ($n = 7, C = 2, \frac{t}{\theta_0} = 0.6$) for given confidence level $p^\star = 0.75$, acceptance number C=2, $\lambda = 2$, $\alpha = 2$ c = 2

$\frac{\theta}{\theta_0}$	2	4	6	8	10	12
OC	0.6741	0.9282	0.9746	0.9883	0.9937	0.9963

Table 5 shows that if the true average life is twice the required mean lifetime ($\frac{\theta}{\theta_0} = 2$) the producer's risk is approximately 0.3259. The producer's risk is almost equal to 0.0063 when the true true average life is greater than or equal to 10 times the specified average life.

From Table 4, we can get the values of the ratio $\frac{\theta}{\theta_0}$ for different choices of C and $\frac{t}{\theta_0}$ in order to assert that the producer's risk was less than 0.05. For example if $p^\star = 0.75$, $\frac{t}{\theta_0} = 0.8$, C=2, Table 4 gives a reading of 5.461. This means the product can have an average life of 5.461 times the specified average lifetime in order that under the above acceptance sampling plan the product is accepted with probability of at least 0.95.

Practical example: Consider the following ordered failure times of the release of a software given in terms of hours(T) from the starting of the execution of the software denoting the times at which the failure of the software is experienced, it was presented by Wood (1996). This data can be regarded as an ordered sample of size 10 with observations ($t_i; i = 1, 2, ..., 10$): 519, 968, 1430, 1893, 2490, 3058, 3625, 4422, 5218, 5823. Let the specified average life be 1000 hours and the testing time be 600 hrs, this leads to ratio of $\frac{t}{\theta_0}$= 0.6 with corresponding n and C as 10, 2 from Table 1 for $p^\star = 0.95$. Therefore, the sampling plan for the above sample data is (n=10,C=2,$\frac{t}{\theta_0}$=0.6). Based on the 10 observations, we have to decide whether to accept the product or reject it. We accept the product only, if the number of failures before 600 hrs is less than or equal to 2. However, the confidence level is assured by the sampling plan only if the given life times follow HLMOL distribution. In order to confirm that the given sample is generated by lifetimes following at least approximately the HLMOL distribution, we have compared the sample quantiles and the corresponding population quantiles and found a satisfactory agreement. Thus, the adoption of the decision rule of the sampling plan seems to be justified. In the sample of 10 units, there is a 1 failure at 519 hours before t = 600 hours. Therefore we accept the product.

In the above example there is only one failure at 519 corresponding to the ASP for HLMOL (n=10,C=2,$\frac{t}{\theta_0}$=0.6) with confidence $p^\star$=0.95. If we compare it to the sampling plans suggested by Kantam, Rosaiah, and Rao (2001), Jose and Joseph (2018), Jose and Sebastian (2011), Rosaiah and Kantam (2005), Ravikumar, Kantam, and Durgamamba (2016) and Al-Nasser, Al-Omari, Bani-Mustafa, and Jaber (2018) correspnding to n=10, C=3, and $p^\star$=0.95. We can see that the termination time t in HLMOL sampling plan is smaller than the others.

4. Stress-strength reliability and its estimation

In this section, we derive and estimate the stress-strength reliability $R = P(X > Y)$. Let X and Y be two independent random variables with HLMO-X distribution with parameters c_1 and $\lambda = 1$, and HLMO-Y distribution parameters c_2 and $\lambda = 1$, that is, $X \sim$ HLMO-$X(1, C_1)$

and $Y \sim$ HLMO-$Y(1, C_2)$. Then, the stress strength reliability R is given by

$$
\begin{aligned}
R = P(X > Y) &= \int_0^\infty P(X > Y|Y = y) \;\; r(y) \; dy \\
&= \int_0^\infty \frac{2c_1\bar{F}(y)}{1 + (2c_1 - 1)\bar{F}(y)} \; \frac{2c_2 f(y)}{[1 + (2c_2 - 1)\bar{F}(y)]^2} \; dy \\
&= \int_0^1 \frac{4c_1c_2v}{[1 + (2c_1 - 1)v][1 + (2c_2 - 1)v]^2} \; dv \\
&= \frac{c_1c_2}{(c_2 - c_1)^2} \int_0^1 \frac{1 - 2c_1}{1 - (1 - 2c_1)v} - \frac{1 - 2c_2}{1 - (1 - 2c_2)v} - \frac{2c_2 - 2c_1}{[1 - (1 - 2c_2)v]^2} \; dv \\
&= \frac{c_1c_2}{(c_2 - c_1)^2} \Bigg[- ln(2c_1) + ln(2c_2) - (2c_2 - 2c_1) \frac{1 - 2c_2}{2c_2[1 - 2c_2]} \Bigg] \\
&= \frac{c_1/c_2}{(1 - c_1/c_2)^2} \Bigg[- ln\Big(\frac{c_1}{c_2}\Big) + \frac{c_1}{c_2} - 1 \Bigg] \qquad (25)
\end{aligned}
$$

4.1. Maximum likelihood estimation of R

The stress strength reliability R is the function of the parameters c_1 and c_2, respectively. Therefore, for maximum likelihood estimate (MLE) of R, we need to obtain the MLE of the parameters c_1 and c_2.

Suppose $x_1, x_2, ..., x_m$ is a random sample of size m from the HLMOL distribution with parameters $\lambda = 1$, α, θ and c_1, and $y_1, y_2, ..., y_n$ is a random sample of size n from the HLMOL distribution with parameters $\lambda = 1$, α, θ and c_2, and let α and θ be known.

Therefore, the log-likelihood function of the observed samples is given by

$$
\begin{aligned}
\ell(c_1, c_2) \propto \; & m\, ln\Big(\frac{2\alpha}{\theta}\Big) + m\, ln(c_1) + (\alpha - 1)\sum_{i=1}^{m} ln\Big(1 + \frac{x_i}{\theta}\Big) - 2\sum_{i=1}^{m} ln\Bigg[\Big(1 + \frac{x_i}{\theta}\Big)^{\alpha} + 2c_1 - 1\Bigg] \\
& + n\, ln\Big(\frac{2\alpha}{\theta}\Big) + n\, ln(c_2) + (\alpha - 1)\sum_{j=1}^{n} ln\Big(1 + \frac{y_j}{\theta}\Big) - 2\sum_{j=1}^{n} ln\Bigg[\Big(1 + \frac{y_i}{\theta}\Big)^{\alpha} + 2c_2 - 1\Bigg] \qquad (26)
\end{aligned}
$$

So, the MLEs of c_1 and c_2, say $\hat{c_1}$ and $\hat{c_2}$, respectively, can be obtained as the solutions of the nonlinear equations

$$
\frac{\partial \ell}{\partial c_1} = \frac{m}{c_1} - \sum_{i=1}^{m} \frac{4}{(1 + \frac{x_i}{\theta})^{\alpha} + 2c_1 - 1} = 0 \qquad (27)
$$

$$
\frac{\partial \ell}{\partial c_2} = \frac{n}{c_2} - \sum_{j=1}^{n} \frac{4}{(1 + \frac{y_j}{\theta})^{\alpha} + 2c_2 - 1} = 0 \qquad (28)
$$

Then the MLE of R is

$$
R = \frac{\hat{c_1}/\hat{c_2}}{(1 - \hat{c_1}/\hat{c_2})^2} \Bigg[- ln\Big(\frac{\hat{c_1}}{\hat{c_2}}\Big) + \frac{\hat{c_1}}{\hat{c_2}} - 1 \Bigg]
$$

The elements of Fishers information matrix are

$$I_{11} = -E\left(\frac{\partial^2 \ell}{\partial c_1^2}\right) = \frac{m}{c_1^2} - 8mE\left(\frac{1}{\left(1+\frac{X}{\theta}\right)^{\alpha} + 2c_1 - 1}\right)$$
$$= m\left(\frac{1}{c_1^2} - 16c_1\int_{2c_1}^{\infty}\frac{1}{u^4}du\right)$$
$$= m\left(\frac{1}{c_1^2} - \frac{2}{3c_1^2}\right)$$
$$= \frac{m}{3c_1^2} \tag{29}$$

$$I_{12} = I_{21} = -E\left(\frac{\partial^2 \ell}{\partial C_1 c_2}\right) = 0 \tag{30}$$

$$I_{22} = -E\left(\frac{\partial^2 \ell}{\partial c_2^2}\right) = \frac{n}{c_2^2} - 8mE\left(\frac{1}{\left(1+\frac{Y}{\theta}\right)^{\alpha} + 2c_2 - 1}\right)$$
$$= \frac{n}{3c_2^2} \tag{31}$$

Theorem 4.1. *As* $m \to \infty$ *and* $n \to \infty$*, then* $[\sqrt{m}(\hat{c_1} - c_1), \sqrt{n}(\hat{c_2} - c_2)] \xrightarrow{d} N_2\big(0, A^{-1}(c_1, c_2)\big)$ *where,*

$$A = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{22} \end{pmatrix}$$

and

$$a_{11} = \lim_{m,n\to\infty} \frac{I_{11}}{m} = \frac{1}{3c_1^2}, \quad a_{22} = \lim_{m,n\to\infty} \frac{I_{22}}{m} = \frac{1}{3c_2^2}$$

Proof. We can use the asymptotics properties of MLEs to prove it. □

To obtain the asymptotic 100(1-α) % confidence interval for R, we proceed as follows

$$b_1(c_1, c_2) = \frac{\partial R}{\partial c_1} = \frac{c_2}{(c_2 - c_1)^3}\left[-2(c_2 - c_1) - (c_1 + c_2)\ ln\ \left(\frac{c_1}{c_2}\right)\right]$$

,

$$b_2(c_1, c_2) = \frac{\partial R}{\partial c_2} = \frac{c_1}{(c_2 - c_1)^3}\left[2(c_2 - c_1) + (c_1 + c_2)\ ln\ \left(\frac{c_1}{c_2}\right)\right] = -\frac{c_1}{c_2}\ b_1(c_1, c_2)$$

.

Then,

$$V(\hat{R}) = V(\hat{c_1})\ b_1^2(c_1, c_2) + V(\hat{c_2})\ b_2^2(c_1, c_2)$$
$$= c_1^2\ b_1^2(c_1, c_2)\ \left(\frac{3}{m} + \frac{3}{n}\right).$$

Thus we have the following result.
As $m \to \infty$, $n \to \infty$, $\frac{\hat{R} - R}{c_1\ b_1(c_1, c_2)\ \sqrt{\frac{3}{m} + \frac{3}{n}}} \xrightarrow{d} N(0, 1)$ and the asymptotic 100(1-α) % confidence interval for R is given by

$$\hat{R} \pm Z_{(\alpha/2)}\ \hat{c_1}\ b_1(\hat{c_1}, \hat{c_2})\ \sqrt{\frac{3}{m} + \frac{3}{n}}$$

where $Z_{(\alpha/2)}$ is the $(1 - \alpha/2)^{th}$ percentiles of the standard normal distribution.
Hence, the asymptotic 95% confidence interval for R is given by

$$\hat{R} \pm 1.96\ \hat{c_1}\ b_1(\hat{c_1}, \hat{c_2})\ \sqrt{\frac{3}{m} + \frac{3}{n}}$$

4.2. Simulation study for R

Here, we mainly present some simulation experimentes to study the performance of the MLE estimator and confidence interval for R. The simulation experiment was repeated N=10000 times each with different sample sizes, (m,n)=(15,15), (20,20), (25,20), (25,25), (25,30), (30,20),(30,25), (30,30). The values of c_1 and c_2 were combinations of c_1=0.2, 0.5, 0.6 and c_2=0.4, 0.8, 0.5. We fixed the values of α and θ as, $\alpha = 4$ and $\theta = 1$. In this simulation study, we computed four measures: the average bias (Bias), average mean square error(MSE), average length of the asymptotic 95% confidence intervals and coverage probability of R.

Table 6: Bias and MSE of the simulated estimates of R for $\alpha = 4$ and $\theta = 1$

	(c_1, c_2)					
	Bias			MSE		
(m,n)	(0.2, 0.4)	(0.5, 0.8)	(0.6, 0.5)	(0.2, 0.4)	(0.5, 0.8)	(0.6, 0.5)
(15,15)	0.0394	0.0356	-0.0141	0.0085	0.0077	0.0068
(20,20)	0.0383	0.0355	-0.0140	0.0067	0.0066	0.0055
(25,20)	0.0398	0.0379	-0.0103	0.0062	0.0063	0.0052
(25,25)	0.0370	0.0353	-0.0129	0.0055	0.0059	0.0049
(25,30)	0.0354	0.0348	-0.0160	0.0051	0.0058	0.0047
(30,20)	0.0416	0.0411	-0.0075	0.0059	0.0064	0.0049
(30,25)	0.0386	0.0381	-0.0109	0.0053	0.0059	0.0046
(30,30)	0.0346	0.0345	-0.0125	0.0048	0.0056	0.0044

Table 7: Average confidence length and coverage probability of the simulated estimates of R for $\alpha = 4$ and $\theta = 1$

	(c_1, c_2)					
	Average confidence length			Coverage probability		
(m,n)	(0.2, 0.4)	(0.5, 0.8)	(0.6, 0.5)	(0.2, 0.4)	(0.5, 0.8)	(0.6, 0.5)
(15,15)	0.3945	0.4010	0.4029	0.9593	0.9773	0.9791
(20,20)	0.3437	0.3487	0.3506	0.9624	0.9688	0.9795
(25,20)	0.3272	0.3315	0.3327	0.9622	0.9635	0.9747
(25,25)	0.3085	0.3125	0.3142	0.9621	0.9591	0.9715
(25,30)	0.2954	0.2994	0.3013	0.9605	0.9571	0.9701
(30,20)	0.3156	0.3195	0.3204	0.9612	0.9556	0.9741
(30,25)	0.2960	0.2998	0.3011	0.9590	0.9526	0.9717
(30,30)	0.2818	0.2855	0.2862	0.9576	0.9485	0.9666

The Bias and MSE of are presented in Table 6. The average confidence lengths and coverage probabilities are reported for 95% confidence intervals using exact MLE and asymptotic distribution of R in Table 7. When $c_1 < c_2$, the bias is positive and when $c_1 > c_2$, the bias is negative. The equal (m=n) and unequal (m$\neq$ n) choices of sample sizes are taken to evaluate the estimates of R. From this extensive study, it has been observed that the Bias decreases with increasing sample size n and fixed sample size m, also Bias increases with increase sample size m and fixed n. In general, Bias, MSE and length of the confidence interval decreases as the sample size increases. It verifies the consistency property of the MLE of R. For small sample sizes (m,n), the coverage probabilities for the MLE's are slight less than nominal value, with the increase of sample sizes (m,n), they more close to the nominal value. We also observe that there is no substantial difference in the Bias, MSE, average confidence lengths and coverage probabilities of R for different choices of the parameters.

4.3. Practical data example for R

In this subsection, We consider the real-life data sets of the waiting times (in minutes) before

service of the customers of two diffrent banks A and B, given by Ghitany, Atieh, and Nadarajah (2008). We are interested in estimating the stress-strength reliability $R = P(X > Y)$ where X (Y) denotes the customer service time in Bank A (B). The data sets are given below
Bank A: X(m=100)
0.8, 0.8, 1.3, 1.5, 1.8, 1.9, 1.9, 2.1, 2.6, 2.7, 2.9, 3.1, 3.2,3.3, 3.5, 3.6,4.0, 4.1, 4.2, 4.2, 4.3, 4.3, 4.4, 4.4, 4.6, 4.7, 4.7, 4.8, 4.9, 4.9, 5.0, 5.3, 5.5, 5.7, 5.7, 6.1, 6.2, 6.2, 6.2, 6.3, 6.7, 6.9, 7.1, 7.1, 7.1, 7.1, 7.4, 7.6, 7.7, 8.0, 8.2, 8.6, 8.6, 8.6, 8.8, 8.8, 8.9, 8.9, 9.5, 9.6, 9.7, 9.8, 10.7, 10.9, 11.0, 11.0, 11.1, 11.2, 11.2, 11.5, 11.9, 12.4, 12.5, 12.9, 13.0, 13.1, 13.3, 13.6, 13.7, 13.9, 14.1, 15.4, 15.4, 17.3, 17.3, 18.1, 18.2, 18.4, 18.9, 19.0, 19.9, 20.6, 21.3, 21.4, 21.9, 23.0, 27.0, 31.6, 33.1, 38.5.
Bank B: Y (n=60)
0.1, 0.2, 0.3, 0.7, 0.9, 1.1, 1.2, 1.8, 1.9, 2.0, 2.2, 2.3, 2.3, 2.3, 2.5, 2.6, 2.7, 2.7, 2.9, 3.1, 3.1, 3.2, 3.4, 3.4, 3.5, 3.9, 4.0, 4.2, 4.5, 4.7, 5.3, 5.6, 5.6, 6.2, 6.3, 6.6, 6.8, 7.3, 7.5, 7.7, 7.7, 8.0, 8.0, 8.5, 8.5, 8.7, 9.5, 10.7, 10.9, 11.0, 12.1, 12.3, 12.8, 12.9, 13.2, 13.7, 14.5, 16.0, 16.5, 28.0.

We fitted the HLMOL distribution for each dataset. Let us first assume that $X \sim$ HLMOL$(\lambda = 1, \alpha = 4, \theta = 7, c_1)$ and $Y \sim$ HLMOL$(\lambda = 1, \alpha = 4, \theta = 7, c_2)$. We used the Anderson-Darling (A-D), Cramer-von Mises and Kolmogorov-Smirnov (K-S) statistics to test the goodness-of-fit and found that the HLMOL distribution is good fitted. The values of A-D, Cramer-von Mises and K-S statistics along P-value are given in Table 8

Table 8: Statistic(P-value) of different goodness-of-fit tests for the data sets

	A-D	Cramer-von Mises	K-S
Bank A(X)	0.2466(0.9721)	0.0311(0.9729)	0.0451(0.987)
Bank B(Y)	0.3349(0.9095)	0.0542(0.852)	0.0748(0.89)

The MLEs of the unknown parameters are $\hat{c_1}$=10.5963, $\hat{c_2}$=3.9109. Replacing the parameters by the estimates we get the MLE of the stress-strength reliability R as 0.6608 and the 95% confidence interval of R is (0.5771, 0.7445).

5. Conclusion

The paper considers the applications of HLMOL distribution in the fields of time series modeling, ASP and stress-strength analysis. In time series modeling, different autoregressive minification processes of order one are developed. These can be used for modelling time series data from different contexts. We developed a ASP for HLMOL distribution by assuming the lifetime of products following HLMOL distribution. For fixed confidence level, the minimum sample size to assert the ratio of specified mean life and the maximum test duration are calculated. The OC values with OC curves and minimum ratio of mean life to the specified life are tabulated. The results are illustrated using a data set. It is shown that the suggested ASP is useful in minimizing the producer's risk. Also, the proposed ASP is more economical than some of the existing ASPs.

In stress-strength analysis, we derive and estimate the stress-strength reliability parameter R based on two independent samples from HLMOL distribution with different parameters. The results for estimation of R by MLE is reported. From the simulation results, it is observed that as the sample size (m,n) increases the Bias, MSE and average confidence length decreases and, the performance of the coverage probability is satisfactory. Also, a real-life data analysis is presented for illustrative purpose.

References

Al-Nasser AD, Al-Omari AI, Bani-Mustafa A, Jaber K (2018). "Developing Single-acceptance Sampling Plans Based on a Truncated Lifetime Test for an Ishita Distribution." *STATISTICS*, **19**(3), 393–406.

Alzaatreh A, Lee C, Famoye F (2013). "A New Method for Generating Families of Continuous Distributions." *Metron*, **71**(1), 63–79.

Arnold BC, Robertson CA (1989). "Autoregressive Logistic Processes." *Journal of Applied Probability*, pp. 524–531.

Birnbaum ZW, McCarty RC, *et al.* (1958). "A Distribution-Free Upper Confidence Bound for $Pr(Y < X)$, Based on Independent Samples of X and Y." *The Annals of Mathematical Statistics*, **29**(2), 558–562.

Birnbaum ZW, *et al.* (1956). "On a Use of the Mann-Whitney Statistic." In *Proceedings of the Third Berkeley Symposium on Mathematical Statistics and Probability, Volume 1: Contributions to the Theory of Statistics*. The Regents of the University of California.

Ghitany ME, Atieh B, Nadarajah S (2008). "Lindley Distribution and Its Application." *Mathematics and Computers in Simulation*, **78**(4), 493–506.

Gillariose J, Tomy L (2018). "A New Life Time Model: The Generalized Rayleigh-Truncated Negative Binomial Distribution." *International Journal of Scientific Research in Mathematical and Statistical Sciences*, **5**(6), 1–12.

Jose KK, Joseph J (2018). "Reliability Test Plan for the Gumbel-uniform Distribution." *Stochastics and Quality Control*, **33**(1), 71–81.

Jose KK, Naik SR, Ristić MM (2010). "Marshall–Olkin q-Weibull Distribution and Max–min Processes." *Statistical Papers*, **51**(4), 837–851.

Jose KK, Sebastian R (2011). "Marshall–Olkin Gumbel-maximum Distribution." Submitted.

Jose KK, Tomy L, Thomas SP (2018). "On a Generalization of the Weibull Distribution and Its Application in Quality Control." *Stochastics and Quality Control*, **33**(2), 113–124.

Kantam RRL, Rosaiah K, Rao GS (2001). "Acceptance Sampling Based on Life Tests: Log-logistic Model." *Journal of Applied Statistics*, **28**(1), 121–128.

Krishnarani SD, Jayakumar K (2008). "A Class of Autoregressive Processes." *Statistics & Probability Letters*, **78**(12), 1355–1361.

Marshall AW, Olkin I (1997). "A New Method for Adding a Parameter to a Family of Distributions with Application to the Exponential and Weibull Families." *Biometrika*, **84**(3), 641–652.

Pillai RN (1991). "Semi-pareto Processes." *Journal of Applied Probability*, pp. 461–465.

Ravikumar MS, Kantam RRL, Durgamamba AN (2016). "Acceptance Sampling Plans for Burr Type X Distribution." *International Journal of Advanced Technology in Engineering and Science*, **4**(8), 36–44.

Rosaiah K, Gadde SR, Kalyani K, Kumar DCU (2018). "Odds Exponential Log-logistic Distribution: An Economic Reliability Test Plan." *International Journal of Science and Research (IJSR)*, **7**, 1653–1660.

Rosaiah K, Kantam RRL (2005). "Acceptance Sampling Based on the Inverse Rayleigh Distribution." *Stochastics and Quality Control*, **20**(2), 277–286.

Sim CH (1986). "Simulation of Weibull and Gamma Autoregressive Stationary Process." *Communications in Statistics-Simulation and Computation*, **15**(4), 1141–1146.

Tavares LV (1980). "An Exponential Markovian Stationary Process." *Journal of Applied Probability*, **17**(4), 1117–1120.

Tomy L, Jose M (2020). "T-Marshall Olkin X Family of Distributions with Properties and Applications." *STATISTICA*. Submitted.

Wood A (1996). "Predicting Software Reliability." *Computer*, **29**(11), 69–77.

Yeh HC, Arnold BC, Robertson CA (1988). "Pareto Processes." *Journal of Applied Probability*, pp. 291–301.

Affiliation:

Lishamol Tomy
Department of Statistics, Deva Matha College
Kuravilangad, Kerala, 686633, India
E-mail: lishatomy@gmail.com

Meenu Jose
Department of Statistics, Carmel College Mala
Thrissur, Kerala, India
E-mail: meenusgc@gmail.com

Austrian Journal of Statistics http://www.ajs.or.at/
published by the Austrian Society of Statistics http://www.osg.or.at/
Volume 51 *Submitted:* 2020-10-27
January 2022 *Accepted:* 2021-02-18